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Astérisque, tome 63 (1979), p. 169-177

<http://www.numdam.org/item?id=AST_1979__63__169_0>
THE WEIL GROUP AS AUTOMORPHISMS
OF THE LUBIN-TATE GROUP

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Introduction:

Let \( L \) be a finite extension of \( \mathbb{Q}_p \), with maximal abelian extension \( L_{ab} \); then the canonical monomorphism \( \alpha \) of \([5, \S 3]\) maps the multiplicative group of \( L \) onto an open dense subgroup \( W(L_{ab}/L) \) of the Galois group of \( L_{ab} \) over \( L \). These modified Galois \([\text{or } W-\text{]}\) groups can be defined more generally, and behave very much like Galois groups \([8, \text{ appendix II}]\), but for some purposes they are more convenient.

For example, there is a representation of \( W(L_{ab}/L) \) on the \( \mathbb{Q}_p \)-vector space \( L \), defined by the obvious multiplication map \( L^X \times L \rightarrow L \).

The trace of this representation defines a \( p \)-adic character of \( W(L_{ab}/L) \) and therefore \([\text{via the natural homomorphism from } W(\overline{\mathbb{Q}_p}/L) \text{ to } W(L_{ab}/L)]\) a \( p \)-adic character of \( W(\overline{\mathbb{Q}_p}/L) \). In this note we construct an extension of this character to \( W(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \) when \( L \) is normal over \( \mathbb{Q}_p \).
Our construction uses a theorem of Safarevič: if $d$ is the degree of $L$ over $\mathbb{Q}_p$, and $D$ is a division algebra with center $\mathbb{Q}_p$, and $P$ is an invariant of $D$ over $\mathbb{Q}$, then $L$ can be embedded as a commutative subfield of $D$; let $N(L)$ be the normaliser of the multiplicative group $L^X$ of $L$ in $D^X$. The canonical morphism $\alpha$ then extends to a canonical isomorphism $\nu : N(L) \to W(L_{ab}/\mathbb{Q}_p)$, and the composition of $\nu^{-1}$ with the reduced trace from $D$ to $\mathbb{Q}_p$ defines a character of $W(L_{ab}/\mathbb{Q}_p)$ and therefore of $W(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$.

In §1 we identify $N(L)$ with a group of "extended automorphisms" of the Lubin-Tate group of $L$: this action defines a cocycle (and thus a representation) $\omega$ of $N(L)$, whose trace is the character described above.

The present work was motivated by the construction of a (topological) spectrum which admits $N(L)$ as a group of automorphisms, such that the representation defined on its $n^{th}$ homotopy group is the $n^{th}$ tensor power of $\omega$ [6]. However, the result of §1 suggests the hope of a constructive proof of the Weil-Safarevič theorem [which might shed some light on the interpretation of $W(L_{ab}/\mathbb{Q}_p)$ as a group of automorphisms [9]] and could therefore be of wider interest.

I wish to thank the US Academy of Sciences and the Steklov Institute of Mathematics for their support of this research, and Yu. I. Manin [resp. Ramesh Gangolli and Han Sahl] for interesting conversations during its early [resp. late] stages. It is a pleasure also to thank the organisers of the Journées de géométrie algébrique for some exciting days in Rennes.

§1, proof of the main result

1.1. A continuous homomorphism $\phi : A[[T]] \to A[[T]]$ of commutative
formal power series rings will be called an extended endomorphism, if

1) $\phi(T)$ lies in the ideal generated by $T$, and

2) the image of the composition $A \xrightarrow{\phi} A[[T]] \to A[[T]]$ lies in $A$.

Consequently $\phi \left( \sum a_i T^i \right) = \sum \phi(a_i) \phi(T)^i$.

Note that the composition of two extended endomorphisms is another, and that the tensor product $\phi \otimes_A \psi$ maps $A[[T \otimes_A T]]$ to itself by $(\phi \otimes_A \psi) \left( \sum a_{ij} T_i \otimes T_j \right) = \sum \phi(a_{ij}) \psi(T)^i \otimes_A \psi(T)^j$.

