Henry Hermes

Lie algebraic methods for the control of infinite dimensional nonlinear evolution equations

Astérisque, tome 75-76 (1980), p. 125-131

<http://www.numdam.org/item?id=AST_1980__75-76__125_0>
INTRODUCTION.

Let $\mathcal{A}$ and $\mathcal{B}$ be densely defined operators acting within a Banach space $B$ of $\mathbb{R}^n$ valued functions. We consider systems of the form

$$\frac{dv_t}{dt} = \mathcal{A}(v_t) + u(t) \mathcal{B}(v_t), \quad v_0 = \varphi \in B$$

where the control $u$ is Lebesgue measurable with values $|u(t)| \leq 1$, and unique solutions of (1) are assumed to exist for small $t > 0$. An observation of (1) is a continuous linear map $g: B \to \mathbb{R}^k$.

Let $v_t$ denote the reference solution, at time $t$, corresponding to control $u \equiv 0$, and $A(t, \varphi) \subset B$ be the set of points attainable at time $t$ by all solutions of (1). We shall study two questions. The first, that of local controllability, is to derive computable conditions to determine when $g(v_t) \in \text{interior } g(A(t, \varphi))$ for small $t > 0$. This includes the study of $\mathbb{R}^n$ controllability, say for delay equations, but not function space controllability, e.g. see [1],[4]. The second question is that of finite dimensional realizations. Specifically, if $t \mapsto g(v_t(u))$ is a solution $t \mapsto v_t(u)$ of (1), we say $g$ admits a strong differential realization on $\mathbb{R}^k$. We shall give an example of an observation of a linear controlled parabolic equation which admits a strong (bilinear) differential realization on $\mathbb{R}^2$. This realization has "singular arcs" which may be studied by high order methods using Lie Theory. This illustrates the difficulties which may occur in "internally

* This research was supported by NSF grant MCS 76-04419-A01.
controlled" systems of partial differential equations, i.e. systems where the control is not a forcing function or boundary value.

I. - LIE PRODUCTS OF OPERATORS.

Throughout, the symbol $s$ will be used as the independent variable for a function $v \in B$. The infinitesimal generators we will consider shall be assumed to have the form

$$\frac{d}{ds} v_t(s) = f_s((T^s v)(s))$$

where for each $s$, $T^s$ is a closed linear operator acting within $B$ (and these may change with the value $s$) while $f^s : \mathbb{R}^n \to \mathbb{R}^n$ is real analytic. We assume, throughout, that our operators generate (at least locally) strongly continuous one parameter semi-groups, i.e., that $dv_t/dt = C(v_t)$, $v_o = \varphi$ has a unique solution for small $t > 0$.

For example, consider the delay equation on $\mathbb{R}^n$; $dx(t)/dt = W(x(t-1))$, $x(s) = \varphi(s)$, $-1 \leq s \leq 0$ where $\varphi \in C[-1,0]$ and $W : \mathbb{R}^n \to \mathbb{R}^n$ is real analytic. Let

$$C(t, \varphi)$$

denote a solution at time $t$ and define $v_t(\varphi)(s) = \zeta(t+s, \varphi)$, $-1 \leq s \leq 0$. The map $t \to v_t(\varphi) \in C[-1,0]$ is a strongly continuous semi-group with infinitesimal generator

$$\left( C v_t \right)(s) = \begin{cases} \frac{d}{ds} v_t(s) = \frac{d}{dt} v_t(s), -1 \leq s < 0 \\ W((S v)_t(0)) = W(v_t(-1)), s = 0 \end{cases}$$

where $S$ is the unit shift operator (i.e. $(Sv)(t) = v(t-1)$). Here, referring to (2), $f^s(\lambda) = \lambda$ and $T^s = d/ds$ if $-1 \leq s < 0$ while $f^s(\lambda) = W(\lambda)$ and $T^s = S$ if $s = 0$. In the Banach space $B = C[-1,0]$, $v_t(\varphi)$ satisfies the equation

$$\frac{d}{dt} v_t(\varphi) = C(v_t), v_o = \varphi.$$ Let $C$ be an operator of the form (2) and $f'$ denote the derivative of $f$. We define $((DC(v)(s))w)(s) = f^s((T^s v)(s))(T^s w)(s)$ and the Lie Product of operators $C, \delta$ of the form (2) as $[[C, \delta] v](s) = ((DC(v)(s))(\delta v))(s) - ((D\delta(v)(s))(Cv))(s)$. As expected, if $C, \delta$ are linear operators then $[C, \delta]$ is just the commutator $C\delta - \delta C$. Furthermore, the above concepts easily generalize to operators of the form $\delta v = \sum_{s=1}^{m} f^s((T^s v)(s)) \delta v_{n,s}$; the details and examples of computations can be found in [4],[5]. We introduce the notation $(\text{ad} C, \delta) = [C, \delta]$ and inductively

