Astérisque

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Almost $A(mod\beta)$ – invariant subspaces

Astérisque, tome 75-76 (1980), p. 239-248
<http://www.numdam.org/item?id=AST_1980__75-76__239_0>

ALMOST A(mod $\mathcal{B}$)- INVARIANT SUBSPACES

by

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0.- ABSTRACT. In this paper we define two new concepts in the context of continuous time finite dimensional time-invariant linear systems. They are those of 'almost' [A(mod $\mathcal{B}$)-] invariant' and 'almost controllability' subspaces. It is shown that there exists a supremal almost invariant and a supremal controllability subspace contained in any given subspace of the state space, and an algorithm for their computation will be given. A feedback characterization of these subspaces is derived. The paper ends with an application of these ideas to the disturbance decoupling problem.

1.- INTRODUCTION.- One of the main innovative and useful circle of ideas which has been put forward in linear system theory in the last decade has been, without any doubt, the development of the notions of A(mod $\mathcal{B}$)- invariant and controllability subspaces. These notions have shown to be very useful on the theoretical level in making apparent the ' fine structure' of multivariable linear systems but, more importantly, they have been instrumental in solving a wide variety of some very convincing control theoretic questions (disturbance decoupling, output stabilization, tracking and regulation, decoupling, etc.). In this paper we develop some related notions. Our study is set in the spirit and the language of the 'geometric' approach as developed by Wonham [1, ch.4, § 5]. Due to the space limitation, we can only provide the proofs of the main results. More details will appear in [2].

2.- Consider the linear system $\Sigma : \dot{x} = Ax + Bu$ with $x \in \mathbb{R}^n =: X$, $u \in \mathbb{R}^m =: U$ and A and B matrices of appropriate dimensions, called the system matrix and the input matrix, respectively. We will consider Σ as a family of trajectories in state space X, i.e., $\Sigma := \{x : \mathbb{R} \to X \mid$ x absolutely continuous and

$\exists\, u : \mathbb{R} \to U$ such that $\dot{x} = Ax + Bu \}$. Our definition of Σ basically comes down to the requirement : $\dot{x}(t) - Ax(t) \in \mathcal{B} := \mathrm{Im}\, B \quad \forall t$. We will write Σ as $\Sigma(A, B)$ when we want to emphasize its dependence on A and B. It is easily seen that for all F and all nonsingular R there holds : $\Sigma(A, B) = \Sigma(A+BF, BR)$. Thus the set of all trajectories is 'feedback invariant'.

3.- For any given Σ we will consider certain subspaces of X. The first set of definitions is standard and is included for completeness. The second set of definitions is new.

3.1.- Definitions.- A linear subspace $V \subset X$ is said to be [A(mod $\mathcal{B}$)-] invariant if $\forall\, x_o \in V$, $\exists\, x : \mathbb{R} \to X$ such that : (i) $x \in \Sigma$; (ii) $x(0) = x_o$; (iii) $x(t) \in V \quad \forall t$. A linear subspace $\mathcal{R} \subset X$ is said to be a controllability subspace if $\forall x_o, x_1 \in R$, $\exists\, T > 0$ and $x : \mathbb{R} \to X$ such that : (i) $x \in \Sigma$; (ii) $x(0) = x_o$, $x(T) = x_1$; (iii) $x(t) \in \mathcal{R} \quad \forall t$.

Invariant subspaces, controllability subspaces, and their applications are studied in much detail in the book of Wonham [1]. We now generalize these notions:

3.2.- Definitions.- A linear subspace $V_a \subset X$ is said to be almost [A(mod $\mathcal{B}$)-] invariant if $\forall\, x_o \in V_a$ and $\varepsilon > 0$ $\exists\, x_\varepsilon : \mathbb{R} \to X$ such that : (i) $x_\varepsilon \in \Sigma$; (ii) $x_\varepsilon(0) = x_o$; (iii) $\inf_{x \in V_a} \|x_\varepsilon(t) - x\| =: d(x_\varepsilon(t), V_a) \leq \varepsilon \quad \forall t$. A linear subspace $\mathcal{R}_a \subset X$ is said to be almost a controllability subspace if $\forall x_o, x_1 \in R_a$ $\exists\, T > 0$ such that $\forall \varepsilon > 0$, $\exists\, x_\varepsilon : \mathbb{R} \to X$ satisfying : (i) $x_\varepsilon \in \Sigma$; (ii) $x_\varepsilon(0) = x_o$, $x_\varepsilon(T) = x_1$; (iii) $d(x_\varepsilon(t), \mathcal{R}_a) \leq \varepsilon \quad \forall t$.