If $F(X,Y) \in A[[X,Y]]$ is a [one-parameter, commutative] formal group law over $A$, then the extended endomorphism $\phi$ of $A[[T]]$ will be called an extended endomorphism of $F$ provided that the diagram

$$
\begin{array}{ccc}
A[[T]] & \xrightarrow{\Delta_F} & A[[T \otimes_A T]] \\
\phi \otimes_A \psi & & \phi \otimes \psi
\end{array}
$$

[defined by $\Delta_F(T) = F(T \otimes_A T)$] is commutative.

If $\text{Aut}^*(F)$ denotes the group of extended automorphisms of $F$ [under composition], then it follows from 1) and 2) that there is an exact sequence

$$1 \rightarrow \text{Aut}_A(F) \rightarrow \text{Aut}^*_A(F) \rightarrow \text{Aut}_{(\text{rings})}(A) \rightarrow 1$$

with the terminal group consisting of the continuous ring-automorphisms of $A$; the usual automorphisms of $F$ over $A$ [4.1.6]2] define the group $\text{Aut}_A(F)$.

1.2. We write $\mathcal{O}_L$ for the valuation ring of $L$, and $^{\wedge}\mathcal{O}_L$ for the valuation ring of the completion $^\wedge L$ of a maximal unramified extension $L_{nr}$ of $L$; if $I$ denotes the residue field of $^{\wedge}\mathcal{O}_L$, and $\mathcal{I}$ is the union of the finite fields, then $^{\wedge}\mathcal{O}_L \cong \mathcal{O}_L \otimes_{W(I)} W(\mathcal{I})$. 

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If \( \pi \in \mathbb{Q}_L \) is a uniformising element, and \( q \) is the cardinality of \( \mathcal{L} \), then the series
\[
\log_{\pi}(T) = \sum_{i \geq 0} \pi^{-1} q^i
\]
defines a formal group law \( F_{\pi}(X,Y) = \log_{\pi}^{-1}(\log_{\pi}(X) + \log_{\pi}(Y)) \) for which the map \( \frac{X}{\pi} \circ \pi \mapsto [a]_{\pi}(T) = \log_{\pi}^{-1}(a \cdot \log_{\pi}(T)) \in \text{Aut}_{\pi}(F_{\pi}) \) is a bijection \([1]\). By "the" Lubin-Tate group of \( L \), we mean the class of formal group laws over \( \mathbb{Q}_L \) isomorphic to \( F_{\pi} \) for some (and hence any) choice of \( \pi \). If \( \pi_0, \pi_1 \) are two choices of uniformising element, then \([5\text{, lemma } 2]\) there exists an invertible series \( \phi_0(T) \in \mathbb{Q}_L[[T]] \) such that
i) \( \phi_0 \) is an isomorphism of \( F_{\pi_0} \) with \( F_{\pi_1} \), and
ii) if \( \sigma \) is the automorphism of \( \mathbb{L} \) defined by the Frobenius operation \( x \mapsto x^q \) on the residue field, then
\[
\sigma(\phi_0(T)) = \phi_0([\pi_0^{-1} \pi_1^q](T)).
\]

We denote the formal group law over \( \mathcal{L} \) defined by reducing \( F_{\pi} \) modulo the maximal ideal \( \mathfrak{m}_L \) of \( \mathbb{Q}_L \) by \( F_{\pi} \): its height equals the degree of \( L \) over \( Q_p \) \([1\text{, lemma } 9]\).

1.3. Now the ring of endomorphisms of a group law of height \( d \) over an algebraically closed field of characteristic \( p \) is the valuation ring \( \mathbb{Q}_D \) of a division algebra \( D \) with center \( Q_p \) and invariant \( d^{-1} \in Q/Z = Br(Q_p) \) \([4\text{, I } \S 7.42]\), and the normalised ordinal valuation of an element of \( \mathbb{Q}_D \) is its height as a power series. It follows that the sequence of 1.1 can be continued to the right as
\[
1 \longrightarrow \mathbb{Q}_D^X \longrightarrow \text{Aut}_{\pi}(F_{\pi}) \longrightarrow \mathbb{Z}(\mathbb{I}/F_{\pi}) = \mathbb{L} \longrightarrow 1
\]
to construct a lifting of the Frobenius endomorphism \( \alpha_0(x) = x^p \) of \( \mathbb{I} \),
let $\phi \in \mathcal{O}_E$ be an endomorphism of height 1 [so $\phi(T) = \phi(T^p)$] with $\phi$ an invertible series; then $\phi = \phi_R^\infty \phi_0$ has the desired property.

