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Downcrossings and the Markov property of local time

Astérisque, tome 52-53 (1978), p. 89-115

<http://www.numdam.org/item?id=AST_1978__52-53__89_0>
Paul Lévy suggested that the Brownian local time at zero could be gotten as a limit of the number of downcrossings of the interval \((0, \epsilon)\) multiplied by \(\epsilon\), as \(\epsilon \to 0\). This was proved in (3), and a shorter proof has been given in (1).

The method is interesting for several reasons, not the least of which is that it works equally for any continuous local martingale. We refer the reader to (2) for a treatment of local time for semi-martingales from this viewpoint.

Our aim in this article is to use the local-time-as-a-limit-of-upcrossings to approach the Markov properties of the local time discovered by F. Knight and D. Ray.

This approach is quite computational, but in fact the computations are conceptually simple and usually come down to summing an appropriate geometric series. It has the advantage of providing the exact distributions of the quantities involved. In addition, the downcrossings themselves have some rather curious properties, so while the path to the final results may the tedious in some stretches, it affords unexpected views in others.

In sections one to three we study the Brownian local time in the simplest case, in which the process starts at 1 and is stopped when it first hits 0. We
then treat the general case in section four, in which we treat the local time for a general regular diffusion. This is perhaps not the most efficient way to do it, because all the ideas needed in the general case are already needed in the special case, making the second half of the article somewhat of a repetition of the first. However, we have let the first part stand, feeling that the theorems are most easily understood in an uncluttered special case, and leaving us free to concentrate on finding the infinitesimal generator in the most general case.

A historical note: the fundamental theorems on the local time which we prove - Theorems 3.3 and 4.1 - are due to Knight and Ray, and to Ray respectively. The only things with any claim to novelty in the article are the approach and the theorems on the downcrossing processes.

1. THE UPCROSSING AND DOWNCROSSING PROCESSES

Let $X$ be a diffusion on the line with a possibly finite lifetime $\zeta$. We suppose that $X$ is canonically defined on the space $(\mathbb{G}, \mathcal{F}, P)$ of right-continuous functions on $[0, \infty)$ to $\mathbb{R} \cup \{\delta\}$ - where $\delta$ is the cemetery - which are continuous until the lifetime $\zeta$ and equal $\delta$ from then on. Let $\theta_t$ be the translation operator, $P_t$ the semi group, and $P^X$ the distribution of $X$ given that $X_0 = x$. The first-hitting time of a point $a$ is

$$T_a = \inf\{t > 0 : X_t = a\}$$

Let $x < y$ and define stopping times $S_0, S_1, \ldots$ by

$$S_0 = T_y, \quad S_1 = S_0 + T_x \circ \theta_{S_0}, \quad S_2 = S_1 + T_y \circ \theta_{S_1},$$

and by induction

$$S_{2n} = S_{2n-1} + T_y \circ \theta_{S_{2n-1}}, \quad S_{2n+1} = S_{2n} + T_x \circ \theta_{S_{2n}}.$$

Then $X_{S_n}$ equals $x$ for odd $n$, and $y$ for even $n$. It makes an upcrossing of $(x, y)$ between $S_{2n-1}$ and $S_{2n}$, and a downcrossing between $S_{2n}$ and $S_{2n+1}$, so the total number of downcrossings, $D_{xy}$, of $(x, y)$ is
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\[ D_{xy} = \sup \{ n : S_{2n-1} < \infty \}. \]

Let \( Y_t = X_t \wedge T_y \) and \( Z_t = X_t \wedge T_x \), and define \( Y_n = Y_n \wedge S_{2n-1} \), and \( Z_n = Z_n \wedge S_{2n-2} \), \( n = 1, 2, \ldots \). We call \( Y_n \) and \( Z_n \) the \( n \)th upcrossing and downcrossing processes respectively.

If \( X_0 \in (x, y) \), it is not clear whether we should call the process from \( t=0 \) until \( t=S_o \) an upcrossing or a downcrossing. We avoid this inessential ambiguity by assuming that \( X_0 = a \) for some real \( a \), and we will only consider intervals which do not contain \( a \).

This agreed, we set \( Y_0(t) = X_t \wedge T_y \) in case \( a < x \), whereas if \( a \geq y \), we set \( Z_0(t) = X_t \wedge T_x \). We then define the upcrossing field \( \mathcal{U}_{xy} \) by

\[ \mathcal{U}_{xy} = \sigma \{ Y_n, n = 1, 2, \ldots \} \]

Remarks

1°). If \( (x, y) \) and \( (x', y') \) are intervals not containing \( z \), with \( x \leq x' \) and \( y \leq y' \), then \( \mathcal{U}_{xy} \subseteq \mathcal{U}_{x'y'} \). This is because it is impossible for an upcrossing of \( (x, y) \) to occur during a downcrossing of \( (x', y') \): each upcrossing of \( (x, y) \) occurs either during an upcrossing of \( (x', y') \) or, if \( a < x' \), before time \( T_y \). Thus all upcrossings of \( (x, y) \) are made by the upcrossing processes \( Y_0, Y_1, \ldots \) of the upper interval. (By the same token, all downcrossings of \( (x', y') \) are made by the downcrossing processes of the lower interval \( (x, y) \)).

2°). \( D_{xy} \) is \( \mathcal{U}_{xy} \)-measurable. Indeed

\[ \{ D_{xy} \geq n \} = \{ S_{2n-1} < \infty \} = \{ Y_n(0) \in \mathbb{R} \} \in \mathcal{U}_{xy} \]

3°). Warning: in case \( a > y \), the process \( Z_1 \) is a sub-process of \( Z_0 \), for both end at time \( T_x \).

The processes \( Z_n \) and \( Y_n \) are defined only on the sets \( \{ S_{2n-2} < \infty \} \) and \( \{ S_{2n-1} < \infty \} \) respectively. If \( D_{xy} = \infty \), it follows from the strong Markov
property that all of the processes $\mathbf{Z}_n$ and $\mathbf{Y}_n$ will be independent. If $D_{xy}$ is finite, we have the following conditional statement.