Thus, whereas an invariant subspace is a subspace in which a trajectory can remain, an almost invariant subspace only requires this trajectory to remain arbitrarily close.

4. We will be interested in all invariant, etc., subspaces contained in a given linear subspace $\mathcal{K}$ of X. These will be denoted by $\mathfrak{V}(\mathcal{K})$, $\mathfrak{R}(\mathcal{K})$, $\mathfrak{V}_a(\mathcal{K})$, and $\mathfrak{R}_a(\mathcal{K})$, respectively, but we will drop $\mathcal{K}$ when $\mathcal{K} = X$. Notice that $\mathfrak{V}$, etc., are defined in terms of trajectories and thus feedback invariant.

5.- Clearly $\mathfrak{R}(\mathcal{K}) \subset \mathfrak{B}(\mathcal{K}) \subset \mathfrak{B}_a(\mathcal{K})$ and $\mathfrak{R}(\mathcal{K}) \subset \mathfrak{R}_a(\mathcal{K}) \subset \mathfrak{B}_a(\mathcal{K})$. An example of an almost controllability subspace which is not invariant is $\mathfrak{B}$. Note that it is important in the definition of almost controllability spaces that T is to be independent of ε . An example of a subspace $\mathfrak{L}$ in which there exists, $\forall \varepsilon > T$, a T_ε and a $x_\varepsilon \in \Sigma$ such that $x_\varepsilon(0) = x_o$, $x_\varepsilon(T_\varepsilon) = x_1$ and $d(x_\varepsilon(t), \mathfrak{L}) \leqslant \varepsilon \;\; \forall t$, is $\operatorname{span}\{\begin{bmatrix}1\\0\end{bmatrix}\}$, with Σ defined by $A = \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}$ and $B = \begin{bmatrix}1\\0\end{bmatrix}$.

6.- Our first theorem is an easy consequence of the definitions :

Theorem.- The families $\mathfrak{B}(\mathcal{K})$, $\mathfrak{R}(\mathcal{K})$, $\mathfrak{B}_a(\mathcal{K})$, and $\mathfrak{R}_a(\mathcal{K})$ are all closed under subspace addition, i.e., $V_1, V_2 \in \mathfrak{B}(\mathcal{K}) \Rightarrow V_1 + V_2 \in \mathfrak{B}(\mathcal{K})$. Hence there exists an element $V^*_{\mathcal{K}} \in \mathfrak{B}(\mathcal{K})$ such that $V \in \mathfrak{B}(\mathcal{K}) \Rightarrow V \subset V^*_{\mathcal{K}}$. Similarly there exist analogously defined subspaces $\mathfrak{R}^*_{\mathcal{K}}$, $V^*_{a,\mathcal{K}}$ and $\mathfrak{R}^*_{a,\mathcal{K}}$.

7.- The following algorithm will play a very essential role in the sequel :

$$\boxed{S^{\mu+1}_{\mathcal{K}} := \mathcal{K} \cap (A\, S^{\mu}_{\mathcal{K}} + \mathfrak{B}) \;\; ; \;\; S^{o}_{\mathcal{K}} = \{0\}} \qquad \text{(ACSA)}$$

We will call it the almost controllability subspace algorithm. This algorithm has been studied by Wonham [1, p.106] who calls it the 'controllability subspace algorithm' . However, in view of 15, it would seem that our nomenclature is more appropriate.

Some properties of (ACSA) are given in the following theorem. A sequence of subspaces $\{\mathfrak{B}_i\} = \{\mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_n\}$ will be called a chain in $\mathfrak{B}$ if $\mathfrak{B} \supset \mathfrak{B}_1 \supset \mathfrak{B}_2 \supset \ldots \supset \mathfrak{B}_n$. Denote $A_F := A+BF$, and $\underline{n} := \{1, 2, \ldots, n\}$.