This shows moreover that $\text{Aut}_I^*(\mathbb{F}_p)$ is isomorphic to the profinite completion of the multiplicative group $I^\times$ of $\mathbb{F}_p$ under the correspondence which sends the endomorphism $\phi$ (which can be written as $\phi(T) = \phi_0(T^p)$ with $\phi_0$ invertible and $p \neq 0$) to the extended automorphism $\phi_0^\infty \phi_0$. It suffices to see that the conjugation of a series $\alpha \in \mathcal{O}_p^\times$ by $\phi$ in $\text{Aut}_I^*(\mathbb{F}_p)$ agrees with its conjugation by $\alpha$ in $I$, or that $P\phi_0^{-1} \alpha \phi_0^{-1} = \alpha \phi$, where $P(T) = T^p$; this is an elementary exercise in the composition of power series.

1.4. It follows similarly that if $L$ is a normal extension of $\mathbb{Q}_p$, then $\text{Aut}_{\mathcal{O}_L}^*(\mathbb{F}_|_L)$ is a central topological extension of the Galois group $G(\mathbb{L}_{nr}/\mathbb{Q}_p)$ by the multiplicative group $\mathcal{O}_L^\times$. To see that the final homomorphism of the sequence in 1.1 is onto, note that if $v_0 = v$ is a uniformising element and $\sigma \in G(\mathbb{L}_{nr}/\mathbb{Q}_p)$ then $\sigma = v(\pi)$ is another and $\sum_{1 \leq i \leq s} a_i T^i \mapsto \sum_{1 \leq i \leq s} v(a_i)(\phi_0^1(T))i$ defines a (noncanonical) lift of $g$ to an extended automorphism. Since any automorphism of a formal group law over an integral domain of characteristic 0 is determined by its leading coefficient, the group $G(\mathbb{L}_{nr}/\mathbb{Q}_p)$ acts on the subgroup $\mathcal{O}_L^\times$ via the canonical homomorphism to $G(\mathbb{L}/\mathbb{Q}_p)$.

1.5. Now an extended automorphism of $\mathbb{L}_\mathcal{O}[[T]]$ maps the ideal $\mathbb{L}_\mathcal{O}[[T]]$ to itself, so an extended automorphism $\phi$ of $\mathbb{F}_p$ defines an extended automorphism of $\mathbb{F}_p$, which we will denote by

$$\rho : \text{Aut}_{\mathcal{O}_L}^*(\mathbb{F}_|_L) \to \text{Aut}_{\mathbb{F}_p}^*(\mathbb{F}_p).$$

Since the reduction of a usual automorphism of $\mathbb{F}_p$ is a usual automorphism of $\mathbb{F}_p$, we have a commutative diagram.
Now the final vertical arrow fits in an exact sequence

\[ 1 \rightarrow I(L/Q_p) \rightarrow G(L_{nr}/Q_p) \rightarrow G(\mathcal{I}/F_p) \rightarrow 1 \]

which defines the inertia group of \( L \) over \( Q_p \), and the homomorphism \( \rho_0 \) is injective since \( F^\pi \) is of finite height. It follows that \( \rho_0 \) is injective, for \( I(L/Q_p) \) acts effectively on \( G_{L}^X \).

It will simplify matters to pull our commutative diagram back along the dense embedding \( \mathbb{Z} \rightarrow \hat{\mathbb{Z}} \) : the effect is to replace \( (D^X)^G \) with \( D^X \), \( G(L_{nr}/Q_p) \) with the open dense subgroup \( W(L_{nr}/Q_p) \), and \( \text{Aut}^*(F^\pi) \) with an open dense subgroup which we will denote \( \text{Aut}^0 \); the original diagram can be recovered by profinite completion.