PROPOSITION 1.1

Let $N \geq 1$ and suppose $a \in \mathcal{A}(x,y)$. Then, conditioned on the event 

$$
\{D_{xy} = N\},
$$

we have that

i). $\mathbf{Z}_1, \ldots, \mathbf{Z}_N$ are iid and independent of $\mathbf{U}_{xy}$;

ii). $\mathbf{Z}_0$ is independent of $\mathbf{Z}_1, \ldots, \mathbf{Z}_N$ and of $\mathbf{U}_{xy}$;

iii). each $\mathbf{Z}_i$ has the distribution of $\mathbf{X} \uparrow \mathbf{T}_x$, conditioned on $\{T_x < \infty\}$.

Proof: let $\Lambda_1, \ldots, \Lambda_N$ and $\Gamma_1, \ldots, \Gamma_N$ be events in path space, that is, elements of $\mathcal{B}$. We will write $\{Y_i \in \Gamma_i\}$ instead of $\{\omega : Y_i(\omega) \in \Gamma_i\}$. Let

$$
\Gamma = \bigcap_{i=1}^N \{Y_i \in \Gamma_i\}.
$$

We must show that

$$
P^a(\{\mathbf{Z}_i \in \Lambda_i, i=1, \ldots, N ; \Gamma | D_{xy} = N\})
= P^a(\Gamma | D_{xy} = N) \prod_{i=1}^N P^Y(\mathbf{X} \uparrow \mathbf{T}_x \in \Lambda_i | T_x < \infty).
$$

Now $D_{xy} = N$ iff $S_0 < \infty, \ldots, S_{2N-1} < \infty, S_{2N+1} = \infty$. Remembering that

$$
\mathbf{Z}_j = \mathbf{Z}_0 \uparrow S_{2j-2} \text{ and } Y_j = Y_0 \uparrow S_{2j-1},
$$

we can write the left-hand side of (1.1) as

$$
P^a(D_{xy} = N)^{-1} P^a(S_0 < \infty, \mathbf{Z}_0 \uparrow S_0 \in \Lambda_1, S_1 < \infty, Y_0 \uparrow S_1 \in \Gamma_1, \ldots)
\cdots \mathbf{Z}_0 \uparrow S_{2N-2} \in \Lambda_N, S_{2N-1} < \infty, Y_0 \uparrow S_{2N-1} \in \Gamma_N, S_{2N+1} = \infty\}

Apply the strong Markov property successively to $S_{2N-1}, S_{2N-2}, \ldots, S_1$, noting that $\mathbf{X} \uparrow S_{2j} = y$ and $\mathbf{X} \uparrow S_{2j-1} = x$.

$$
P^a(D_{xy} = N)^{-1} P^a(S_0 < \infty) P^Y(\mathbf{Z} \in \Lambda_1, S_1 < \infty) P^X(\mathbf{Y} \in \Gamma_1, S_0 < \infty) \cdots
\cdots P^Y(\mathbf{Z} \in \Lambda_N, S_0 < \infty) P^X(\mathbf{Y} \in \Gamma_N, S_1 = \infty)
$$
Collect all terms containing $Z$

\[
\prod_{i=1}^{N} p^Y\{Z \in \Lambda_i, S_1 < \infty\} \left[ \prod_{i=1}^{N-1} \left( P^X\{Y \in \Gamma_i, S_0 < \infty\} \prod_{j=N}^{1} p^X\{Y \in \Gamma_j, S_{j-1} < \infty\} \right) \right]
\]

Taking $\Lambda_1 = \ldots = \Lambda_N = \Omega$ above, we identify the term in brackets as $P^A(D_{xy} = N) P^Y\{S_1 < \infty\}^{-N}$. But $Z_t = X_t \wedge T_x$, so

\[
P^Y\{X_t \wedge T_x \in \Lambda_1, T_x < \infty\} = P^Y\{Z \in \Lambda_1, S_1 < \infty\} P^Y\{S_1 < \infty\}^{-1},
\]

and we conclude that the above equals the right-hand side of (1.1), which is therefore proved. The statement for $Z_0$ follows upon replacing $Z_1$ by $Z_0$ above and making the obvious modifications to account for the fact that $Z_0$ starts from a rather than $y$.

2. THE BASIC CALCULATIONS

We will be encountering geometric distributions fairly often in what follows, so it is convenient to record the following.

**Lemma 2.1**

Let $X$ be an integer-valued random variable with

\[
P\{X \geq n\} = \begin{cases} 1 & n=0 \\ c \cdot n^{-1} & n=1,2,\ldots & c \leq 1, \ r < 1. \end{cases}
\]

Then $E(X) = \frac{c}{1-r}$, $\text{Var}(X) = \frac{c(1-r-c)}{1-r}$

and $E(e^{sX}) = \frac{1-c+(c-r)e^s}{1-r e^s}$.

Most of the calculations we will make below involve nothing deeper than the following lemma.
Lemma 2.2

Let \( b < x' < y' \) and let \( \{M_t, t \geq 0\} \) be a continuous martingale with initial value \( z > b \), such that \( T_b < \infty \) a.s. Let \( D_{x',y'} \) be the number of downcrossings of \( (x',y') \) before \( T_b \). Then

i). \( P(D_{x',y'} \geq n) = \frac{y' \wedge z - b}{y' - b} \left( \frac{x' - b}{y' - b} \right)^{n-1} \), \( n \geq 1 \)

ii). \( E(D_{x',y'}) = \frac{y' \wedge z - b}{y' - x'} \)

iii). \( \text{Var}(D_{x',y'}) = \frac{y' \wedge z - b}{y' - x'} \)

iv). \( E(e^{s D_{x',y'}}) = 1 - \frac{(y' \wedge z - b)(1 - e^s)}{y' - b - (x' - b)e^s}, \text{Re} \ s < \frac{y' - b}{x' - b} \)

Proof: once (i) is proved, (ii), (iii), and (iv) follow from Lemma 2.1.

