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# ON QUASI-REACHABLE REALIZATIONS OF A POLYNOMIAL RESPONSE

by

Eduardo D. SONTAG

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ABSTRACT. - This paper studies the class of quasi-reachable realizations of a fixed polynomial response. This class is described as a complete lattice. Various subclasses are explored in detail with respect to the induced order, and examples are given.

## I. - INTRODUCTION AND NOTATIONS.

Polynomial response maps are basically those given by discrete-time Volterra series. A class of natural realizations for such maps is given by systems whose states evolve in n-dimensional Euclidean space via first order polynomial difference equations, but various facts suggest that this class must be "completed" in order to allow for more arbitrary schemes as state-spaces. To shorten the exposition, most of the definitions and basic results of SONTAG [1976,1979] will not be repeated ; these will be quoted as "PRM (a)", where "(a)" refers to the numbering in these references.

We recall some of the basic notations and definitions : fixing an infinite field $k$ , a "k-space" means the set of all k-points of an affine k-reduced k-scheme ; "k-space morphisms" or "polynomial maps" are scheme-morphisms. An abstract system $\Sigma = (X, P, h, x^*)$ is given by a ("state-space") set $X$ , a ("transition") map $P: X \times U \to X$ , an "output" map $h : X \to Y$ , and an "initial state" $x^*$ in $X$ . For simplicity, $U$ (input-values space) will be $k^m$ and $Y$ (output-values space) = $k^p$ for some $m$, $p$, and $P(x^*, 0) = x^*$ (equilibrium initial state). A morphism $T : \Sigma_1 \to \Sigma_2$ will be a map $X_1 \to X_2$ with $T(x_1^*) = x_2^*$ , $T(P(x, u)) = P(T(x), u)$ , and $h_2(T(x)) = h_1(x)$ , for all $x$, $u$ .
A k-system has $X$ a k-space and $P$, $h$ polynomial maps ; a k-system morphism is a $T$ as above which is polynomial (so an isomorphism is a polynomial

change of variables in the state-space). An <u>almost-variety</u> $X$ is a k-space with $A(X)$ (=algebra of polynomial functions on $X$ ) = a subalgebra of a finitely generated k-algebra (if $A(X)$ is itself f.g., $X$ is a <u>variety</u>) ; an [<u>almost</u>-] <u>polynomial</u> system has $X$ an [almost-] variety. For an algebra $A$, $X(A)$ is the k-space of k-points of $A$ (= homomorphisms $A \to k$ ) ; both $A(.)$ and $X(.)$ are naturally seen as functors, giving an equivalence between the categories of k-spaces and k-reduced algebras. The <u>input space</u> $\Omega$ was introduced in PRM as a "completion" of the set of finite-support sequences $U[z]$ ; a system is <u>quasi-reachable</u> when the reachability map is dominating, i.e. when reachable states are dense in $X$.

Attention is restricted here to quasi-reachable systems. If a given system is not quasi-reachable, it may be restricted to the closure of its reachable set ; this operation is rather straightforward, and restricting to such systems simplifies matters considerably. Isomorphism classes of quasi-reachable systems are ordered in a natural way, as studied below.

## II. - <u>THE LATTICE</u> QR(f) .

All systems appearing in this paper will be assumed to be quasi-reachable realizations of a fixed polynomial response map $f$.

(2.1). DEFINITION. - $\Sigma_2 \leq \Sigma_1$ means that there exists a k-system morphism $T : \Sigma_1 \to \Sigma_2$.

The above defines a pre-order among systems, which will become a partial order when isomorphic systems are identified.

(2.2). LEMMA. - <u>If</u> $T_i : \Sigma_1 \to \Sigma_2$ , $i = 1, 2$ , <u>are morphisms, then</u> $T_1 = T_2$ . <u>Furthermore, the</u> $T_i$ <u>are dominating</u>.

<u>Proof</u> : Since $\Sigma_1$ is quasi-reachable, the abstractly canonical state-space $X_{ac}$ is dense in $X_1$ . Thus by an argument as in PRM (7.7), $T_1 = T_2$ on $X_{ac}$. The equality follows by continuity. A similar argument proves the last statement.