$$(\text{ad}^{k+1} C, \delta) = [C, (\text{ad}^k C, \delta)].$$
Now consider the equation (1) with $\mathcal{A}$, $\mathcal{B}$ operators of the form (2). Let $\eta_t(\varphi)$ denote the solution, at time $t$, of $\frac{dv}{dt} = \mathcal{A}(v_t)$, $v_0 = \varphi$, and $D\eta_t(\varphi)$ the differential of the map $\varphi \to \eta_t(\varphi)$. In [4] we showed the

**DECOMPOSITION THEOREM.** - Assume the maps $t \to \eta_t(\varphi)(s)$ and $t \to D\eta_t(\varphi)(s)$ are real analytic for all $s$ and that $D\eta_t(\varphi) \to \text{id}$ in the strong operator topology as $t \to 0$. Then a sufficient condition that the composition $\eta_t(\psi_t(\varphi,u))$ be a solution of equation (1) is that $\psi_t(\varphi,u)$ satisfy

\[
\frac{dv}{dt} = u(t) \sum_{\nu=0}^{\infty} (-t)^\nu \mathcal{V} \cdot (\text{ad} \mathcal{A}, \mathcal{B})(v_t), \quad v_0 = \varphi.
\]

Such decomposition theorems provide the key to the applications of Lie theory to differential equations and control systems, [6], [7].

Let $g : B \to \mathbb{R}^k$ be continuous and linear, $\mathcal{C}$ an operator of the form (2), $\xi_t(\varphi)$ the solution at time $t$ of $\frac{dv}{dt} = \mathcal{C}(v_t)$, $v_0 = \varphi$ and define

$g_*(\mathcal{C}) = \lim_{t \to 0} \frac{d}{dt} g(\xi_t(\varphi))$. Analogous to the case of control systems on manifolds, we associate with (1) the set of operators

\[
\mathcal{J}^1 = \{ (\text{ad}^\nu \mathcal{A}, \mathcal{B}) : \nu = 0, 1, \ldots \}
\]

and let $\mathcal{J}^1(\varphi)$ denote the elements of $\mathcal{J}^1$ evaluated at $\varphi$.

**THEOREM (Local Controllability).** - The observation $g : B \to \mathbb{R}^k$ of system (1) is locally controllable along the reference solution $\eta_t(\varphi)$ corresponding to $u \equiv 0$ (i.e. $g(\eta_t(\varphi)) \in \text{int} \mathcal{G}(A(t,\varphi))$ for small $t > 0$) if $\dim \text{span} g_*(\mathcal{J}^1(\varphi)) = k$.

**Proof:** Form $\tilde{B} = B \times \mathbb{R}^k$; let $w(t) = g(v_t) \in \mathbb{R}^k$ and $\tilde{\mathcal{A}} = \begin{bmatrix} \mathcal{A} & v_t \\ w(t) & g_*(\mathcal{A}v_t) \end{bmatrix}$. The "augmented system" on $\tilde{B}$ is

\[
\begin{bmatrix}
\frac{dv}{dt} \\
\frac{dw}{dt}
\end{bmatrix} = \begin{bmatrix}
\mathcal{A} & v_t \\
\mathcal{B} & w(t)
\end{bmatrix}
+ u(t) \begin{bmatrix}
\mathcal{A} & v_t \\
\mathcal{B} & w(t)
\end{bmatrix}
+ \begin{bmatrix}
\varphi \\
\varphi
\end{bmatrix}.
\]
Define $\mathcal{J}^{1} \begin{bmatrix} \phi \\ g(\phi) \end{bmatrix} = \begin{bmatrix} (\text{ad}^\nu \hat{A}, \hat{a}) \\ g(\phi) \end{bmatrix} : \nu = 0,1, \ldots \}$ and $\pi : \hat{B} \to \mathbb{R}^k$ be projections to $\mathbb{R}^k$, specifically for $\psi \in \hat{B}$, $\psi' \in \mathbb{R}^k$, $\pi \begin{bmatrix} \psi \\ y \end{bmatrix} = y$. Let $\hat{A}(t, \varphi) \subset \hat{B}$ denote the elements attainable at time $t$ by solutions of (i) corresponding to all admissible controls. The reference solution of (i) corresponding to $u = 0$ is $\begin{bmatrix} \eta_t \\ g(\eta_t) \end{bmatrix} \in \hat{A}(t, \varphi)$. Clearly if $v_t(u)$ denotes a solution of (i) at time $t$ corresponding to control choice $u$, then $\begin{bmatrix} v_t(u) \\ g(v_t(u)) \end{bmatrix}$ is a solution of (i). It follows that

(ii) $\pi(\hat{A}(t, \varphi)) = g(A(t, \varphi))$.