To prove (i), note that for \( M \) to make one downcrossing of \( (x',y') \), it must first reach \( y' \) before hitting \( b \), an event which is certain if \( z > y' \) and which has probability \( z - b/y' - b \) if \( z < y' \). Once at \( y' \), it is sure to make at least one downcrossing.

Next, in order to make \( n+1 \) downcrossings, it must first make \( n \), with probability \( p_n \), say. It will then be at \( x \), and must first return to \( y \) without hitting \( x \), and event of probability \( x' - b/y' - b \). It is then sure to make at least one more downcrossing, since it eventually reaches \( b \). Thus

\[ p_{n+1} = p_n \frac{x' - b}{y' - b}. \]

Since \( p_1 = y' \wedge z - b/y' - b \), (i) follows by induction.

We now specialize our diffusion \( X \). Let \( X \) be a Brownian motion with \( X_0 = 1 \), stopped when it first hits the origin. (In fact, it is not necessary that \( X \) be a Brownian motion; everything we do below is valid if \( X \) is any diffusion on natural scale).

If \( x < y \) and if \( D_{xy}(t) \) is the number of downcrossings of \( (x,y) \) in the interval \([0,t]\), then \( 2(y-x) D_{xy}(t) \) converges a.s. as \( y+x \) to the standard
Brownian local time at $x$. ([1], [2], and [3]).

Rather than looking at this for a fixed $t$, we want to look at this at the time $T_0$, the first hit of zero. Accordingly, we will write $D_{xy}$ instead of $D_{xy}(T_0)$ for the total number of upcrossings of $(x,y)$ by $X$, and we will define

$$M_{xy} = (y-x) D_{xy}$$

(We are normalizing by $y-x$ instead of $2(y-x)$, so $M_{xy}$ will converge to half the standard Brownian local time as $y \downarrow x$).

Before doing this let us note that we can derive many of the properties of $D_{xy}(t)$ from those of $D_{xy}(T_0)$. Indeed, $D_{xy}(t) = D_{xy}(T_0)$ if $t > T_0$, and, on $\{t < T_0\}$,

$$D_{xy}(t) = D_{xy}(T_0) - D_{xy}(T_0) \circ \theta_t - \gamma,$$

where $\gamma = 0$ or $1$ is there to account for the possibility that $t$ falls during a downcrossing, which would not be counted in either $D_{xy}(t)$ or $D_{xy}(T_0) \circ \theta_t$.

But now if $\lim_{y \uparrow x} (y-x) D_{xy}(T_0)$ exists a.s., so does $\lim_{y \uparrow x} (y-x) D_{xy}(T_0) \circ \theta_t$ by the Markov property, hence so does $\lim_{y \uparrow x} D_{xy}(t)$. Similarly, if this limit is continuous in $x$ at time $T_0$, it must be continuous at time $t$, too.

**PROPOSITION 2.3**

Let $(x,y)$ and $(x',y')$ be intervals not containing 1, such that $x < x'$ and $y < y'$. Then

(i) $M_{xy}$ is $L^p$-bounded for $y$ in compacts, all $p > 0$;

(ii) $M_{x',y'}$ is conditionally independent of $\mathcal{U}_{xy}$ given $M_{xy}$;

(iii) $E[M_{x',y'} | \mathcal{U}_{xy}] = M_{xy} + y' \wedge 1 - y \wedge 1$;

(iv) $\text{Var}[M_{x',y'} | \mathcal{U}_{xy}] = (x'-x + y'-y) M_{xy} + (y' \wedge 1 - y \wedge 1)(x'-y + y' - y' \wedge 1)$

**Proof**: once (iii) has been proved, (i) follows, since by Lemma 2.2 (iv), all moments of $M_{xy}$ exist, and so for $p \geq 1$, if $x < y \leq k$, Proposition 2.3 (iii) implies:
To prove (ii), consider the cases \( y \leq 1 \) and \( y > 1 \) separately. Suppose first that \( y > 1 \) and let \( Z_1, Z_2, \ldots \) be the successive downcrossing processes, as defined in §1. Each downcrossing of \((x',y')\) takes place during a downcrossing of \((x,y)\); it is not possible for one to happen during an upcrossing. Thus, if \( V_1, V_2, \ldots \) are the downcrossings of \((x',y')\) by \( Z_1, Z_2, \ldots \) respectively, then

\[
(2.2) \quad D_{x'y'} = V_1 + V_2 + \ldots
\]

But the \( Z_i \) are iid (Proposition 1.1) and independent of \( U_{xy} \), and the \( V_i \) are functions of the \( Z_i \), so that \( D_{x'y'} \) must be conditionally independent of \( U_{xy} \) given \( D_{xy} \), which proves (ii) in this case. Moreover, the exact distribution of the \( V_i \) are given by Lemma 2.2, with \( z=y \) and \( b=x \).

\[
(2.3) \quad E(V_i) = \frac{y-x}{y'-x}, \quad \text{Var}(V_i) = \frac{(y-x)(x'-x+y'-y)}{(y'-x')^2}
\]

Since \( D_{xy} \) is \( U_{xy} \)-measurable

\[
E(D_{x'y'} | U_{xy}, D_{xy} = N) = \sum_{i=1}^{N} E(V_i | D_{xy} = N)
\]

\[
= N \frac{y-x}{y'-x} = \frac{y-x}{y'-x} \cdot D_{xy}
\]

Since \( M_{xy} = (y-x)D_{xy} \),

\[
(2.4) \quad E(M_{x'y'} | U_{xy}) = M_{xy} \quad \text{if} \quad y > 1.
\]

