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Invariant Functionals and Polynomial Growth

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by

J. W. Jenkins

Recall that a locally compact group $G$ is said to be amenable if there is an element $p \in L^\infty(G)^*$ such that i) $p > 0$, ii) $\langle p, \delta_g \ast \psi \rangle = \langle p, \psi \rangle$ for all $g \in G$ and $\psi \in L^\infty(G)$ (here $\delta_g$ is the unit mass at $g$ and $\ast$ denotes convolution), and iii) $\langle p, \chi_A \rangle = 1$, where $\chi_A$ denotes the characteristic function of a subset $A \subseteq G$. In this note we consider the situation in which the "normalization condition" iii) is replaced by iii)\' $\langle p, \theta \rangle = 1$, where $\theta$ is a given non zero, non negative element of $L^\infty(G)$. The first results on this problem were obtain by Rosenblatt [6].

We begin with the following observation : if $G$ is not compact and $K$ is a compact subset of $G$ then there is an infinite sequence of elements $\{g_i\} \subseteq G$ such that for any positive integer $N$,

$$\chi_G \geq \sum_{i=1}^{N} \delta_{g_i} \ast \chi_K.$$

If $p$ is an element of $L^\infty(G)^*$ that satisfies i), ii), iii)\' for $\theta = \chi_K$, then for all positive integers $N$,

$$\langle p, \chi_G \rangle \geq \sum_{i=1}^{N} \langle p, \delta_{g_i} \ast \chi_K \rangle = N.$$

This contradiction shows that, in general, given a $\theta$, the desired functional can exist only on a proper subspace of $L^\infty(G)$. Hence we begin by considering the smallest translation invariant subspace of $L^\infty(G)$ containing $\theta$.

Definition 1.- A locally compact group is said to be type $A$ if for any non zero, non negative $\theta \in L^\infty(G)$, there is a linear functional $p$
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defined on

\[ S_\theta = \{ \sum \alpha(g) \delta_g * \theta \mid \text{supp } \alpha \text{ finite} \} \]

such that i) \( p > 0 \), ii) \( \langle p, \delta_g * \psi \rangle = \langle p, \psi \rangle \) for \( g \in G \), \( \psi \in S_\theta \), iii) \( \langle p, \theta \rangle = 1 \).

In order to state our first theorem we need to recall the notion of polynomial growth. Fix a left Haar measure on \( G \) and denote the measure of a subset \( A \) of \( G \) by \( |A| \). For any positive integer \( n \), let \( A^n = \{ a_1 a_2 ... a_n \mid a_i \in A \} \). \( G \) is said to have polynomial growth if for every compact neighborhood of \( e \) in \( G \), \( U \), there is a polynomial \( p \) such that \( |U^n| \leq p(n) \) for \( n = 1, 2, 3, ... \). The following theorem was first proved in [6] for discrete groups and in [3] for arbitrary groups.

**Theorem 2.** A locally compact group with polynomial growth is type \( A \).

**Proof.** We define a linear functional \( p \) on \( S_\theta \) by setting

\[ \langle p, \sum \alpha(g) \delta_g * \theta \rangle = \sum \alpha(g) \]. It is obvious that \( p \) satisfies conditions ii) and iii) of definition 1. It only remains to show that \( p \) is positive, i.e. if \( \sum \alpha(g) \delta_g * \theta > 0 \) then \( \sum \alpha(g) > 0 \). For this purpose, let \( U = U^{-1} \) be a compact neighborhood of \( e \) in \( G \) containing the support of \( \alpha \) and such that \( \langle \chi_U, \theta \rangle > 0 \). Set

\[ A_n = \int_{U^n} \theta(s) \, ds \]

and note that \( 0 < A_n \leq ||\theta||_{\infty} |U^n| \). Hence \( \lim(A_n)^{1/n} = 1 \). Thus there is a subsequence \( \{n_k\} \) such that \( A_{n_k-1} / A_{n_k+1} \rightarrow 1 \). Now if \( g \in \text{supp } \alpha \) then \( U^{n-1} \subseteq gU^n \subseteq U^{n+1} \), and hence, for such \( g \),

\[ A_{n-1} \leq \int_{gU^n} \theta(s) \, ds \leq A_{n+1} \].