By a slight abuse of notation, the same letter will be used for a system and for its isomorphism class. Let $QR(f)$ denote the set of all isomorphism classes of quasi-reachable realizations of $f$ ; then $QR(f)$ inherits the preorder $\leqslant$ ; in fact :

(2.3). COROLLARY. - $QR(f)$ <u>is partially ordered by</u> $\leqslant$ .

<u>Proof</u> : If $T : \Sigma_1 \to \Sigma_2$ and $S : \Sigma_2 \to \Sigma_1$ are morphisms, then $TS : \Sigma_2 \to \Sigma_2$ must be equal to the identity morphism, by (2.2). Similarly, $ST$ is the identity. So $T$ is an isomorphism.

Recall that $\Sigma_{free}(f)$ is the system having the input space $\Omega$ as its state-space, and with transitions extending the concatenation operation on input sequences ; see PRM (6.10). By PRM (8.2), the reachability map $g : U[z] \to X_\Sigma$ extends to a polynomial map $g^\Omega$ from $\Omega$ to $X_\Sigma$ , for any k-system $\Sigma$ . If $\Sigma$ realizes $f$ , then $g$ induces an abstract-system morphism from the system with $X = U[z]$ , $P := $ concatenation, and $h, x^*$ as in $\Sigma_{free}(f)$ , into $\Sigma$ . Thus $g^\Omega$ induces a k-system morphism from $\Sigma_{free}(f)$ into $\Sigma$ . Since on the other hand, by PRM (11.3), the canonical realization $\Sigma_f$ is terminal among quasi-reachable ones, it follows that :

(2.4). PROPOSITION. - $\Sigma_{free}(f)$ <u>is the (unique) largest, and</u> $\Sigma_f$ <u>the (unique) smallest, element of</u> $QR(f)$ .

If $T : \Sigma_1 \to \Sigma_2$ is a dominating k-system morphism, $A(T)$ gives $A(X_2)$ as a subalgebra of $A(X_1)$ , with "co-transitions" $A(P_1)$ and "co-output map" $A(h_1)$ extending $A(P_2)$ , $A(h_2)$ . Conversely, given any k-subalgebra $A$ of $A(X_1)$ such that :

$$(2.5) \qquad A \text{ includes } A_f$$

(note that $A_f$ if a subalgebra of $A(X_1)$ , by the quasi-reachability assumption) and

$$(2.6) \qquad A(P_1)(A) \text{ is included in } A \otimes \tilde{U}, \tilde{U} := A(U) ,$$

then the restriction of $A(P_1)$ to $A$ , together with the restriction of $A(x_1^*)$

to A and $A(h_1)$ (seen as a homomorphism into A ), define a system $\Sigma_2$ with $A(X_2) = A$ and $\Sigma_2 \leq \Sigma_1$ . Furthermore, A determines a unique such $\Sigma_2$ (up to isomorphism), since $A(P_2)$, $A(h_2)$, $A(x^*_2)$ are given necessarily by the above procedure.

Thus, the (isomorphism classes of) systems less or equal than $\Sigma_1$ are in a one-to-one correspondence with the algebras satisfying (2. 5) and (2. 6). Furthermore, this correspondence preserves orderings, when the subalgebras A are ordered by inclusion. But (2. 5) and (2. 6) are preserved under intersections, and if a family $A_i$ satisfies (2. 6) then the algebra generated by the union of the $A_i$ again satisfies (2. 5), (2. 6). Translating these facts into the partial order for systems, and applying them for $\Sigma_1 = \Sigma_{free}(f)$ :

(2. 7). THEOREM. - QR(f) is a complete lattice.

Although the technicalities are very different, the above is formally very similar to the result for linear responses over rings presented in SONTAG [1977].

## III. - SOME RELEVANT SUBLATTICES.

The lattice QR(f) is too "large", in that it contains realizations of arbitrary dimension. Certain sublattices described below are much more interesting ; it is a remarkable fact that there seems to be no way to study any of these lattices without in some way first introducing QR(f) . In this section, f will be assumed to be finitely realizable.

(3. 1). DEFINITION. - MD(f) denotes the (isomorphism classes of) minimal-dimension realizations of g , viewed as a partially-ordered subset of QR(f) .

(3. 2). THEOREM. - MD(f) is a complete sublattice of QR(f) .