Theorem.- (i) The sequence $\{S^{\mu}_{\mathcal{K}}\}$ is monotone nondecreasing ; moreover $S^{\dim \mathcal{K}}_{\mathcal{K}} = S^{\infty}_{\mathcal{K}} := \lim_{\mu \to \infty} S^{\mu}_{\mathcal{K}}$ and $S^{\mu+1}_{\mathcal{K}} = S^{\mu}_{\mathcal{K}} \Rightarrow S^{\mu}_{\mathcal{K}} = S^{\infty}_{\mathcal{K}}$;

(ii) $S^{\infty}_{\mathcal{K}} = \inf\{\mathfrak{L} \subset \mathcal{K} \mid \mathfrak{L} = \mathcal{K} \cap (A\mathfrak{L} + \mathfrak{B})\}$;

(iii) $S^{\infty}_{\mathcal{K}} = \eta^*_{\mathcal{K}} := \sup\{\mathfrak{L} \subset \mathcal{K} \mid \exists F$ and a chain $\{\mathfrak{B}_i\}$ in $\mathfrak{B}$ such that $\mathfrak{L} = \mathfrak{B}_1 + A_F \mathfrak{B}_2 + \ldots + A_F^{n-1} \mathfrak{B}_n\}$;

(iv) $S_{\mathcal{K}}^{\mu} = \sup\{\mathcal{L} \subset \mathcal{K} \mid \exists F$ <u>and a chain</u> $\{\mathcal{B}_i\}$ <u>in</u> $\mathcal{B}$ <u>such that</u> $\mathcal{L} = \mathcal{B}_1 + A_F\mathcal{B}_2 + \ldots + A_F^{\mu-1}\mathcal{B}_n\}$.

<u>Proof</u> : (i) and (ii) are shown in [1, p.110-111]. To show (iii) and (iv) notice first that it is easily seen recursively on μ that every space of the form $\mathcal{B}_1 + A_F\mathcal{B}_2 + \ldots + A_F^{\mu-1}\mathcal{B}_n$ is included in $S_{\mathcal{K}}^{\mu}$. It thus suffices to prove that $S_{\mathcal{K}}^{\mu}$ can itself be written in this form. This will again be shown recursively. It is obviously true for $\mu = 1$. Assume now that $S_{\mathcal{K}}^{\mu}$ is of this form and notice that we may as well take $S_{\mathcal{K}}^{\mu} = \mathcal{B}_1 \oplus A_F\mathcal{B}_2 \oplus \ldots \oplus A_F^{\mu-1}\mathcal{B}_{\mu}$. Let $\mathcal{B}'_i$ be such that $\mathcal{B}_i \oplus \mathcal{B}'_j = \mathcal{B}_{i-1}$ with $\mathcal{B}_o := \mathcal{B}$. Thus $S_{\mathcal{K}}^{\mu+1} = \mathcal{K} \cap (A\, S_{\mathcal{K}}^{\mu} + \mathcal{B}) = \mathcal{K} \cap (S_{\mathcal{K}}^{\mu} + \mathcal{B}'_1 + A_F\mathcal{B}'_2 + \ldots + A_F^{\mu-1}\mathcal{B}'_{\mu} + A_F^{\mu}\mathcal{B}_{\mu})$. Now, $S_{\mathcal{K}}^{\mu} \subset \mathcal{K}$ and thus there exist linearly independent vectors ℓ_i $(i \in \underline{\ell})$ in $\mathcal{B}'_1 + A_F\mathcal{B}'_2 + \ldots + A_F^{\mu-1}\mathcal{B}'_{\mu} + A_F^{\mu}\mathcal{B}_{\mu}$ such that $S_{\mathcal{K}}^{\mu+1} = S_{\mathcal{K}}^{\mu} \oplus \operatorname{span}\{e_1, e_2, \ldots, e_{\ell}\}$. In fact, since $\mathcal{K} \cap (\mathcal{B}'_1 + A_F\mathcal{B}'_2 + \ldots + A_F^{\mu-1}\mathcal{B}'_{\mu}) = \{0\}$, every e_i is of the form $e_i = A_F^{\mu} b_i + \sum_{k=1}^{\mu} A_F^{k-1} b'_{i,k}$ with $b_1, \ldots, b_{\ell} \in \mathcal{B}_{\mu}$ linearly independent and $b'_{i,k} \in \mathcal{B}'_k$. Define now $x_{i,1} := b_i$, $x_{i,2} := b'_{i,\mu} + A_F b_i$, $\ldots$, $x_{i,\mu} := b'_{i,2} + A_F b'_{i,3} + \ldots + A_F^{\mu-1} b_i, x_{i,\mu+1} := b'_{i,1} + A_F b'_{i,2} + \ldots + A_F^{\mu} b_i = e_i$. Notice that $x_{i,k+1} = A_F x_{i,k} + b'_{i,\mu-k}$ and that, since the $A_F^{k-1} b_i$'s are linearly independent, so are the $x_{i,k}$'s $(k \in \underline{\mu}\ ,\ i \in \underline{\ell})$. Define now the matrix F'' by $F'' : x_{i,k} \to r_{i,k}$ with $r_{i,k}$ such that $Br_{i,k} = b'_{i,\mu-k}$. Thus $BF''x_{i,k} = b_{i,\mu-k}$, whence $x_{i,k} = (A_F + BF'')^{k-1} b_i$. This yields the desired expression for $S_{\mathcal{K}}^{\mu+1}$ with $\mathcal{B}_{\mu+1} = \operatorname{span}\{b_1, \ldots, b_{\ell}\}$ and the new $F' := F + F''$ on $\operatorname{span}\{x_{i,k}\}$ $(k \in \mu\ ,\ i \in \ell)$ and $F' := F$ on $\mathcal{B}''_1 \oplus A_F\mathcal{B}''_2 \oplus \ldots \oplus A_F^{\mu-1}\mathcal{B}''_{\mu}$ with $\{\mathcal{B}''_i\}$ a chain in $\mathcal{B}_1$ satisfying $\mathcal{B}''_i \oplus \mathcal{B}_{\mu+1} = \mathcal{B}_i$. ∎