Recalling that \( \sum \alpha(g) \delta_g * \theta > 0 \), we have
\[ 0 \leq \int_{U} \sum_{\alpha(g)} \delta_{g} \ast \theta(s) \, ds = \sum_{\alpha(g)} \int_{U} \delta_{g} \ast \theta(s) \, ds. \]

Dividing the inequality by \( A \) and taking the limit on \( k \) we have, since \( \text{supp} \alpha \) is finite, \( \sum_{\alpha(g)} > 0 \).

There is a partial converse to theorem 2.

**Theorem 3.** Suppose that \( G \) is a finitely generated, discrete, solvable group or that \( G \) is a connected group, and that \( G \) is type A. Then \( G \) has polynomial growth.

Proof. Suppose that \( G \) a finitely generated, discrete, solvable group or that \( G \) is a connected group, and that \( G \) does not have polynomial growth. Then by [4] and [6], there exist elements \( a, b \in G \) and a compact neighborhood \( U \) of \( e \) in \( G \) such that if \( [a, b] \) denotes the semi group generated by \( a \) and \( b \), we have i) \( a[a, b] \cap b[a, b] = \emptyset \) and ii) \( sU \cap tU = \emptyset \) if \( s, t \in [a, b] \), \( s \neq t \). Let \( V = [a, b] \). Then

\[ \text{supp} \delta_{a} \ast \chi_{V} \cap \text{supp} \delta_{b} \ast \chi_{V} \subseteq aV \cap bV \]

\[ = (a[a, b] \cap b[a, b]) V = \emptyset. \]

Thus \( \chi_{V} > \delta_{a} \ast \chi_{V} + \delta_{b} \ast \chi_{V} \). Hence there can not be a functional \( p \) on \( S_{\chi} \) satisfying the conditions of definition 1.

\( S_{\theta} \) is the smallest subspace of \( L^{\infty}(G) \) on which we might hope to find a translation invariant positive functional normalized with respect to \( \theta \). By contrast, the largest subspace is

\[ L_{\theta} = \{ \psi \in L^{\infty}(G) \mid |\psi| \leq \mu \ast \theta, \mu \in M(G) \}, \]

where \( M(G) \) denotes the algebra of all bounded, regular Borel measures on \( G \). Further, instead of asking only for invariance with respect to convolution by point masses, one could require that \( < p, \mu \ast \psi > = \mu(G) < p, \psi > \) for all \( \psi \in L_{\theta} \) and all \( \mu \in M(G) \). We will construct such functionals \( p \), inductively, obtaining at each stage a
functional invariant with respect to a larger subalgebra of $M(G)$. This construction also requires continuity of the functionals with respect to the following norm, $\| \cdot \|_\Theta$, on $L_\Theta$.

$$\| \psi \|_\Theta = \inf \{ \| \mu \| \mid \| \psi \|_\Theta \leq \mu \star \theta, \mu \in M(G) \} .$$

**Définition 4.** Let $A$ be a subalgebra of $M(G)$. $G$ is said to have $A$-invariance if for all $0 \neq \theta \in L^\infty(G)$ for which $0 \leq \theta \leq f \star \theta$ for some $f \in L^1(G)$, there is a $p \in L^*_\Theta$ such that

i) $p \geq 0$,

ii) $< p, \mu \star \psi > = \mu(G) < p, \psi >, \mu \in A, \psi \in L_\Theta$,

iii) $< p, \mu \star \psi > = \int < p, \delta_x \star \psi > d\mu(x), \mu \in M(G), \psi \in L_\Theta$,

iv) $< p, \theta > = 1$.

The following theorem and proof were inspired by Ludwig [5].