1.6. It remains to identify the image of \( \rho \). We observe first that because \( F^\pi \) has coefficients in \( I \), the extended automorphism \( \rho([\pi]) = \sigma^d = \sigma^d \) commutes with elements of \( \rho_0(\mathcal{G}_{L}^X) \) in \( D^X \). It follows that \( \rho_0(\mathcal{G}_{L}^X) \) and \( \sigma^d \) generate a (normal) subgroup of \( \text{Aut}^0 \) isomorphic to \( L^X \), and that the image of \( \rho \) is therefore contained in the normaliser \( N(L) \) of \( L^X \) in \( D^X \). But now the Weyl group of \( L^X \) in \( D^X \) is \( G(L/Q_p) \) if \( L \) is normal [8, appendix III §7] so we have a commutative diagram

\[
\begin{array}{cccccc}
1 & \rightarrow & \mathcal{L}_L^X & \rightarrow & \text{Aut}^*(F^\pi) & \rightarrow & G(L_{nr}/Q_p) & \rightarrow & 1 \\
& & \downarrow \rho_0 & & \downarrow \rho & & \downarrow & & \\
1 & \rightarrow & \mathcal{L}_D^X & \rightarrow & (D^X)^G & \rightarrow & G(\mathcal{I}/F_p) \cong \hat{\mathbb{Z}} & \rightarrow & 1
\end{array}
\]
with exact rows and columns. If \( x \in N(L) \) then there is some \( y \) in \( \Aut^0 \) such that \( z = y^{-1}x \) lies in \( L^X \), so \( x = yz \) lies in \( \Aut^0 \).

This completes the proof of

1.7. **proposition:** The morphism \( \rho \) maps an open dense subgroup of \( \Aut^X_\Delta(F_n) \) onto the normaliser \( N(L) \) of \( L^X \) in \( D^X \).

§2. some corollaries

2.1. If \( \delta \in N(L) \), then we write \( \overline{\rho}^{-1}(\delta)(T) = \overline{w}(\delta)T + \) higher order terms for the action of the extended automorphism \( \overline{\rho}^{-1}(\delta) \) on the formal parameter \( T \); here \( \overline{w}(\delta) \) is a unit of \( \overline{\Delta^X} \). The composition \( N(L) \xrightarrow{\overline{\rho}^{-1}(\delta)} W(L_{ab}/Q_p) \xrightarrow{w} W(L_{nr}/Q_p) \) defines an action of \( N(L) \) on \( \Delta^X \) which we will denote by juxtaposition. With this notation, we have

\[
\overline{w}(\delta\delta_1) = \overline{\delta_1}(\overline{w}(\delta)) \cdot \overline{w}(\delta_1);
\]

in other words, \( \overline{w} \) is a crossed antihomomorphism from \( N(L) \) to \( \overline{\Delta^X} \).

Note that if \( \delta \in \overline{\Delta^X} \), then \( \overline{w}(\delta) = \delta^{-1} \) [7, III§A4].

An extended automorphism of \( F_n \) defines an extended automorphism of \( F_n \otimes_{\Delta^X} H \), and it follows that

\[
\overline{\rho}^{-1}(\delta)(T) = \log_{\delta^{-1}_{\Delta^X}}(\overline{w}(\delta) \cdot \log_{\Delta^X}(T));
\]

consequently the crossed antihomomorphism \( \overline{w} \) specifies the action of \( N(L) \) on \( \overline{\Delta^X}[[T]] \).

2.2. The 1-cocycle \( \delta \mapsto \overline{w}(\delta^{-1}) \) of \( N(L) \) with values in the right \( N(L) \) module \( (\overline{\Delta^X})^{op} \) [defined by \( x^{op} \delta = (\overline{\delta^{-1}})^{op}x \)] defines a class in a continuous cochain cohomology group isomorphic to \( H^1_c(W(L_{ab}/Q_p);(\overline{\Delta^X})^{op}) \).