The \( V_i \) are independent, so their variances add, and

\[
\text{Var}(D_{x'y'} | U_{xy}, D_{xy} = N) = N \frac{(y-x)(x'-x+y'-y)}{(y'-x')^2},
\]

so that

\[
(2.5) \quad \text{Var}(M_{x'y'} | U_{xy}) = (x'-x+y'-y)M_{xy} \quad \text{if} \quad y > 1
\]

In case \( y < 1 \), the same analysis holds except that there can now be some downcrossings of \((x',y')\) before it reaches \( y \). This means that the process
must be taken into account, and $Z_1$ ignored (see remark 3, §1) since all
downcrossings of $Z_1$ are also downcrossings of $Z_\circ$. Thus

$$D_{xy} = V_0 + V_2 + V_3 + \ldots .$$

As before, $D_{x'y'}$ will be conditionally independent of $U_{xy}$, so that

$$E(D_{x'y'} | U_{xy} , D_{xy} = N) = E(V_0) + (N-1) E(V_2).$$

Taking $z=1$ in Lemma 2.2.

$$E(V_0) = \frac{y'y' - x}{y'y' - x'} , \ Var(V_0) = \frac{(y'y' - x)(x'-x+y'-y'A_1)}{(y'y' - x')^2} .$$

Since the expectation and variance of $V_2,...,V_N$ are given by (2.5)

$$E(D_{x'y'} | U_{xy}) = \frac{y'y' - y}{y'y' - x'} + \frac{y-x}{y'-x} D_{xy} ,$$

$$\ Var(D_{x'y'} | U_{xy}) = \frac{(y - y'A_1)(x'-y+y'-y'A_1)}{(y'y' - x')^2} + \frac{(y-x)(x'-x+y'-y'A_1)}{(y'y' - x')^2} D_{xy} .$$

In terms of $M_{xy}$, if $y > 1$ this is

$$E(M_{x'y'} | U_{xy}) = y'A_1 - y + M_{xy} \ (2.6)$$

$$\ Var(M_{x'y'} | U_{xy}) = (y - y'A_1)(x'-y+y'-y'A_1) + (x'-x+y'-y'A_1) + (x'-x+y'-y) D_{xy} \ (2.7)$$

Equations (2.4)-(2.7) prove (ii) and (iii). q e d

Remark: $M_{xy}$, $0 \leq x < y$, $1 \notin (x,y)$ is a two-parameter process. If we par-
tially order its parameter set by "$<$", where $(x,y) < (x',y')$ if $x \leq x'$,
y \leq y', then Proposition 2.3 tells us that it is a Markov process in the sense that
$M_{x'y'}$ is conditionally independent of $U_{xy}$ - and hence of the "past" before
$(x,y)$ - given $M_{xy}$. It is natural to ask whether this process satisfies Levy's
Markov property: that is, given a nice subset $A$ of the parameter set, is it
true that the process $M_{xy}$ for $(x,y) \in A$ is independent of $M_{xy}$ for $(x,y)$
outside of $A$, given $M_{xy}$, $(x,y) \in A \cup A$? The answer is no. While it is not hard
to show that this is true for sets $A$ of the form $\{(x,y) : 0 \leq x < y \leq y_0\}$, it
doesn't hold for those of the form $\{(x,y) : x \leq x_0 , 0 \leq x < y \leq y_0\}$. 

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3. THE RESULTS

With the basic calculations out of the way, we can draw some conclusions. The following, for example, is an immediate consequence of Proposition 2.3.

THEOREM 3.1

\( \{M_{xy} - y A 1, U_{xy}, 0 \leq x < y, 1 \notin (x, y)\} \) is a two-parameter martingale. In particular, it is a martingale in either parameter when the other is fixed.

Since \( M_{xy} - y A 1 \) is an \( L^p \)-bounded martingale in \( y \) for each fixed \( x \), the martingale convergence theorem allows us to define the local time \( L_x \) at \( x \) by

\[
L_x = \lim_{y \to x} M_{xy}.
\]

Define \( R_x = \bigcap_{y > x} U_{xy} \). Note that the fields \( U_x \) are increasing and that \( L_x \) is \( U_x \)-measurable. Then we have:

COROLLARY 3.2

\( \{L_x - x A 1, U_x, x \geq 0\} \) is a martingale, locally bounded in \( L^p \) for all \( p > 0 \), whose associated increasing process is

\[
A_x = 2 \int_0^x L_y dy.
\]

Proof: set \( N_x = L_x - x A 1 \); it is an \( L^p \)-bounded martingale for all \( p > 1 \) (just let \( y \downarrow x \) in Theorem 3.1). We can then go to the limit in Proposition 2.3 (iv) to see that

\[
E\{N_x^2 - N_x^2 | U_x\} = \text{Var}\{L_x^2 | U_x\}
\]

\[
= 2(x' - x) M_{xy} + (x' - x A 1)(x' - x + x' - x A 1).
\]

To identify \( A_x \) as the associated increasing process, note that

\[
E\left[ 2 \int_x^{x'} L_y dy | U_x \right] = 2 \int_x^{x'} \left( E\{N_y | U_x\} + y A 1 \right) dy
\]

\[
= 2(x' - x) N_x + 2 \int_x^{x'} y A 1 dy
\]

\[
= 2(x' - x)(L_x - x A 1) + 2 \int_x^{x'} y A 1 dy.
\]
which equals the right-hand side of (3.1), verifying that \( N_x \) is a martingale.

We are now in a position to prove the Markov property of the local time \( L_x \) as a function of \( x \). The following is the simplest case of theorems by Ray and Knight.