Since $\text{span} \mathcal{J}^{1} \begin{bmatrix} \phi \\ g(\phi) \end{bmatrix}$ is the "first order local set of directions" in which one can proceed via solutions of (i), from the inverse function theorem (see [8], chap. I. § 5, in particular corollaries 1, 1s) one has

(iii) $g(\eta_t) = \pi \begin{bmatrix} \eta_t \\ g(\eta_t) \end{bmatrix} \in \text{int} \pi(\hat{A}(t, \varphi)) = \text{int} g(A(t, \varphi))$

for small $t > 0$ if $\text{dim} \text{span} \mathcal{J}^{1} \begin{bmatrix} \phi \\ g(\phi) \end{bmatrix} = k$. To complete the proof one need only show that $\pi(\mathcal{J}^{1} \begin{bmatrix} \phi \\ g(\phi) \end{bmatrix}) = g(\mathcal{J}^{1} \begin{bmatrix} \phi \\ g(\phi) \end{bmatrix})$ which is a straightforward calculation using induction.

As an application, we consider a tension controlled vibrating string. Let $\varrho$ denote density, and $s = s(t)$ tension. The one dimensional equation of the vibrating string is $\varrho \frac{\delta^2 w}{\delta t^2} = \delta / \delta x (s(t) \delta w / \delta x)$, where $w$ measures deflection from the rest position. Choose $u_o$ as a nominal value of $s(t)/\varrho$, $u > 0$ such that $u - u > 0$ and the control $u(t) = s(t)/\varrho - u_o$, with $|u(t)| \ll u$. Let $\delta w / \delta x = v_1$ and $\delta w / \delta t = v_2$ to obtain the first order system

(i) \[
\begin{align*}
\delta v_1 / \delta t = \delta v_2 / \delta x, & \quad v_1(x) = \varphi_1(x), & \quad 0 \leq x \leq \ell \\
\delta v_2 / \delta t = u_o \delta v_1 / \delta x + u(t) \delta v_1 / \delta x, & \quad v_2(x) = \varphi_2(x), & \quad 0 \leq x \leq \ell
\end{align*}
\]

where we assume the string is clamped at both ends so $\varphi_2(0) = \varphi_2(\ell) = 0$. Take $C_o[0, \ell]$ to be those functions in $C[0, \ell]$ which vanish at $0$ and $\ell$ and $B = C[0, \ell] \times C_o[0, \ell]$. The boundary data $w(t, 0) = w(t, \ell) = 0$ is implicit in $B$. 

128
if the constants chosen in recovering \( w \) from \( v_1, v_2 \) are properly chosen.

This is assumed. With 
\[
\mathbf{a} = \begin{bmatrix}
0 & \partial / \partial x \\
\partial / \partial x & 0
\end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix}
0 & 0 \\
\partial / \partial x & 0
\end{bmatrix},
\]

(ii) 
\[
\mathcal{J}^1(\phi) = \{ (\text{ad}^\nu \mathbf{a}, \mathbf{b}) : \nu = 0, 1, \ldots \} = \left\{ \begin{bmatrix}
0 \\
\partial \phi / \partial x
\end{bmatrix}, \begin{bmatrix}
\partial^2 \phi / \partial x^2 \\
-2 \partial \phi / \partial x
\end{bmatrix}, \begin{bmatrix}
2 \partial^2 \phi / \partial x^3 \\
-2 \partial^2 \phi / \partial x^3
\end{bmatrix}, \begin{bmatrix}
2 \partial^3 \phi / \partial x^4 \\
-2 \partial^3 \phi / \partial x^4
\end{bmatrix}, \begin{bmatrix}
-3 \partial^2 \phi / \partial x^5 \\
2 \partial^3 \phi / \partial x^5
\end{bmatrix}, \begin{bmatrix}
-3 \partial^2 \phi / \partial x^5 \\
2 \partial^3 \phi / \partial x^5
\end{bmatrix}, \ldots \right\}.
\]

One may now readily check specific initial data and observations for local controllability. For example, if \( \phi^1(x) = (\pi/\ell) \cos(\pi x/\ell) \), \( \phi^2(x) = 0 \) the solution with \( u(t) \equiv 0 \) (i.e. the reference solution) is \( w(t, x) = \cos(\sqrt{\nu} \pi t/\ell) \sin(\pi x/\ell) \), i.e. at each time \( t \), the position of the string is a scalar multiple of \( \sin(\pi x/\ell) \).