8.- The following feedback characterizations of invariant and controllability subspaces may be found in [1] :

8.1. $V \in \mathfrak{V} \Leftrightarrow \exists F$ such that $A_F V \subset V \Leftrightarrow A V \subset V + \mathcal{B}$

8.2. $R \in \mathfrak{R} \Leftrightarrow \exists F, G$ such that $R = \langle A_F \mid BG \rangle$.

Here $\langle A \mid \mathcal{B} \rangle := \mathcal{B} + A\mathcal{B} + \ldots + A^{n-1}\mathcal{B}$ is the set of states reachable from 0 along trajectories of Σ .

The analogous characterizations of almost invariant and almost controllability subspaces are more difficult to obtain but shed a great deal of light on the nature of these subspaces :

Theorem. - (i) $V_a \in \mathfrak{V}_a$ iff $\exists V \in \mathfrak{V}$ and $\mathfrak{R}_a \in \mathfrak{R}_a$ such that $V_a = V + R_a$;
(ii) $\mathfrak{R}_a \in \mathfrak{R}_a$ iff $\exists F$ and a chain $\{\mathfrak{B}_i\}$ in $\mathfrak{B}$ such that $\mathfrak{R}_a = \mathfrak{B}_i + A_F \mathfrak{B}_2 + \ldots + A_F^{n-1} \mathfrak{B}_n$.

Let $f : \mathbb{R}^+ \to \mathbb{R}$ denote the step function, $\delta := \dot{f}$ the Dirac delta, $f^{(2)}$ its derivative, etc. If we use $u(t) = r\,\delta(t)$ as 'input' b Σ we obtain $x(0^+) = x(0^-) + Br$; if we use $x(t) = r\,f^{(2)}(t)$ as 'input', we obtain $x(0^+) = x(0^-) + ABr$, and if we use $x(t) = r\,f^{(i)}(t)$, we obtain $x(0^+) = x(0^-) + A^{i-1} B r$. By using smooth approximations we get a trajectory which will stay close to $x(0^-)$+ span $\{Br, ABr, \ldots, A^{i-1} Br\}$. Thus the result in the above theorem may be explained by considering $\dot{x} = A_F x + Bx$ and using inputs of the form $\Sigma\, a_i f^{(i)}(t)$. Remark, however, that it is necessary that one should be allowed to move in the directions $Br, ABr, \ldots, A^{i-2} Br$ in order to be able to use $f^{(i)}$ in order to move in the direction $A^{i-1} Br$.

