Astérisque

AST

Pages préliminaires

Astérisque, tome 77 (1980), p. 1-9

<http://www.numdam.org/item?id=AST_1980__77__1_0>

Many problems in analysis have natural formulations as questions of continuity of linear operators defined on spaces of functions or distributions. Such questions can often be answered by relatively straightforward techniques if they can first be reduced to the study of the operator on an appropriate class of simple elements which, in some convenient sense, generate the entire space. For example, a linear operator mapping the Lebesgue space L^1 into a Banach space is continuous if and only if it is bounded on characteristic functions. The selection of an orthonormal basis which diagonalizes or nearly diagonalizes an operator on the Lebesgue space L^2 offers us a similar type of approach.

The main these of the two papers in this volume is a description of a decomposition into simple building blocks of elements in generalizations of the classical Hardy spaces. In contrast to the Lebesgue spaces, these generalized Hardy spaces are not rearrangement invariant. Rather, they consist of functions (or distributions) which satisfy both size and cancellation conditions. In the case of holomorphic or harmonic functions the cancellation is implicit in the differential equation which the functions satisfy. In the case of the "real variable" theory, the cancellation properties are different, but are still sufficient to allow estimates substantially more subtle than those based solely on size considerations.

The history of the classical Hardy spaces and their modern generalizations is rich and we will not summarize it here (the reader interested in this history could start with the book of Stein and Weiss [10], the survey article by Coifman and Weiss [3], the survey talks in [1] or the Proceedings of the 1978 AMS Summer Institute [11]). Much of the interest in Hardy spaces arose from the observation that they provide a useful substitute for the Lebesgue space L^1. Indeed, many naturally occurring operators in harmonic analysis and in the theory of differential equations which are not bounded on the Lebesgue space L^1 are bounded on the Hardy space H^1. The following is an example of such an operator. Let us start with a function $f(x)$ defined on the real axis in the complex plane. Let $F(z)$ be its holomorphic projection to the upper half plane given by the Cauchy integral

$$F(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f(t)}{t-z}\,dt$$

with $z = x + iy$, $y > 0$. Now consider the boundary values $g(x) = \lim_{y \to 0+} F(x + iy)$. The mapping from f to $g \equiv Cf$ is a continuous linear operator from L^p to L^p for $1 < p < \infty$. While this mapping is not continuous at the end points $p = 1$ and $p = \infty$, it is a continuous map on H^1, which is a large subspace of L^1. In fact, H^1 was first defined to be the subspace of L^1 on which the operator C is bounded. It turns out that many other operators, apparently quite different from C, are also bounded on H^1 (but not on L^1).

A breakthrough in the understanding of this space H^1 and its important generalizations to n dimensions was made by C. Feffermen and E.M. Stein in [4]. There, they present various descriptions of these spaces and their duals. Using their ideas it is possible to describe H^1 in terms of basic building blocks called atoms:

Definition. A function $a(x)$ (defined on $\mathbb{R}^n$) is called an atom (actually, a 1-atom) if its support is contained in a ball B, $\|a\|_\infty \leq 1/|B|$ ($|B|$ is the Lebesgue measure of B) and $\int_{\mathbb{R}^n} a(x)dx = 0$.

We can now define the Hardy spaces in terms of these building blocks:

Definition. A function f (defined on $\mathbb{R}^n$) belongs to the Hardy space $H^1(\mathbb{R}^n)$ if there is a sequence of numbers λ_j satisfying $\sum_1^\infty |\lambda_j| < \infty$ and a sequence of atoms a_j so that $f = \sum_1^\infty \lambda_j a_j$. The H^1 norm of f is defined to be the infimum of the expressions $\sum|\lambda_j|$ with respect to all possible representations of f of the type just described.

It is a theorem (see [6]) that this "atomic" space is, indeed, the classical space $H^1(\mathbb{R}^n)$.

The atomic theory of Hardy spaces on $\mathbb{R}^n$ is presented in [3] and [6].

Several comments are in order. First, since $\int|a| \leq 1$ for any atom a, the space $H^1(\mathbb{R}^n)$ is a subspace of L^1. In fact, it is a subspace of L^1 consisting of functions which have a certain subtle type of cancellation. If the condition that atoms have mean value zero were dropped from the definition then the resulting space would be $L^1(\mathbb{R}^n)$. Secondly, for many purposes, the atoms are the natural elementary building blocks on which to analyze various operators. Any linear or sublinear estimate that is obtained for atoms (often by means of a simple argument) extends to all of H^1. For example, it is relatively straightforward to show that pseudodifferential operators (of order zero) map atoms into $L^1(\mathbb{R}^n)$. Hence, such operators map H^1 into L^1. Since such operators are also bounded from L^2 to L^2 one can then use the theory of interpolation of operators to conclude that these operators are also bounded from L^p to L^p for $1 < p < 2$ (that is, one can interpolate between H^1 and L^2 and obtain L^p. This is another important sense in which H^1 is a suitable and natural substitute for L^1). Finally, as was mentioned above, the spaces $H^1(\mathbb{R}^n)$ were originally defined by the requirement that operators similar to the operator C described earlier be bounded. If the spaces are defined that way, then what we have offered as a definition is, in fact, one of the deepest results of the subject.