Take as observation \( g^1(v_t) = v_t(\ell/2) \), \( g^2(v_t) = v_t^2(\ell/2) \), i.e. respectively the angle of deflection of the string at \( \ell/2 \) and the velocity of the point at \( \ell/2 \).

Computing shows rank \( g^1(\phi^1) = 1 \), not 2, hence the theorem does not imply local controllability of this observation. Physically this is expected since the initial data was chosen so that \( g^1(v_t) \), the angle of deflection at \( \ell/2 \), would be zero for all \( t \). On the other hand, the velocity of the point at \( \ell/2 \), i.e. \( g^2(v_t) \), can be locally controlled via tension.

One may show that for any integer \( k \) there exists initial data \( \phi^1, \phi^2 \) and an \( \mathbb{R}^k \) valued observation \( g \) which is locally controllable along the reference via tension.

II. - FINITE DIMENSIONAL DIFFERENTIAL REALIZATIONS.

Consider a special case of equation (1) i.e.

\[
\frac{dv_t}{dt} = Cv_t + u(t) b, \quad v_0 = \varphi \in \mathcal{B}
\]

where \( C \) is linear and \( b \in \mathcal{B} \). By differentiating \( g(v_t(u)) \) with respect to \( t \) it follows that if there exists a mapping \( C : \mathbb{R}^k \) such that
then a strong differential realization of $g$ exists. If $C$ has $\ker g$ as an invariant subspace, equation (6) may be used to define $C_\ast g$. This rather severe restriction leads one to "change the order of events". Given an equation of the form (8), suppose $C$ has an invariant subspace, $S$, of co-dim $k$. Then any observation $g$ having $S$ as kernel does admit a strong differential realization. A specific example with entails a slightly more general evolutionary equation than (5) is

Example 2.1. - Consider the parabolic partial differential equation

\begin{equation}
\begin{aligned}
\frac{\partial v}{\partial t} &= \frac{\partial^2 v}{\partial x^2} + u(t) \left( \frac{\partial v}{\partial x} + \sin x \right) \\
v_0(x) &= 2 \cos x + 2 .
\end{aligned}
\end{equation}

Let $S^1$ be the one sphere parametrized by $-\pi \leq x \leq \pi$ and consider equation (7) on $S^1_x [0, \infty)$. Let $B = \{ w \in L^2[-\pi, \pi] : w(-\pi) = w(\pi) = 0 \}$, $\mathcal{A} v_t = \frac{\partial^2 v_t}{\partial x^2}$, $\mathcal{B} v_t = \frac{\partial v_t}{\partial x}$ and $b \in B$ be given by $b(x) = \sin x$. For the observation we choose $g = (g_1, g_2)$ with $g_1(v_t) = \frac{1}{\pi} \int_{-\pi}^{\pi} v_t(x) \sin x \, dx$, $g_2(v_t) = \frac{1}{\pi} \int_{-\pi}^{\pi} v_t(x) \cos x \, dx$. Then $S = \text{span} \{1, \sin x, \cos x : x = 2, 3, \ldots \} = \ker g$ while codim $S = 2$ and $\mathcal{A}, \mathcal{B} : S \rightarrow S$. To compute the strong differential realization, let $v_t(x) = \sum_{j=0}^{\infty} \sigma_j(t) \sin jx + \sum_{j=0}^{\infty} \gamma_j(t) \cos jx$ and form the equation $d/dt g(v_t) = g_\ast \mathcal{A} v_t + u(t) g_\ast \mathcal{B} v_t + u(t) g_\ast b$. Letting $\sigma_1(t) = \gamma_1(t)$, $\gamma_1(t) = \gamma_2(t)$ the differential realization on $\mathbb{R}^2$ is

\begin{equation}
\begin{aligned}
\dot{y}_1 &= -y_1 + u(t) (1-y_2) , \quad y_1(0) = 0 \\
\dot{y}_2 &= -y_2 + u(t) y_1 , \quad y_2(0) = 2 .
\end{aligned}
\end{equation}

If we consider the problem, for (7), of finding that measurable control $u$ with $|u(t)| \leq 1$ such that $g(v_t) = (0,1)$ in minimum time $t$, this leads to the problem of reaching $(0,1)$ in minimum time by a solution of (8). The method of [9, §22] may be used to show that the optimal solution of this latter "bilinear problem" is $y_1(t) \equiv 0$, $y_2(t) = 2e^{-t}$, a singular arc obtained with control $u(t) \equiv 0$. \hfill \_\_\_ \_ \_ \_ \_ \_
REFERENCES


Henry HERMES
Department of Mathematics
University of Colorado
Boulder, COLORADO 80309 (U.S.A.)