9.- We now proceed towards a proof of Theorem 8. Consider first the following proposition which is proven in [3, Lemma 6] :

Proposition. - Consider the single input system $\Sigma : \dot{x} = Ax + bu$ with $U = \mathbb{R}$. Then given any $x_o, x_1 \in \mathfrak{B} := \mathrm{Im}\, b$, $T \neq 0$, and $\varepsilon > 0$ there exists $x_\varepsilon \in \Sigma \cap C^\infty$ such that $x_\varepsilon(0) = x_o$, $x_\varepsilon(T) = x_1$, and $d(x_\varepsilon(t), \mathfrak{B}) \leq \varepsilon$ $\forall t$.

10.- The following proposition yields the sufficiency of Theorem 8 in the single input case :

Proposition. - Consider the single input system $\Sigma : \dot{x} = Ax + bu$ with $U = \mathbb{R}$. Then $R_i := \mathfrak{B} + A\mathfrak{B} + \ldots + A^{i-1}\mathfrak{B}$ is almost a controllability subspace.

Proof : It is easy to see that it suffices to consider the controllable case and that we may pick the basis in X to our convenience. Moreover, since $\Sigma(A + bf, b) = \Sigma(A, b)$ and $R_i(A, b) = R_i(A + bf, b)$ it follows that we may as well start with a system in 'feedback canonical form' :

$\dot{x}_1 = x_2, \ldots, \dot{x}_i = x_{i+1}, \ldots, \dot{x}_{n-1} = x_n$, $\dot{x}_n = u$. We need to show that $\text{span}\{e_{n-i+1}, \ldots, e_n\}$, where e_i denotes the i-th standard basis vector in $\mathbb{R}^n$, is almost a controllability subspace. Define for any subspace $\mathcal{L}$, $\Sigma/\mathcal{L} := \{z\ \mathbb{R} \to X/\mathcal{L} \mid \exists x \in \Sigma$ such that $z(t) = x(t)/\mathcal{L}\ \forall t\}$. Note that in general $\Sigma/\mathcal{L}$ will not be a system (its paths need not have the 'state property'). However $\Sigma/\mathfrak{R}_{i-1}$ is a system in the sense we have defined it if we restrict attention trajectories in C^∞. By identifying the associated A and b matrices and using Proposition 9 on $\Sigma/\mathfrak{R}_{i-1}$ it follows that for any $z_o \in X/\mathfrak{R}_{i-1}$, $T \neq 0$, and $\varepsilon > 0$, there exists $z_\varepsilon \in \Sigma/\mathfrak{R}_{i-1}$ such that $z_\varepsilon(0) = z_o$, $z_\varepsilon(T) = 0$, and $d(z_\varepsilon(t), \mathfrak{R}_i/\mathfrak{R}_{i-1}) \leq \varepsilon\ \ \forall t$. This implies that for all $x_o \in \mathfrak{R}_i$, $T \neq 0$, and $\varepsilon > 0$, there exists $x_\varepsilon \in \Sigma$ such that $x_\varepsilon(0) = x_o$, $x_\varepsilon(T) \in \mathfrak{R}_{i-1}$, and $d(x_\varepsilon(t), \mathfrak{R}_i) \leq \varepsilon\ \ \forall t$. From $\mathfrak{R}_i$ we thus move to $\mathfrak{R}_{i-1}$, from $\mathfrak{R}_{i-1}$ to $\mathfrak{R}_{i-2}$, etc., to $\{0\}$. This yields, by moving from $x_o \in \mathfrak{R}_i$ to 0 and from 0 to $x_1 \in \mathfrak{R}_i$ (by taking $T < 0$ in the above argument), the claim of the proposition. ∎