**THEOREM 3.3** (D. Ray, F. Knight)

The process \( \{L_x, x \geq 0\} \) is a continuous strong Markov process on \([0,\infty)\), absorbed at zero, and having infinitesimal generator \( \mathcal{E} \) if \( f \in \mathcal{D}(\mathcal{E}) \cap C^2 \),

\[
\mathcal{E}f(t) = \begin{cases} 
  t f''(t) + f'(t) & \text{if } 0 < t < 1 \\
  t f''(t) & \text{if } t \geq 1.
\end{cases}
\]

**Proof**: it is nearly clear from Proposition 2.3 that \( L \) is a Markov process, but to pass from nearly clear to clear will take some work. We will do this by brute force, and calculate the characteristic function of \( L_x \). We claim that if \( 0 < x < x' \) and \( \Re s \leq 0 \)

\[
E(e^{sL_x} | U_x) = \frac{1+(x'-x\wedge 1)s}{1-(x'-x\wedge 1)s} e^{\frac{sL_x}{1-(x'-x)s}}
\]

The Markov property follows immediately since, as the right-hand side of (3.3) depends only on \( L_x, L_{x'} \), must be conditionally independent of \( U_x \), and hence of \( L_x'' \) for \( x'' \leq x \), given \( L_x \).

To see (3.3), let \( \epsilon > 0 \), and consider the intervals \((x, x+\epsilon)\) and \((x', x'+\epsilon)\), where \( x+\epsilon < x' \). Let \( D_0, V_1, \ldots \) be the downcrossing processes of the lower interval \((x, x+\epsilon)\). Suppose \( x \geq 1 \). Then

\[
D_{x', x'+\epsilon} = V_1 + V_2 + \ldots
\]

and, given \( D_x = N, V_1, \ldots, V_N \) are iid and independent of \( U_{x, x+\epsilon} \). Apply Lemma 2.2 (iii) with \( b=x, z=x+\epsilon \) and \( y'=x'+\epsilon \).
\[ (3.4) \ E\{ e^{S\mathbf{M}_{\chi',\chi'+\epsilon}} | \mathcal{U}_{\chi,\chi+\epsilon} \} = E\{ e^{S\mathbf{D}_{\chi',\chi'+\epsilon}} | \mathcal{U}_{\chi,\chi+\epsilon} \}. \]

On \( \{ D_{\chi \chi'+\epsilon} \} = N \) this is
\[ = E\{ e^{\epsilon S V_1} \} = \left[ 1 - \frac{\epsilon (1-e^{\epsilon s})}{\epsilon + (x'-x)(1-e^{s})} \right]^{1/\epsilon} \mathbf{M}_{\chi,\chi+\epsilon}. \]

As \( \epsilon \to 0, M_{\chi,\chi+\epsilon} \to L_{\chi} \) while the quotient is
\[ \frac{-s\epsilon}{1-(x'-x)s} + o(\epsilon), \]
so the right-hand side of (3.3) tends to
\[ (3.5) \ e \frac{sL_{\chi}}{1-(x'-x)s}. \]

In case \( x' < 1 \), we reason as before that \( D_{\chi',\chi'+\epsilon} = V_0 + V_2 + \ldots \) so that
\[ e^{S\mathbf{M}_{\chi',\chi'+\epsilon}} | \mathcal{U}_{\chi \chi'} = \frac{E\{ e^{\epsilon S V_0} \}}{E\{ e^{\epsilon S V_1} \}} \frac{E\{ e^{\epsilon S V_1} D_{\chi',\chi'+\epsilon} \}}{E\{ e^{\epsilon S V_1} \}}. \]

The expression involving \( V_1 \) is just as above, with its limit given by (3.5). Let \( z=1 \) in Lemma 2.2 (iii) to see that the quotient is
\[ \frac{E\{ e^{\epsilon S V_0} \}}{E\{ e^{\epsilon S V_1} \}} = \frac{1-(x'-x'A_1)s}{1-(x'-x)s} + o(\epsilon). \]

which, together with (3.5), gives (3.3).

To see \( L_{\chi} \) is continuous, note that, if
\[ \varphi(s) = E\{ e^{s L_{\chi} - L_{\chi}} | \mathcal{U}_{\chi} \} = \frac{1+(x'-x'A_1)s}{1-(x'-x)s} e^{\frac{(x'-x)s^2 L_{\chi}}{1-(x'-x)s}}, \]
then
\[ E\{ (L_{\chi} - L_{\chi})^4 | \mathcal{U}_{\chi} \} = \varphi^{(4)}(0). \]
We can compute the fourth derivative, which is most easily done by considering the cases \( x < x' \leq 1 \) and \( x > 1 \) separately rather than by differentiating the formula as is, and we find that in both cases,
\[ \varphi^{(4)}(0) = 4(x'-x)^2 L_{\chi}^2 + \text{higher powers of } (x'-x). \]
Since $L_x$ is $L^p$-bounded for $x$ in compacts for all $p$, we conclude that for $0 < x < x' < k$, there is a constant $c$ such that $E((L_x - L_x)^2) < c(x' - x)^2$.

This implies by a well-known theorem of Kolmogorov that there is a version of $L_x$ which is continuous in $x$.

Now we can calculate the transition function $P_z(u, v)$ from (3.2). Indeed, (3.2) says that if $f(v) = e^{sv}$, $s < 0$, that if $z = x' - x$ and if $x' < 1$, for instance,

$$P_z f(u) = \frac{1}{1 - zs} e^{su} \frac{1}{1 - zs},$$

which is continuous in $u$. It follows that $u \rightarrow P_z f(u)$ is continuous if $f$ is a linear combination of exponentials, and, since these are dense in $C_o$, for all $f \in C_o$. In short, $P_z$ is a Feller semi-group, and the process is strongly Markov on $0 < x < 1$. Similarly, it is also a Feller process with a different semi-group on $x \geq 1$.