Proof : Minimal realizations correspond to those subalgebras A of the algebra of Volterra series which satisfy (2. 5) and (2. 6) together with the additional condition that A is algebraic over $A_f$ . This is again a complete lattice.

(3.3). REMARK.- By PRM (9.3), if $\Sigma$ is a realization of dimension n then the n-step reachability map is dominating. By the arguments in PRM (12.12), it follows that two minimal realizations are isomorphic if and only if $A(g_n)(A)$ is the same subalgebra of $A(U^n)$ for both of them, where $n = \dim A_f$. This permits calculations to be carried out explicitly, in $A(U^n)$.

Realizations in MD(f) are characterized by the fact that their observation fields are algebraic over the canonical observation field $Q_f$. Another important subclass of realizations is :

(3.4). DEFINITION.- A realization $\Sigma$ of f is quasi-canonical iff $Q(\Sigma)$ is equal to $Q_f$. The poset of quasi-canonical realizations is QC(f).

(Note that the natural inclusion of the observation algebra $A_f = A(X_f)$ in $A(\Sigma)$ extends to an inclusion of $\mathbf{Q}_f$ in $Q(\Sigma)$, for any quasi-reachable realization $\Sigma$ ).

A dominating k-space morphism $T : X \to Z$ is birational when $A(Z)$ has the same quotient field as $A(X)$, (identifying via $A(T)$). The meaning of (3.4) will be clarified by the algebraic :

(3.5). LEMMA.- Let X, Z be almost-varieties, $T : X \to Z$ dominating. Assume that the field k is algebraically closed and has characteristic zero. Then T is birational if and only if there is a (Zariski) open set $Z_1$ in Z such that the fibre $T^{-1}(z)$ has precisely one element, for each z in $Z_1$.

Proof : The argument is essentially that in PRM (4.6). By DIEUDONNE [1974, Section 5.3], the varieties $X_1$, $Z_1$ can be choosen to be normal (i.e., $A(X_1)$, $A(Z_1)$ are integrally closed). If T is birational, n = m in PRM (4.6) and the restriction map $X_1 \to Z_1$ is finite and onto ; furthermore, s = cardinality of fibres = 1 by DIEUDONNÉ [1972, Prop. 5.3.2]. Conversely, if fibres have generically a single point then the argument in PRM (4.6) proves that n = m, so $Q(X)$ is algebraic over $\mathbf{Q}(Z)$, with separable degree one ; since char k = 0 , they are equal.

The above is a straightforward generalization of a result well-known for varieties. Since $\Sigma_f$ may be nonpolynomial, however, (3.5) is needed in order to conclude, for $k$ as in (3. 5) :

(3. 6). PROPOSITION. - The (quasi-reachable) almost-polynomial system $\Sigma$ is in $QC(f)$ if and only if there exists an open (hence dense) subset $X_1$ of its state-space $X$ such that no two states in $X_1$ are indistinguishable.

Proof : Immediate from (3. 5), by considering the canonical morphism $T: \Sigma \to \Sigma_f$ .

This justifies the terminology "quasi-canonical" = quasi-reachable plus "quasi-observable" in the above sense. Such systems have been suggested before in the context of minimality of discrete-time nonlinear systems ; see PEARLMAN [1977] . (A related concept appears implicitly in the last section of HERMANN and KRENER [1977] ). The "if" in (3. 6) is not true, for general $f$ , over the reals, but it is valid for restricted kinds of systems (classes of multilinear systems, etc . ).

(3. 7). THEOREM. - $QC(f)$ is a complete sublattice of $QR(f)$ .

Proof : As in the previous cases.

In particular, there exists a largest quasi-canonical realization $\Sigma^f$ . Explicitly, $\Sigma^f$ can be obtained by intersecting $Q_f$ with the algebra of Volterra series (this gives $A^f$ , the algebra of functions on the state-space $X^f$ ) , and restricting the maps defining $\Sigma_{free}(f)$ . That $A^f$ indeed satisfies (2. 6) follows from the more general result :

(3. 8). LEMMA. - If $\Sigma_2 \leq \Sigma_1$ , then (with the notations in (2. 5) (2. 6)), $A := Q(A_2) \cap A_1$ satisfies (2. 6).