11.- From Proposition 10, it is easy to prove that every space of the form $\mathfrak{R}_a = \mathfrak{B}_1 + A_F \mathfrak{B}_2 + \ldots + A_F^{n-1} \mathfrak{B}_n$ with $\{\mathfrak{B}_i\}$ a chain in $\mathfrak{B}$ is almost a controllability subspace. Indeed, $\mathfrak{R}_a$ may then be written as $\mathfrak{R}_a = \sum_{i=1}^m \mathfrak{R}_i$ with each $\mathfrak{R}_i$ of the form $\mathfrak{B}_i + A_F \mathfrak{B}_i + \ldots + A_F^{\mu_i - 1} \mathfrak{B}_i$ and $\mathfrak{B}_i \in \mathfrak{B}$. Now, each $\mathfrak{R}_i$ is almost controllable. Consequently, since $\mathfrak{R}_a$ is closed under addition, $\mathfrak{R}_a \in \mathfrak{R}_a$.

12.- We will now show that every almost controllability subspace is of the form $\mathfrak{B}_1 + A_F \mathfrak{B}_2 + \ldots + A_F^{n-1} \mathfrak{B}_n$ for some F and some chain $\{\mathfrak{B}_i\}$ in $\mathfrak{B}$. The proof is based on the following lemma :

<u>Lemma</u>.- <u>Assume</u> $\mathfrak{R}_a \in \mathfrak{R}_a$, <u>with</u> $\mathfrak{R}_a \cap \mathfrak{B} = \{0\}$. <u>Then</u> $\mathfrak{R}_a = \{0\}$.

<u>Proof</u> : Assume $0 \neq x_o \in \mathfrak{R}_a$. Then there exists, for any $\varepsilon > 0$, an $x_\varepsilon \in \Sigma$ such that $x_\varepsilon(0) = x_o$ and $d(x_\varepsilon(t), \mathfrak{R}_a) \leq \varepsilon$. Let $X = \mathfrak{B} \oplus Z$, with $Z \supset \mathfrak{R}_a$, and write $x_\varepsilon = (b_\varepsilon, z_\varepsilon)$. Note that $z_\varepsilon(0) = z_o \neq 0$. Now, z_ε satisfies a differential equation of the form $\dot{z}_\varepsilon = A_1 z_\varepsilon + B_1 b_\varepsilon$ with $\|b_\varepsilon(t)\| \leq \varepsilon\ \ \forall t$. Clearly, for ε sufficiently small, it will be impossible to transfer z_o to 0 with b_ε as control in T units of time (T independent of ε !), as required for almost controllability. This contradiction establishes the lemma. ∎

13.- The claim made in the beginning of section 12 can now be established along the following lines : let $\mathcal{R}_a \in \mathfrak{R}_a$, and consider $S^{\infty}_{\mathcal{R}_a}$ as defined in section 7.

Now, $\Sigma/S^{\infty}_{\mathcal{R}_a}$ defines a system and $\mathcal{R}_a/S^{\infty}_{\mathcal{R}_a}$ is almost controllable relative to it. This is easily seen by writing $S^{\infty}_{\mathcal{R}_a}$ as $\mathcal{B}_1 \oplus A_F \mathcal{B}_2 \oplus \ldots \oplus A_F^{n-1} \mathcal{B}_n$ and considering the compatible feedback canonical for m [1, p.123]. In addition, since $S^{\infty}_{\mathcal{R}_a}$ is the maximal subspace of $\mathcal{R}_a$ which may be written in this form, $S^{\infty}_{\mathcal{R}_a} = \mathcal{R}_a \cap (A S^{\infty}_{\mathcal{R}_a} + \mathcal{B})$ which implies that the input matrix of $\Sigma/S^{\infty}_{\mathcal{R}_a}$ intersected with $\mathcal{R}_a/S^{\infty}_{\mathcal{R}_a}$ is $\{0\}$. Hence, by Lemma 12, $\mathcal{R}_a/S^{\infty}_{\mathcal{R}_a} = \{0\}$ and thus $\mathcal{R}_a = S^{\infty}_{\mathcal{R}_a}$. The result then follows from Proposition 7.