We have just defined $H^1(\mathbb{R}^n)$ as the space of scalar combinations of basic building blocks that are localized and satisfy a size condition as well as a cancellation condition. This type of approach can be extended to a very large range of other contexts (see [3], [8], [2], [7]). There are situations, however, in which the restriction to functions supported on balls is unnatural or inconvenient. Two such situations are presentsd in the two papers in this volume. In one, atoms are a bit too simple to use for proving that certain operators map a Hardy space into itself; in the other, the building blocks are holomorphic (or harmonic) functions and, thus, cannot have support in balls.

Let us begin by describing the first situation. It is fairly clear that, if $a(x)$ is an atom on $\mathbb{R}^1$, then $Ca \in L^1(\mathbb{R}^1)$ (use the L^2-theory near the support of a and use the cancellation property near infinity). It is not completely

clear, however, from this atomic definition that Ca is, in fact, in $H^1(\mathbb{R}^1)$ (although it is true). Ca is not an atom since it does not have compact support. Nevertheless, it does have mean zero and has rapid decay at infinity. One can show that Ca is a sum of atoms which have as their supports the support B of a and the successive doubles of B. Furthermore, the coefficients in this sum of atoms decrease as fast as the terms of a geometric series. This configuration, an element of $H^1(\mathbb{R})$ which can be realized as a sum of "neatly stacked" atoms, which we call a molecule, occurs frequently. In fact, many of the operators to which we have alluded, that are bounded on H^1 but not on L^1, have the property that they map atoms uniformly into molecules. This fact (and its generalizations, for example to H^p, $p < 1$) is one of the major themes of the second paper in this volume. The fundamental size and cancellation conditions which characterize molecules are captured by the following:

Definition. A function $M \in L^2(\mathbb{R}^n)$ is a molecule centered at 0 (for $H^1(\mathbb{R}^n)$) if

$$\left(\int_{\mathbb{R}^n} |M(x)|^2 dx\right)\left(\int_{\mathbb{R}^n} |M(x)|^2 |x|^{n+1} dx\right)^n \leq C \quad \text{and} \quad \int_{\mathbb{R}^n} M(x)dx = 0 .$$

It is a direct verification that an atom is a molecule. Conversely, as indicated above, every molecule can be written as a neatly stacked sum of atoms.

The definition just given is well suited to analysis by Fourier transform techniques. For instance, if $n = 1$, then $\hat{M}$, the Fourier transform of the molecule M, is characterized by

$$\left(\int_{\mathbb{R}} |\hat{M}(\xi)|^2 d\xi\right)\left(\int_{\mathbb{R}} |\hat{M}'(\xi)|^2 d\xi\right) \leq C , \quad \hat{M}(0) = 0 .$$

It is straightforward to check that if a is an atom and m is a function satisfying $\|m\|_\infty + \|\xi m'(\xi)\|_\infty < \infty$ then $\hat{M} = m\hat{a}$ satisfies the above condition. These facts form the outline of the proof that these multiplier operators map H^1 boundedly into itself.

Let us now pass to the other situation where the building blocks consist of

holomorphic (or harmonic) functions (and, hence, cannot be atoms). The methods used to obtain the decompositions in terms of these building blocks do not involve the theory of maximal functions developed by Fefferman and Stein. Rather, the decomposition is based on properties of the Bergman kernel for homogeneous domains. Notwithstanding this difference, the decomposition turns out to involve molecules of the type described above. The development of functions in terms of these building blocks is the main theme of the first paper in this volume. We shall now describe these results in more detail.

Let $B(z, \xi) = (1 - \bar{\xi}z)^{-2}$, where z and ξ belong to the unit disc D of the complex plane. The following result is proved in the first paper:

Theorem. There exists a set of points $\{\xi_j\}$ in D with the following property: A holomorphic function $f(z)$ is in $L^1(D)$ (i.e., $\int_D |f| dxdy < \infty$) if and only if

$$f = \sum_{j=1}^{\infty} \lambda_j \frac{[B(z,\xi_j)]^2}{B(\xi_j,\xi_j)}$$

for scalars λ_j with $\Sigma|\lambda_j| < \infty$.