Finally, the generator is easily determined by a stochastic integral argument. If $f$ is bounded and twice continuously-differentiable, then

$$f(L_{x+h}) - f(L_x) = \int_x^{x+h} f'(L_y) dL_y + \frac{1}{2} \int_x^{x+h} f''(L_y) dA_y$$

where $A_y$ is the increasing process of Corollary 3.2. This is

$$= \int_x^{x+h} f'(L_y) d(L_y - y A_1) + \int_x^{x+h} f'(L_y) dy A_1 + \int_x^{x+h} f''(L_y) L_y dy$$

Take the expectation given $U_x$. As $L_y - y A_1$ is a martingale the first integral has zero expectation, so

$$Gf(L_x) = \lim_{h \rightarrow 0} \frac{1}{h} E[f(L_{x+h}) - f(L_x)|U_x] = \lim_{h \rightarrow 0} \frac{1}{h} \left\{ \int_x^{x+h} f'(L_y) dy A_1 + \int_x^{x+h} f''(L_y) L_y dy | U_x \right\}$$

$$= 1_x f''(L_x) + f'(L_x) 1_{\{x < 1\}},$$

proving (3.2). $\quad \text{q.e.d}$
4. THE EXTENSION TO GENERAL DIFFUSIONS

Let $X$ be a regular diffusion on the line with lifetime $\zeta < \infty$, and scale function $s(x)$. We assume that there is an $a$ such that

\begin{equation}
X_0 = a \text{ a.s.}, \quad \text{and } X_{\zeta} = 0 \text{ a.s.}
\end{equation}

and also that $s(0) = 0$. Define

\[ f_x(y, z) = P_y \{ T_z < T_x \} \]

and

\[ p(y, z) = P_y \{ T_z < \infty \} \]

By the definition of the scale function, if $0 \notin [x, z]$,

\begin{equation}
\frac{f_x(y, z)}{f_x}$\frac{s(y) - s(x)}{s(z) - s(x)}$.
\end{equation}

If $x < y < z$, $p(z, x) = p(z, y) p(y, x)$; if $y > 0$, $p(z, y) = 1$, for by (4.1), the process must pass $y$ in order to die at the origin. We have treated the case where $\zeta = \inf \{ t : X_t = 0 \}$ in the first part of this article, so we will now suppose that $\zeta > T_0$. Thus, set

\[ p(x) = \begin{cases} 
1 & x > 0 \\
 p(0, x) & x \leq 0.
\end{cases} \]

Then for $y > x$,

\begin{equation}
p(y, x) = p(x) / p(y)
\end{equation}

(There is a similar function, $p_+$, such that if $x < y$, $p(x, y) = p_+(y) / p_+(x)$, but we will not need to use it).

In order to relate $s$ and $p$, notice that if $x-h < x < 0$, the process can pass from $x$ to $x-h$ either by going there directly, or by first going to zero, then passing to $x-h$. Thus

\[ p(x, x-h) = f_0(x, x-h) + f_{x+h}(x, 0) p(0, x-h). \]

Use (4.2), (4.3) and a little algebra to see that
Let \( h \to 0 \) to see that

\[
\frac{dp}{ds} = \frac{p(1-p)}{s} \quad x < 0
\]

which has the solution

\[
p(x) = \begin{cases} 
1 & \text{if } x > 0 \\
1 & \text{if } x < 0 \\
\frac{1}{1-cs(x)} & \text{if } x < 0.
\end{cases}
\]

Since \( p(x) \) decreases as \( x \) decreases, \( c > 0 \). The parameter \( c \) determines the amount of time \( X \) spends at 0 before \( \zeta \). If \( \zeta \) is the first hitting time of 0, \( c=\infty \); if \( c=0 \), the process is never killed, and if \( 0 < c < \infty \), the process spends some time at the origin before \( \zeta \).

Let \( a^+ \) and \( a^- \) be the positive and negative parts of \( a \), and define

\[
\delta(x) = \begin{cases} 
0 & \text{if } x \geq a^+ \\
1 & \text{if } a^- < x < a^+ \\
2 & \text{if } x < a^-.
\end{cases}
\]

As before, let \( D_{xy} \) be the total number of downcrossings of \( (x,y) \) by \( X \). Define the local time \( L_x \) at \( x \) by

\[
L_x = \lim_{y \to x} (y-x)D_x
\]

if it exists. We are going to follow the methods of the first three sections to determine the properties of the local time process. Although we will not be ready to prove it for some time, the final result will be the following, which is a minor modification of a theorem of D. Ray [6].

**Theorem 4.1**

Suppose \( s \in C^{(2)} \), and let \( m=\inf\{X_t\} \). Then the process \( \{L_x, x > m\} \) is an inhomogeneous diffusion on \( \mathbb{R}^+ \) absorbed at 0, with infinitesimal generator \( \mathcal{L}_x : \text{if } f \in C^{(2)} \) is bounded.
\[(4.6) \quad \xi f(u) = uf''(u) + (\delta(x) - \frac{(p^2(x)s'(x))'u}{p^2(x)s'(x)}f'(u), \; u > 0.\]

We will be following the same path in this section that we did in sections 1-3, so we will be able to treat some of the calculations in less detail. However, the total length of the treatment is increased since at each stage we have to handle the three special cases \(x < a^-, a^- \leq x < a^+,\) and \(x \geq a^+\) separately.

The reason these three cases require special handling is apparent if we look at the paths of \(X_t.\) If \(\xi_x = \sup\{t : X_t = x\},\) we can divide the path into three sections: from 0 to \(T_x,\) from \(T_x\) to \(\xi_x,\) and from \(\xi_x\) to \(\zeta.\) The process behaves differently in each of these intervals. If, now, \(x < x',\) and \(x, x' \in (-\infty, 0)\) all three of these time-intervals can contribute to \(L_x';\) if \(0 < x, x' < a,\) there is no contribution from \((T_x, \zeta),\) i.e. \(L_{\zeta} = L_{T_x}\), and if \(x, x' \in (a, \infty),\) there is no contribution from either \((0, T_x)\) or from \((T_x, \zeta).\) Thus, it is not so much that we need to treat the space-intervals \((-\infty, 0)\), \((0, a)\) and \((a, \infty)\) separately as it is that we need to handle \(L_{T_x}, L_{\xi_x} - L_{T_x},\) and \(L_{\zeta} - L_{\xi_x}\) separately.