**Theorem 5.** Suppose $G$ is a connected group with polynomial growth or that $G$ is a compact extension of a nilpotent group. Then $G$ has $M(G)$-invariance.

The proof of this theorem requires the following.

**Lemma 6.** Suppose $G$ has polynomial growth and that $G$ contains normal subgroups $H$ and $K$ with $K \subseteq H$. Suppose further that each element of $H/K$ is contained in a compact neighborhood that is invariant under the inner-automorphisms from $G/K$. Then, if $G$ has $M(K)$-invariance it has $M(H)$-invariance.

Proof. Let $\theta$ be a non zero, non negative element of $L^\infty(G)$ and let $p \in L^*_\Theta$ satisfying i) - iv) of definition 4 with respect to $M(K)$.

Let $\hat{U}$ be a compact neighborhood of $e$ in $H/K$ that is invariant under inner-automorphism from $G/K$. Given $\varepsilon > 0$, we define $f_\varepsilon, \hat{u}(x)$ on $H/K$ by $f_\varepsilon, \hat{u}(x) = (1 + \varepsilon)^{-1}$ if $\hat{x} \in \hat{U}$, $f_\varepsilon, \hat{u}(x) = (1 + \varepsilon)^{-n}$ if $\hat{x} \in \hat{U}^n - \hat{U}^{n-1}, n \geq 2$, and $f_\varepsilon, \hat{u}(x) = 0$ if $\hat{x} \varepsilon < \hat{U}$, the closed
(and open) subgroup of $H/K$ generated by $\hat{U}$. Since $H/K$ has polynomial growth

$$\|f_{\epsilon}, \hat{U}\|_1 = (1 + \epsilon)^{-1}|\hat{U}| + \sum_{n=2}^{\infty} (1 + \epsilon)^{-n} |\hat{U}^n - \hat{U}^{n-1}| < \infty.$$  

Furthermore, for $y \in \hat{U}$ and $x \in <\hat{U}>$

$$|f_{\epsilon}, \hat{U}(\hat{x} \hat{y}) - f_{\epsilon}, \hat{U}(\hat{x})| \leq \epsilon f_{\epsilon}, \hat{U}(\hat{x}).$$

Since $\theta \leq f_0 * \theta$ for some $f_0 \in L^1(G)$,

$$1 = <p, \theta> = <p, f_0 * \theta> = \int_G f_0(x) <p, \delta_x * \theta> dx.$$

Thus, $<p, \delta_x * \theta> > 0$ on some open subset of $G$. Hence, for some $a \in G$, $<p, \delta_a x * \theta> > 0$ for $x$ is some open subset of $<\hat{U}>$. Since $<p, \delta_k * \psi> = <p, \psi>$ for all $k \in K$, we can define $<p, \delta_x * \psi>$ for $x \in H/K$ by setting $<p, \delta_x * \psi> = <p, \delta_x * \psi>$. We define $p'_\epsilon, \hat{U}$ on $L_\theta$ by

$$<p'_\epsilon, \hat{U}, \psi> = \int_{H/K} f_{\epsilon}, \hat{U}(\hat{x}) <p, \delta_{\hat{x}} * \psi> d\hat{x}.$$

Fix $\delta \in H/K$ so that

$$(1 - \epsilon)^{-1} <p'_\epsilon, \hat{U}, \delta_{\hat{U}} * \theta> \sup_{y} <p'_\epsilon, \hat{U}, \delta_{\hat{y}} * \theta> = \alpha,$$

and define $p_{\epsilon}, \hat{U}$ on $L_\theta$ by setting

$$<p_{\epsilon}, \hat{U}, \psi> = \alpha^{-1} <p'_\epsilon, \hat{U}, \delta_{\hat{U}} * \psi>,$$

for all $\psi \in L_\theta$.