14.- In order to establish Theorem 8 we still need to show that every $V_a \in \mathfrak{V}_a$ is of the form $V_a + \mathcal{R}_a$. We will not give this proof in detail : it is a bit more involved then but parallel to the reasoning in 12 and 13. The lemma analogous to Lemma 12 is :

<u>Lemma</u>.- <u>Assume</u> $V_a \in \mathfrak{V}_a$ <u>with</u> $V_a \cap \mathcal{B} = \{0\}$. <u>Then</u> $V_a \in \mathfrak{V}$.

This lemma applied to the system $\Sigma/S^{\infty}_{V_a}$ will establish that $V_a = V + \mathcal{R}^*_{a,V_a}$ with $V \in \mathfrak{V}$.

15.- Theorem 8, Proposition 7, and the results in Wonham [1, ch.4] culminate in the following algorithms :

$$\text{Let} \quad V^{\mu+1}_{\mathcal{K}} = \mathcal{K} \cap A^{-1}(V^{\mu}_{\mathcal{K}} + \mathcal{B}) \ ; \ V^{o} = \mathcal{K} ,$$

$$S^{\mu+1}_{\mathcal{K}} = \mathcal{K} \cap (A\, S^{\mu}_{\mathcal{K}} + \mathcal{B}) \ ; \ S^{o} = \{0\} .$$

$$\text{Then} \quad V^{*}_{\mathcal{K}} = V^{\infty}_{\mathcal{K}} := \lim_{\mu \to \infty} V^{\mu}_{\mathcal{K}}$$

$$\mathcal{R}^{*}_{a,\mathcal{K}} = S^{\infty}_{\mathcal{K}} := \lim_{\mu \to \infty} S^{\mu}_{\mathcal{K}}$$

$$V^{*}_{a,\mathcal{K}} = V^{\infty}_{\mathcal{K}} + S^{\infty}_{\mathcal{K}}$$

$$\mathcal{R}^{*}_{\mathcal{K}} = S^{\infty}_{\mathcal{K}} \cap V^{\infty}_{\mathcal{K}} = S^{\infty}_{V^{*}_{\mathcal{K}}} = S^{\infty}_{V^{\infty}_{\mathcal{K}}} .$$

These algorithms are easy to set up numerically if A, B and $\mathcal{K}$ are numerically specified which is of course indispensable if any of this is going to be relevant in applications.

16.- There are various special types of (almost) invariant subspaces :

Definition.- $V \in \mathfrak{B}$ is said to be a coasting subspace if $\mathfrak{R}^*_V = \{0\}$; $\mathfrak{R} \in \mathfrak{R}_a$ is said to be a sliding subspace if $\mathfrak{R}^*_{\mathfrak{R}_a} = \{0\}$ (equivalently $V^*_{\mathfrak{R}_a} = \{0\}$).

The basic types of almost invariant subspaces are : coasting subspaces $\mathcal{S}$ ($\Sigma \mid \mathcal{S}$ is a flow described by $(A+BF)|\mathcal{S}$ with F such that $(A+BF)\mathcal{S} \subset \mathcal{S}$), controllability subspaces $\mathfrak{R}$ ($\Sigma \mid \mathfrak{R}$ is controllable), and sliding subspaces S (motions arbitrarily close to S are possible but non-zero motions in S are impossible. Moreover, in order to steer closer and closer to S , the inputs need to become more and more 'distribution like' and motions ' slide' along S with greater and greater speed).

It is easy to show that every almost invariant subspace may be decomposed as $V_a = \mathfrak{R}^*_{V_a} \oplus \mathcal{S} \oplus S$ with $\mathfrak{R}^*_{V_a} \oplus S = V^*_{V_a}$ and $\mathfrak{R}^*_{V_a} \oplus S = \mathfrak{R}^*_{a, V_a}$. If $\mathfrak{R}^*_{V_a} = \{0\}$ then this decomposition is unique.

17.- Every almost invariant subspace can be approximated arbitrarily closely by invariant subspaces. This approximation should be understood in the topology of the Grassmann variety $G^a(X)$ of all q-dimensions subspaces of X . Thus $\mathcal{L}_\varepsilon \xrightarrow[\varepsilon\to 0]{} \mathcal{L}$ means that if $\mathcal{L} = \mathrm{Im}\begin{bmatrix} I \\ 0 \end{bmatrix}$ then for ε sufficiently small $\mathcal{L}_\varepsilon = \mathrm{Im}\begin{bmatrix} I \\ B_\varepsilon \end{bmatrix}$ and $\| B_\varepsilon \| \xrightarrow[\varepsilon\to 0]{} 0$.