The following two results generalize Lemma 2.2 and Proposition 2.3.

**Lemma 4.2**

\[P^a[D_{xy} \geq n] = p(a, y)p(y, x)(p(y, x)p(x, y))^{n-1}.\]

**Proof:** to have one downcrossing, the process must first reach \(y,\) then \(x,\) which has probability \(p(a, y)p(y, x).\) If the process has already finished \(n\) downcrossings and is at \(x,\) then to make another, it must first go from \(x\) to \(y,\) and then return to \(x,\) which it does with probability \(p(x, y)p(y, x).\) The lemma follows by induction.
Lemma 4.3

Let \((x,y)\) and \((x',y')\) be intervals with \(x \leq x'\) and \(y \leq y'\). Suppose neither \(a\) nor \(0\) is in the interval \([x,y']\), and let \(V_{xy}\) be the number of downcrossings of the interval \((x',y')\) before time \(T_x\). Then for \(n \geq 1\),

(a) \[P_y\{V_{xy} \geq n \mid T_x < \zeta\} = f_x(y,y')p(y',y)\left[f_x(x',y')p(y',x')\right]^{n-1}\]

(b) If \(a > y'\),

\[P_{a,y}^{+}\{V_{xy} \geq n \mid T_x < \zeta\} = \left[f_x(x',y')p(y',x')\right]^{n-1}\]

(c) If \(0 > y'\),

\[P_{y',y}^{+}\{V_{xy} \geq n \mid T_x \geq \zeta\} = \left[f_x(x',y')p(y',x')\right]^n\]

Proof: In order to have at least one downcrossing the process must first reach \(y'\) without hitting \(x\), then reach \(x'\) before \(\zeta\), which has probability \(p_x(y,y')p(y',x')\). If the process has just completed \(n-1\) downcrossings, it is at \(x'\). To make another, it must first hit \(y'\) without hitting \(x\), then reach \(x'\), which has probability \(r = f_x(x',y')p(y',x')\). Thus by induction

\[(4.7) \ P_y\{V_{xy} \geq n\} = f_x(y,y')p(y',x')r^{n-1}.

Having finished \(n\) downcrossings, the process is again at \(x'\). To have \(T_x < \zeta\), it must go from \(x'\) to \(x\), which has probability \(p(x',x)\). Since \(P_x\{T_x < \zeta\} = p(y,x)\),

\[(4.8) \ P_y\{V_{xy} \geq n \mid T_x < \zeta\} = \frac{f_x(y,y')p(y',x')p(x',x)}{p(y,x)} r^{n-1}.

After simplifying, (e.g. \(p(y',x')p(x',x) = p(x',x)\)) this gives (a).

If \(a > y'\), the same derivation holds except that \(y\) is replaced by \(a\), and the probability of reaching \(y'\) initially is \(p(a,y')\) instead of \(f_x(y,y')\) in (4.7) and (4.8).

To prove (c), note that if the process has just finished \(n\) downcrossings, it is at \(x'\); then \(T_x > \zeta\) if the process never reaches \(x\), an event of probability \(1-p(x',x)\), while \(P_y\{T_x > \zeta\} = 1-p(y,x)\). Combining this with (4.7), we get...
\[ p^y_{x \rightarrow n} \mid T_x > \zeta = \frac{1-p(x',x)}{1-p(y,x)} f_x(y,y') p(y',x') r^{n-1}. \]

But notice that
\[ 1-p(y,x) = f_x(y,x') (1-p(x',x)) \]
since, if the process is at \( y \), in order to have \( T_x > \zeta \), it must first hit \( x' \) - since it is at \( 0 \) at time \( \zeta \) then not hit \( x \). This makes the above expression
\[
= \frac{f_x(y,y')}{f_x(y,x')} p(y',x') = f_x(x',y') p(y',x') r^{n-1} = r^n.
\]

\[ \text{qed} \]

**NOTATION** : we will often write \( \hat{x} \) instead of \( s(x) \).

**COROLLARY 4.3**

Under the same conditions

(a) \[ E^y_{x \rightarrow y', \mid T_x < \zeta} = \frac{p(x)p(y) (\hat{y} - \hat{x})}{p(x')p(y') (\hat{y} - \hat{x}')} ; \]

(b) \[ E^a_{x \rightarrow y', \mid T_x < \zeta} = \frac{p(x) (\hat{x}' - \hat{x})}{p(x')(\hat{y} - \hat{x}')}; \]

(c) \[ E^y_{x \rightarrow y', \mid T_x > \zeta} = \frac{p(x) (\hat{y}' - \hat{x})}{p(y') (\hat{y} - \hat{x}')}; \]

(d) \[ \text{Var}^y_{x \rightarrow y', \mid T_x < \zeta} = \frac{p(x)p(y) (\hat{y} - \hat{x})^2}{(p(x')p(y') (\hat{y}' - \hat{x}'))^2} \]
\[ \cdots [2p(x)p(y') (\hat{y} - \hat{x}) - p(x')p(y') (\hat{y}' - \hat{x}')] - p(x)p(y) (\hat{y} - \hat{x})] \]

(e) \[ \text{Var}^y_{x \rightarrow y', \mid T_x > \zeta} = \text{Var}^a_{x \rightarrow y', \mid T_x < \zeta} = \frac{p(x)^2 (\hat{x}' - \hat{x}) (\hat{y}' - \hat{x})}{p(x')p(y') (\hat{y}' - \hat{x}')^2} \]

**Proof** : in a sense, there is nothing to prove, since (a)-(e) follow from Lemma 4.2 and Lemma 1.1. However, there are some identities which we use to simplify the expressions which we should point out. Let \( r = f_x(x',y') p(y',x') \) and notice that
\[ p(x',x) = f_y(y',x) + f_x(x',y') p(y',x') p(x',x) \]