We first show that $\|p_{\epsilon}, \hat{U}\|_\theta \leq 1$. For this, suppose that $\psi \in L_\theta$ and that $|\psi| \leq \mu * \theta$. Then
\[ |\langle p_\varepsilon, \hat{u}, \psi \rangle| \leq |\langle p_\varepsilon, \hat{u}, \psi \rangle| = \alpha^{-1} \int_{H/K} f_\varepsilon, \hat{u}(\hat{x}) < p, \delta_{axb} \psi > d\hat{x} \]

\[ \leq \alpha^{-1} \int_{H/K} f_\varepsilon, \hat{u}(\hat{x}) < p, \delta_{axb} \psi > d\hat{x} \]

\[ = \alpha^{-1} \int_{G} \int_{H/K} f_\varepsilon, \hat{u}(\hat{x}) < p, \delta_{axb} \psi > d\hat{x} d\mu(z) \]

\[ \leq \int_{G} d\mu(z) \leq \|\mu\| . \]

Also note that if \( \hat{y} \in \hat{U} \), \( \psi \in L_\Theta \), then

\[ |\langle p_\varepsilon, \hat{u}, \delta_{y} \psi \rangle| - |\langle p_\varepsilon, \hat{u}, \psi \rangle| \]

\[ \leq \alpha^{-1} \left| \int f_\varepsilon, \hat{u}(\hat{x}) < p, \delta_{axby} \psi - \delta_{axb} \psi > d\hat{x} \right| \]

\[ = \alpha^{-1} \int |f_\varepsilon, \hat{u}(\hat{x})| < p, \delta_{axb} \psi > d\hat{x} \]

\[ \leq \varepsilon \alpha^{-1} \int f_\varepsilon, \hat{u}(\hat{x}) < p, \delta_{axb} \psi > d\hat{x} \]

\[ = \varepsilon |\langle p_\varepsilon, \hat{u}, \psi \rangle| . \]

Choose a sequence of compact neighborhoods of \( e \) in \( H/K \), \( \hat{U}_n \), that are invariant under the inner-automorphisms from \( G/K \) and such that

\[ \cap_{n=1}^\infty \hat{U}_n = H/K . \]

For each positive integer \( n \), let \( p_n = p_{1/n}, \hat{U}_n \). By \( \omega^* \)-compactness of the unit ball in \( L_\Theta^* \), the sequence \( \{p_n\} \) has a cluster point \( p_\infty \). Clearly \( p_\infty \geq 0 \) and, since

\[ 1 > \langle p_n, \theta \rangle = \langle p_{1/n}, \hat{U}_n, \theta \rangle = |\alpha^{-1} \langle p_{1/n}, \hat{U}_n, \delta_{a_n} \theta \rangle | \geq 1 - 1/n , \]

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\(< p_\infty, \theta > = 1 \). It is also clear that for \( k \in K \) and \( \psi \in L_0 \),

\(< p_\infty, \delta_k \ast \psi > = < p_\infty, \psi > \) and hence that \(< p_\infty, \delta_y \ast \psi > = < p_\infty, \delta_y \ast \psi >

= < p_\infty, \psi > \) for \( y \in H/K \).

If \( \psi \in L_0 \) and \( \mu \in M(G) \), then

\(< p_\infty, \mu \ast \psi > = \lim < p_{1/n}, \check{\psi}_{n}, \mu \ast \psi >

= \lim \int_G < p_{1/n}, \check{\psi}_{n}, \delta_x \ast \psi > d\mu(x) \).

Since

\(|< p_{1/n}, \check{\psi}_{n}, \delta_x \ast \psi >| \leq ||\psi||_0 \),

the dominated convergence theorem gives

\(< p_\infty, \mu \ast \psi > = \int < p_\infty, \delta_x \ast \psi > d\mu(x) \).

Finally, note that if \( \mu \in M(H) \), \( \psi \in L_0 \),

\(< p_\infty, \mu \ast \psi > = \int < p_\infty, \delta_x \ast \psi > d\mu(x)

= \mu(G) < p_\infty, \psi > \).

Therefore \( p_\infty \) satisfies i) - iv) of definition 4.