The Hochschild-Serre spectral sequence of the topological extension

\[
E : 1 \xrightarrow{} W(L_{ab}/L_{nr}) \xrightarrow{} W(L_{ab}/Q_p) \xrightarrow{} W(L_{nr}/Q_p) \xrightarrow{} 1
\]
yields an exact sequence.
\[
\cdots - H^1_c(W(L_{ab}/\mathbb{Q}_p); \hat{\omega}_L^X) - H^0_c(W(L_{nr}/\mathbb{Q}_p); \hat{\omega}_L^X) - H^1_c(\hat{\omega}_L^X; \hat{\omega}_L^X) \cong G(L/\mathbb{Q}_p)
\]

invariants of \( \text{Hom}_c(\hat{\omega}_L^X, \hat{\omega}_L^X) \rightarrow H^0_c(W(L_{nr}/\mathbb{Q}_p); \hat{\omega}_L^X) \cong H^0(G(L/\mathbb{Q}_p); \hat{\omega}_L^X) \)
of terms of low degree. The existence of the cocycle \( \omega \) implies
\[
d_2 = 0; \text{ since } d_2(x) = -x \cup [x] \text{ theorem } 4 \text{ it follows that; the in}
\]
clusion \( \hat{\omega}_L^X - \hat{\omega}_L^X \) induces the zero map from \( H^0_c(W(L_{nr}/\mathbb{Q}_p); \hat{\omega}_L^X) \) to \( H^2_c(W(L_{nr}/\mathbb{Q}_p); \hat{\omega}_L^X) \). A direct proof of this might suggest a construc-tion for \( \omega \).

2.3. The isomorphism \( G(L_{ab}/\mathbb{Q}_p) \) with \( \text{Aut}^* (F_{\pi}) \) defined in §1.7

\( \hat{\omega}_L^X \)
respects an implicit proalgebraic group structure, which may be made explicit by observing that \( G(L_{ab}/\mathbb{Q}_p) \) is isomorphic to the semidirect product \( \text{I}(L_{ab}/\mathbb{Q}_p) \cdot G(\overline{\mathcal{I}}/F_p) \), in which \( \text{I}(L_{ab}/\mathbb{Q}_p) \) is the inertia group of \( L_{ab} \) over \( \mathbb{Q}_p \). In particular, \( \text{I}(L_{ab}/\mathbb{Q}_p) \) admits a continuous action of \( G(\overline{\mathcal{I}}/F_p) \), and may therefore be regarded as a proetale groupscheme over \( F_p \) [2,II§5]. On the other hand \( \hat{\omega}_L^X \) is represented by a group of power series with coefficients in \( \overline{\mathcal{I}} \), and has an obvious structure as proetale groupscheme defined \( \text{a priori} \) over \( \mathcal{I} \), in which the generator of \( G(\overline{\mathcal{I}}/\mathcal{I}) \) acts on \( \hat{\omega}_L^X \) by \( \pi \)-conjugation in \( D^X \). The maximal compact subgroup \( N^0(L) \) of \( N(L) \) inherits this structure.

However, if the uniformising element \( \pi \) of \( \hat{\omega}_L^X \) is chosen to satisfy an Eisenstein equation with coefficients in \( F_{\pi} \), then \( F_{\pi} \) has coefficients in \( F_p \), and \( \hat{\omega}(X) = X^0 \) defines an endomorphism of \( F_{\pi} \) which maps to an \( f \)-th root of \( \pi \) in \( \text{Aut}^*(F_{\pi}) \), where \( q = p^f \). It follows that \( N^0(L) \) is in fact a proetale groupscheme defined over \( F_p \), and is isomorphic as such to \( \text{I}(L_{ab}/\mathbb{Q}_p) \). Consequently the group of \( F_p \)-valued points of \( \text{I}(L_{ab}/\mathbb{Q}_p) \) can be identified with the auto-morphisms of \( F_{\pi} \) defined over \( \hat{\mathcal{I}}_p \), which leads to the
corollary: $I(L_{ab}/Q_p)(F_p) \cong \mathbf{A}_p$

references:
2. M. Demazure, P. Gabriel, Groupes Algébriques I, North Holland

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