The points ξ_j are, roughly, a lattice with respect to the hyperbolic metric on the disc. The representation based on these points is a discrete analog of the following reproducing formula for holomorphic functions in $L^1(D)$:

$$f(z) = c\int_D f(\zeta)\frac{[B(z,\zeta)]^2}{B(\zeta,\zeta)}\, d\zeta \wedge d\bar{\zeta} .$$

The relation of this result to the theory of molecular decomposition is shown by the observation that the function which is $z[B(z,\zeta_j)]^2[B(\zeta_j,\zeta_j)]^{-1}$ on D and 0 when $z \notin D$ is a molecule for $H^1(\mathbb{R}^2)$. Thus, the last theorem can be combined with the molecular theory of Hardy spaces to show that if f is a holomorphic function in $L^1(D)$ and $f(0) = 0$ then the function $F(z)$ given by

$$F(z) = \begin{cases} f(z) & z \in D \\ 0 & z \in D \end{cases}$$

belongs to $H^1(\mathbb{R}^2)$.

Here is a different type of application of these ideas. Let L^2_+ be the space

of square integrable functions on the positive reals. Suppose k is a function on the positive reals and H the linear map defined on L^2_+ by

$$(Hf)(x) = \int_0^\infty k(x + y)f(y)dy .$$

Such operators are (the continuous analogs of) Hankel operators.

We wish to describe conditions on k which insure that the operator H is a trace class map (i.e., a nuclear map) of L^2_+ into iteslf. It is not hard to see that the following are examples of trace class operators of this form: Let z be a point in the upper half plane, $z = x + iy$. Let $e_z(t)$ be the unit vector in L^2_+ given by $e_z(t) = \sqrt{2}\, y^{1/2} e^{izt}$ and H_z the operator defined by $H_z f = \langle f , e_z \rangle \overline{e}_z$. This corresponds to the choice $k(t) = k_z(t) = 2ye^{-i\bar{z}t}$. Obviously, H_z is of trace class. Since absolutely convergent sums of sequences of operators H_z are also trace class operators, we can use these examples to generate a large collection of trace class operators. We claim that, in fact, all trace class Hankel operators can be represented by such sums. Furthermore, only certain types of sums need be considered:

Theorem. There exists a set of points $\{z_j\}$, $j = 1 , 2 , 3 , \ldots ,$ in the upper half plane with the following property: the operator H is of trace class iff $k(t) = \Sigma\lambda_j k_{z_j}(t)$ for some sequence $\{\lambda_j\}$ such that $\Sigma|\lambda_j| < \infty$.

As before, the points $\{z_j\}$ are selected so as to be, roughly, a lattice with respect to the hyperbolic metric.

This theorem extends results of Howland and Rosenblum [5] and gives a new approach to recent results of Peller [10].

The basic idea of the proof is to show that H is of trace class if and only if a certain transform of k is in a certain Bergman space. Once that is shown, the previously described decomposition theorem for functions in the Bergman space can be used. Details and generalizations are presented in the first paper of this volume.

Let us kow turn to a more detailed description of the second paper. This work presents in detail the theory of representation of functions in the Hardy space $H^1(\mathbb{R}^n)$, of distributions in the Hardy spaces $H^p(\mathbb{R}^n)$, $0 < p < 1$, and illustrates how such representations can be used to study certain operators. The notion of molecule introduced by the authors is a highly generalized version of the definition given above. The principal technical result that is proved is that a function (or distribution) belongs to a Hardy space if and only if it can be written as an appropriately normalized sum of molecules. This result is then used to give a systematic treatment of various classes of operators of interest in harmonic analysis. For example, sharp results are obtained for fractional integral operators and for multiplier operators satisfying conditions of the Hörmander type (as explained above, this is done by estimating the images of atoms and showing that they are molecules).

This paper also develops the theory of molecular decomposition for Hardy spaces of functions defined on subsets of $\mathbb{R}^2$ for which the integrability conditions are given with respect to measures other then Lebesgue measure. Although the results are rather technical, there are a number of direct applications. Perhaps the most startling is the one mentioned above: if F on $\mathbb{R}^2$ is holomorphic on D (or, merely, harmonic) and is 0 outside D, then $F \in H^1(\mathbb{R}^2)$ if and only if $F \in L^1(\mathbb{R}^2)$ (provided $F(0) = 0$).

Actually the second paper presents a range of atomic and molecular decompositions for each fixed $H^p(\mathbb{R}^n)$, $0 < p \leq 1$. The different decompositions correspond (among other things) to different descriptions of the dual spaces of $H^p(\mathbb{R}^n)$. Once the different decompositions are shown to yield equivalent norms on $H^p(\mathbb{R}^n)$, one can conclude that the various descriptions of the dual space are equivalent. This yields equivalence relations between various spaces of smooth functions of the Morrey-Companato type.

In a sense, the second paper can be considered to be a sequel of the survey article [3]. In the latter, one is presented with a theory of Hardy spaces in a

very general setting (on "spaces of homogeneous type") that deals with metric and measure theoretic properties many spaces have in common with $\mathbb{R}^n$. As a consequence many of the properties peculiar to $\mathbb{R}^n$ (for example, its differentiable structure and the presence of translations and dilations) were not exploited. The second article in this volume shows how the "finer points" of the theory can be developed in certain particular situations where additional structure is available.

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