Proof of theorem 5. If \( N \) is a normal, nilpotent subgroup of \( G \) such
that \( G/N \) is compact and if \( N = N_1 \supset \ldots \supset N_k = \{e\} \) is the lower center
series for \( N \), then for each \( i = 1, \ldots, k-1 \), \( N_i/N_{i+1} \) has large compact
neighborhoods invariant under the inner-automorphisms form \( G/N_{i+1} \).

If \( G \) is a connected Lie group with polynomial growth then the eigenvalues
of \( \text{Ad}s \) are of modulus 1 for all \( s \in G \) (see [4] or [1], where the follo­
wing fact was first pointed out). Let \( S \) be the solu-radical of \( G \), \( LS \)
its Lie algebra and \( LS_\mathbb{C} \), the complexification of \( LS \). By Lie's theorem,
there is an ordered basis for \( LS_\mathbb{C} \), \( \{X_1, \ldots, X_m\} \) so that the matrix
representation for Ads with respect to this basis is upper triangular for all \( s \in S \). Let \( V_j \) be the subspace spanned by \( \{X_1, \ldots, X_j\} \). Then the action of Ads on \( V_j / V_{j-1} \) is multiplication by \( \alpha_j(s) \) where \( |\alpha_j(s)| = 1 \), i.e. Ads acts by rotation on each space \( V_j / V_{j-1} \). Thus we can find subspaces \( \{0\} = W_0 \subset W_1 \subset \ldots \subset W_n \) of \( L^S \), each invariant under Ads with \( \dim(V_j / V_{j-1}) \leq 2 \), and with Ads acting by rotation on each space \( W_j / W_{j-1} \), \( j = 1, 2, \ldots, n \). Thus \( W_j \) is an ideal in \( L^S \) and if \( S_j \) is the corresponding closed normal subgroup of \( S \), each element of \( S_j / S_{j-1} \) has large compact neighborhoods invariant under the inner-automorphisms from \( S / S_{j-1} \), and also \( G/S \), since \( G/S \) is compact.

If \( G \) is a connected group with polynomial growth, then there is a compact, normal subgroup \( K \) of \( G \) such that \( G/K \) is a Lie group with polynomial growth. The above argument applied to \( G/K \) produced the desired series of subgroups.

In order to show that polynomial growth is not sufficient to imply \( M(G) \)-invariance, as is the case with being type A, we prove the following.

**Theorem 7.** Let \( A \) be a subalgebra of \( M(G) \) that contains the point masses and is closed with respect to involution. If \( G \) has \( A \)-invariance then for each \( f \in A \cap L^1(G) \), \(-1 \notin \text{sp}(f * f^*)\).

Proof. Assume there is an \( f \in A \cap L^1(G) \) such that \(-1 \in \text{sp}(f * f^*)\). Then for all \( g \in L^1(G) \), \( \|g * f * f^* + g + f * f^*\| \geq 1 \), for other wise,

\[
[(g-1)(-f * f^* - 1)]^{-1} = [1 + [(g-1)(-f * f^* - 1) - 1]]^{-1}
\]

exists in \( C \cap L^1(G) \), contradicting the assumption that \(-1 \in \text{sp}(f * f^*)\). Hence, there is a \( \varphi \in L^\infty(G) \) such that \( <g * f * f^* + g, \varphi> = 0 \) for all \( g \in L^1(G) \) and \( <f * f^*, \varphi> = 1 \). Thus \( f * f^* * \varphi = -\varphi \).