Theorem.- Let $V_a \in \mathfrak{B}_a$. Then $\exists\, V_\varepsilon \in \mathfrak{B}$ such that $\lim\limits_{\varepsilon\to 0} V_\varepsilon = V_a$.

We will not give the (simple) proof.

It is of interest to investigate the behaviour of the spectrum and of the feedback gain on the V_ε's as $\varepsilon \to 0$, i.e., to investigate $\sigma(A_{F_\varepsilon} \mid V_\varepsilon)$ and F_ε as $\varepsilon \to 0$, where F_ε is such that $A_{F_\varepsilon} V_\varepsilon \subset V_\varepsilon$. Let $V_a = \mathfrak{R} \oplus \mathcal{S} \oplus S$. Then it is obviously possible to choose $V_\varepsilon = \mathfrak{R} \oplus \mathcal{S} \oplus V_\varepsilon'$ with $\lim\limits_{\varepsilon\to 0} V_\varepsilon' = S$. Thus the spectrum of

V_ε is arbitrary on $\mathcal{R}$, fixed on $\mathcal{S}$, but nothing definite can be said a priori about the spectrum on V'_ε. In any case, it is possible to choose the V'_ε such that : $\min\limits_{\lambda \in \sigma(A_{F_\varepsilon} | V'_\varepsilon)} \mathcal{R}e\{\lambda\} \xrightarrow[\varepsilon \to 0]{} -\infty$. In addition, it is true that : $\min\limits_{\lambda \in \sigma(A_{F_\varepsilon} | V'_\varepsilon)} |\lambda| \xrightarrow[\varepsilon \to 0]{} \infty$ since V'_ε has as its limit a sliding subspace. In addition $\lim\limits_{\varepsilon \to 0} \|F_\varepsilon | V_\varepsilon\| = \infty$. These comments actually show that sliding subspaces are more akin to invariant than to controllability subspaces.

We remark that $\mathfrak{B}_a$ is precisely the closure of $\mathfrak{B}$ but that it need not be true that an almost controllability subspace is the limit of controllability subspaces.

18.- The notions of almost invariant and of almost controllability subspaces as such do not have a direct generalization to discrete time systems : $\Sigma : = x(t+1) = A\,x(t) + B\,u(t)$, $t \in \mathbb{Z}$. The expressions which we found for $V^*_{a,\mathcal{K}}$ and $\mathcal{R}^*_{a,\mathcal{K}}$ however do have interesting interpretations in discrete time. These have to do with the study of subspaces for which each point is reachable from 0 along this subspace and subspaces for which each point may be steered to 0 along this subspace. These have applications to be design of dead-beat controllers and observers.

19.- Application to the disturbance decoupling problem.

The disturbance decoupling problem may be stated as follows : consider the system :

$$\dot{x} = Ax + Bu + Gw \quad ; \quad y = Cx \quad ,$$

where u denotes the control, w the disturbance, and y the controlled output, when does there exists F such that feedback control $u=Fx$ results in a closed loop system in which y is independent of w ? This compelling control theoretic question may be solved very elegantly by means of the theory of invariant subspaces. Indeed it is solvable [1, p. 90] if and only if $V^*_{\ker C} \supset \operatorname{Im} G$.

Using almost invariant subspaces we can now also answer the question whether there exists F such that the effect of w on y becomes arbitrarily small. Indeed :

Theorem.- Let $H_F : t \in \mathbb{R}^+ \to C\, e^{(A+BF)t} G$ and let $\|.\|_p$ denote the $\mathcal{L}_p(0,\infty)$ norm. Given any $\varepsilon > 0$ there exists F_ε such that $\|H_{F_\varepsilon}\|_p \leq \varepsilon \quad \forall\, 1 \leq p \leq \infty$ if and only if $V^*_{a,\ker C} \supset \operatorname{Im} G$.

Other applications include almost disturbance decoupling with stability and almost decoupling. We are presently working on the application of these ideas in singular 'cheap' control and filtering problems.

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