Proof : Since $\Sigma_1$ is quasi-reachable, its transition map $p$ is dominating; thus $A(P)$ is one-to-one. So $A(P)$ extends to a homomorphism from $Q(A_1)$ into $Q(A_1 \otimes \tilde{U})$ , which itself restricts to a homomorphism from $Q(A_2)$ into $Q(A_2 \otimes \tilde{U})$ . Since $A_2$ satisfies (2. 6), the result will follow from :

(3.9) $$Q(A_2 \otimes \widetilde{U}) \cap (A_1 \otimes \widetilde{U}) \subseteq Q(A_2) \otimes \widetilde{U} .$$

To prove this, let a be in the left-hand side, i.e., $a = \Sigma\, c_i \otimes T_i$, where $U = X(k[T_1, \ldots, T_m])$, $c_i$ in $A_1$, and $(\Sigma\, b_i \otimes T_i)\, a = \Sigma\, a_i \otimes T_i$, with the $a_i$, $b_i$ in $A_2$, some $b_i$ nonzero.

The set :

$$V := \{ u \text{ in } U \text{ with } (\Sigma\, b_i \otimes T_i)(u) \neq 0 \}$$

is a proper algebraic subset of $U$, ( see PRM (2.4)) and, for $u$ in $V$, $a(u)$ belongs to $Q(A_2)$. Thus, by PRM (12.11), all $c_i$ are in $Q(A_2)$, as wanted. (The same proof works for more general $U$).

The largest quasi-canonical realization $\Sigma^f$ is therefore obtained, using $\Sigma_1 = \Sigma_{free}(f)$ and $\Sigma_2 = \Sigma_f$ above.

Other natural classes (sub-posets) of realizations are that of all polynomial realizations (X a variety), or of all minimal polynomial realizations, or of all realizations with $X_\Sigma$ = affine space, etc. These classes do not form sublattices, however. In fact, as shown by examples in the next sections, meets or joins of systems of these kinds are not necessarily in any sense "nice". This provides a further justification for considering nonpolynomial k-systems, seen as those needed to "complete" the various posets.

## IV. - FIBRE PRODUCTS.

The lattice operations in $QR(f)$ can be interpreted more concretely than in the previous sections. This is particulary simple with the join, which obviously corresponds to a fibre product construction, i.e., ($\Sigma_1$ join $\Sigma_2$) is given by the product of $\Sigma_1 / \Sigma_f$ and $\Sigma_2 / \Sigma_f$ in the category of all k-system morphisms $\Sigma / \Sigma_f$ from quasi-reachable realizations $\Sigma$ into $\Sigma_f$ (with the standard morphisms $r : \Sigma_1 / \Sigma_f \to \Sigma_2 / \Sigma_f$ corresponding to the $r : \Sigma_1 \to \Sigma_2$ such that $r$ composed with $\Sigma_2 / \Sigma_f$ is $\Sigma_1 / \Sigma_f$). Thus $\Sigma = (\Sigma_1$ join $\Sigma_2)$ has state-space a closed subset of $X_1 \times X_2$ :

(4.1) $$X = \{(g_1(w), g_2(w)) , \quad w \text{ in } U^* \} ,$$

and $P((x_1, x_2), w) := (P(x_1, w), P(x_2, w))$, initial state $(x_1^*, x_2^*)$, and $h(x_1, x_2) := h_1(x_1)$ (or $h_2(x_2)$).

The calculations for the following examples are easy, using (3. 3) ; initial states are zero, and $U = k$, unless otherwise stated.

(4. 2). EXAMPLE. - Let $\Sigma_1$ be

$$\begin{aligned} x(t+1) &= u^2(t) \\ y(t) &= x^3(t) \end{aligned}$$

and $\Sigma_2$ be

$$\begin{aligned} x(t+1) &= u^3(t) \\ y(t) &= x^2(t) \, . \end{aligned}$$

Both realize the same response map $f$ with canonical realization (which is also their meet) :

$$\begin{aligned} x(t+1) &= u^6(t) \\ y(t) &= x(t) \, . \end{aligned}$$

Their product is the system whose state-space is the "cusp" $\{(x_1, x_2)$ in $k^2$ with $x_1^3 = x_2^2\}$ and

$$\begin{aligned} x_1(t+1) &= u^2(t) \\ x_2(t+1) &= u^3(t) \\ y(t) &= x_1^3(t) \, , \end{aligned}$$

which is more complex than the original systems.