Let \( \theta = \|\varphi\|^2 \). Note that for \( h \in L^1(G) \)
\[ |h * \varphi|^2(s) = \left| \int h(t) \varphi(t^{-1}s) \, dt \right|^2 \]
\[ \leq \left[ \int |h(t)|^{1/2} |h(t)|^{1/2} |\varphi(t^{-1}s)| \, dt \right]^2 \]
\[ \leq \int |h(t)| \, dt \int |h(t)| |\varphi(t^{-1}s)|^2 \, dt \]
\[ = \|h\|_1 \|h\| \|\varphi\|^2 . \]

Hence, \( \vartheta = |\varphi|^2 = |f * f^* * \varphi|^2 \leq \|f * f^*\| |f * f^*| * \theta . \)

Thus, there is a functional \( \rho \) on \( h_0 \) satisfying i) - iv) of definition 4 with respect to \( A \). Now, if \( g, h \in L^1(G) \),

\[ (g * \varphi)(h * \varphi) \leq \frac{1}{2} \left\{ |g * \varphi|^2 + |h * \varphi|^2 \right\} \in L_0 . \]

We define a bilinear form \( B \) on \( A \cap L^1(G) \) by

\[ B(g, h) = < \rho, (g * \varphi)(h * \varphi) > . \]

Clearly \( B(g, g) \geq 0 \) for \( g \in A \cap L^1(G) \) and

\[ B(f * f^*, f * f^*) = < \rho, |\varphi|^2 > = 1 . \]

Also, for \( g \in A \cap L^1(G) \),

\[ B(g, g) = < \rho, |g * \varphi|^2 > \leq \|g\|_1 < \rho, |g * \theta > = \|g\|_1^2 . \]

Thus \( B \) is bounded, and if \( g, h \in A \cap L^1(G) \), \( s \in G \),

\[ B(\delta_s * g, \delta_s * h) = < \rho, \delta_s * [(g * \varphi)(h * \varphi)] > = B(g, h) . \]

Hence, for \( g, h, k \in A \cap L^1(G) \)

\[ B(g * h, k) = B(h, g^* * k) . \]

Recalling that \( f * f^* \varphi = -\varphi \), we have

\[ 0 \leq B(f^*, f^*) = -B(f^* * f * f^*, f^*) = -B(f * f^*, f * f^*) = -1 . \]

This contradiction implies that \( -1 \notin \text{sp}(f * f^*) \).
A Banach *-algebra $A$ is said to be symmetric if $-1 \in \sigma_p(aa^*)$ for all $a \in A$. Theorems 5 et 7 show that $L^1(G)$ is symmetric if $G$ is a connected group with polynomial growth or if $G$ is a compact extension of a nilpotent group. This was first proved by Ludwig [5].

Hulanicki [2], considered the following group: let $D$ be a countable direct sum of $\mathbb{Z}_2^\infty$, and let

$$D = \prod_{d \in D} (D)_d = \{ (x_d)_d \in D \mid x_d \in D \}$$

be the direct product of $D$ copies of $D$. Define $t : D \to \text{Aut}(D)$ by $t(c)(x_d)_d \in D = (x_d + c)_d \in D$. Then $D \ltimes_t D$ is a solvable group and for each $s \in D \ltimes_t D$, $s^4 = e$. Hulanicki showed that $L^1(D \ltimes_t D)$ is not symmetric, hence $D \ltimes_t D$ is a solvable group with polynomial growth that does not have $M(D \ltimes_t D)$-invariance. Hence even for solvable groups being type $A$ is a weaker condition than having $M(G)$-invariance.

Let $F(G)$ denote the subalgebra of $M(G)$ consisting of all measures with finite support.

Using the Krein extension theorem, one can easily see that $G$ is type $A$ if and only if $G$ has $F(G)$-invariance. Hence polynomial growth implies $F(G)$-invariance. Using the same techniques as in the proof of theorem 2, one can show that polynomial growth implies $K(G)$-invariance where $K(G)$ denotes the measures with compact support, or even invariance with respect to algebras of measures that vanish sufficiently rapidly at infinity. (In Hulanicki's example, for instance, one selects a sequence of subsets $U_n \subset U_{n+1}$ such that $U \cup_n = D \ltimes_t D$ and requires the measure $\mu$ to satisfies

$$\lim_n |< u_n >| |\mu| (D \ltimes_t D \not\subset < u_n >) = 0.$$  

It is not known how large a subalgebra will give invariance for arbitrary groups with polynomial growth.
References