(4. 3). EXAMPLE. - Here $\Sigma_1$ and $\Sigma_2$ have as state-space the closed set consisting of those vectors $(x_1, x_2, x_3, x_4)$ in $k^4$ with $x_1 x_3 = x_2^2$, and input set $U = k^2$. The equations are, for $\Sigma_1$ :

$$\begin{aligned} x_1(t+1) &= u(t) \\ x_2(t+1) &= u(t)\, v(t) \\ x_3(t+1) &= u(t)\, v(t)^2 \\ x_4(t+1) &= x_2(t) + x_1(t)\, x_2(t)\, u(t) + x_3(t)\, v(t) \\ y(t) &= x_4(t) \end{aligned}$$

and for $\Sigma_2$ :

$$\begin{aligned} x_1(t+1) &= v(t) \\ x_2(t+1) &= u(t)\ v(t) \\ x_3(t+1) &= u(t)^2\ v(t) \\ x_4(t+1) &= x_2(t)+x_3(t)\,u(t)+ x_1(t)\ x_2(t)\ v(t) \\ y(t) &= x_4(t)\ . \end{aligned}$$

Their meet is the canonical realization $\Sigma_f$ with $X_f$ = all 4-vectors with $x_1^3 = x_2\, x_3$ , and :

$$\begin{aligned} x_1(t+1) &= u(t)\ v(t) \\ x_2(t+1) &= u^2(t)\ v(t) \\ x_3(t+1) &= u(t)\ v(t)^2 \\ x_4(t+1) &= x_1(t)+ x_2(t)\ u(t)+ x_3(t)\ v(t) \\ y(t) &= x_4(t)\ . \end{aligned}$$

Here the join turns out to be simpler than all of the above ; it is the system with $X = k^3$ and

$$\begin{aligned} x_1(t+1) &= u(t) \\ x_2(t+1) &= v(t) \\ x_3(t+1) &= x_1(t)\ x_2(t)+ x_1(t)^2\ x_2(t)\ u(t)+ x_1(t)\ x_2(t)^2\ v(t) \\ y(t) &= x_3(t)\ . \end{aligned}$$

## V. - EXAMPLE OF NON POLYNOMIAL k-SYSTEMS.

The purpose of this section is to give an example illustrating how nonpolynomial k-systems arise naturally when studying polynomial realizations in QR(f) .

(5.1). EXAMPLE. - Let f be the response of $\Sigma_0$ , where $X = k^2$ and $x^* = (1,1)'$ and

$$\begin{aligned} x_1(t+1) &= x_1(t)\ x_2(t) \\ x_2(t+1) &= x_2(t)(u(t)+1) \\ y(t) &= x_1(t)\ . \end{aligned}$$

Then, there exists a family a polynomial systems $\Sigma_i$ such that for every non-canonical (quasi-reachable) realization $\Sigma$ , there is an i with $\Sigma$ strictly greater (in QR(f) ) than $\Sigma_i$ . Since the canonical realization is nonpolynomial

(PRM (18.1)), there results in particular an infinite chain

$$\Sigma_1 > \Sigma_2 > \Sigma_4 > \Sigma_8 > \ldots > \Sigma_f ,$$

and $\Sigma_f$ appears in taking the meet of this chain. The $\Sigma_i$ have algebras $K[T_1, T_1, T_2, \ldots, T_1 T_2^{i-1}, T_2^i]$ ; the construction is detailed in SONTAG [1979]. A consequence of this construction is that it is not possible to obtain a "canonical realization theory" for f solely via affine spaces, or even polynomial systems, at least if the existence of a terminal object in the category of quasi-reachable realizations is sought, as usual in system theory.

## VI. - FINAL REMARKS.

Various other classes can be defined as subposets of QR(f) . "Normal" realizations, for instance, form an interesting subclass which give "less singular" state-spaces and permit a much stronger uniqueness result for canonical realizations ; this is explained in SONTAG [1979] . Similarly, results can be easily extended to more arbitrary input and output - values sets, or to systems whose state-spaces are <u>nonaffine</u> schemes, although in this latter case the exposition is technically more complicated.

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