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which just expresses the fact that the process can go from \( x' \) to \( x \) either directly, without hitting \( y' \), or by first hitting \( y' \), then returning to \( x' \), then to \( x \). Thus

\[
1 - r = \frac{f_{y'}(x',x)}{p(x',x)}
\]

If we apply the formulae of Lemma 4.1 with \( r \) as above, and with \( c \) equal to \( f_x(y,y')p(y',y) \), \( l \), and \( r \) respectively, the expectations in (a), (b) and (c) become respectively

\[
\begin{align*}
\text{(4.10a)} & \quad \frac{f_x(y,y')p(y',y)p(x',x)}{f_{y'}(x',x)} \\
\text{(4.10b)} & \quad \frac{p(x',x)}{f_{y'}(x',x)} \\
\text{(4.10c)} & \quad \frac{f_x(x',y)p(y',x')p(x',x)}{f_{y'}(x',x)}
\end{align*}
\]

Turning to the variances, in (d) we write \( 1 + r - c = 2 - (1 - r) - c \) to get

\[
\text{(4.10d)} \quad \left( \frac{p(x',x)}{f_{y'}(x',x)} \right)^2 f_x(y,y')p(y',y) \left[ 2 - \frac{f_{y'}(x',x)}{p(x',x)} - f_x(y,y')p(y',y) \right]
\]

If either \( c = 1 \), or \( c = r \), \( c(1 + r - c) = r \), so the final two variances equal \( r(1 - r)^2 \), i.e.

\[
\text{(4.10e)} \quad \left( \frac{p(x',x)}{f_{y'}(x',x)} \right)^2 f_x(x',y')p(y',x').
\]

If we write (4.10a) – (4.10e) in terms of \( p(x) \) and \( s(x) = \mathbb{R} \), we get (a)–(e).

\[\text{\(q.e.d\)}\]

For \( x < y \), define

\[M_{xy} = p(x)p(y)(s(y) - s(x))D_{xy} \]

We assume that \( a > 0 \), \( a \) being the initial value of the process, and leave it to the reader to make the necessary modifications in case \( a < 0 \). The following generalizes Theorem 3.1.
THEOREM 4.4

Let \( x_0 \leq 0 \). The following are two-parameter martingales which are locally bounded in \( L^p \) for \( 1 \leq p < \infty \).

(a) \( \{M_{xy}, U_{xy}, a \leq x < y\} \);

(b) \( \{M_{xy} - \bar{\gamma}, U_{xy}, 0 \leq x < y \leq a\} \);

(c) \( \{M_{xy} - \int_{x_o}^x p^2(u)du - \int_{y}^y p^2(u)du, U_{xy}, x_o \leq x < y \leq 0\} \),

on the set \( \{\inf_{t} X_t < x_o\} \).

Proof: the \( M_{xy} \) are geometric random variables, hence are in \( L^p \) for fixed \( x \) and \( y \). Once it is established that (a)-(c) are martingales, local boundedness in \( L^p \) is immediate since in case (a), for example, \( |M_{xy}|^p \) will be a sub-martingale, so if \( x < y < k \), \( E(|M_{xy}|^p) \leq E(|M_{k,k+1}|^p) < \infty \).

Let \((x,y)\) and \((x',y')\) be intervals such that \( a \leq x \leq x' \) and \( y \leq y' \). Let \( \mathcal{Z}_1, \mathcal{Z}_2, \ldots \) be the successive downcrossing processes of \((x,y)\) and let \( V_i \) be the number of downcrossings of \((x',y')\) by \( \mathcal{Z}_i \). Both the initial and final values of \( X \) lie below \( x \), so all downcrossings of \((x',y')\) occur during downcrossings of \((x,y)\). Thus

\[
D_{x'y'} = V_1 + V_2 + \ldots
\]

Given that \( D_{xy} = N, V_1, \ldots, V_N \) are iid and independent of \( U_{xy} \), with a distribution given by Lemma 4.3a). Since \( p(y',y) = p(y',x') = 1 \)

\[
E[V_j | D_{xy} = N] = \frac{\bar{\gamma} - \bar{\gamma}}{\bar{\gamma}' - \bar{\gamma}'} N,
\]

so

\[
E[D_{x'y'} | U_{xy}] = \frac{\bar{\gamma} - \bar{\gamma}}{\bar{\gamma}' - \bar{\gamma}'} D_{xy}
\]

and

\[
E[M_{x'y'} | U_{xy}] = (\bar{\gamma} - \bar{\gamma})D_{xy} = M_{xy}.
\]
In case (b), \( a > y' \) so there is at least one downcrossing of \( (x', y') \) before \( T_x \), and we must consider the initial downcrossing process \( \xi_0 \) instead of \( \xi_1 \) (see remark 3, §1). Then

\[
D_{x'y'} = V_0 + V_2 + \ldots.
\]

The distribution of \( V_0 \) is given by Lemma 4.3 (b), and that of \( V_2, \ldots, V_N \) is the same as above. Thus

\[
E[D_{x'y'} | D_{xy} = N] = \frac{\hat{\gamma}' - \hat{\gamma}}{\hat{y}' - \hat{y}} + (N-1) \frac{\hat{\gamma} - \hat{\gamma}}{\hat{y}' - \hat{y}}.
\]

leading to

\[
E[M_{x'y'} | \mathcal{U}_{xy}] = \hat{\gamma}' - \hat{\gamma} + M_{xy}.
\]

Finally, in case (c), \( X_{c-} = 0 > y' \), so the last "downcrossing" of \( (x, y) \) is incomplete. If \( D_{xy} = N \), then \( \xi_{N+1} \) starts from \( y \), and is killed before hitting \( x \), so that

\[
D_{xy} = V_0 + V_2 + \ldots + V_N + V_{N+1},
\]

and the distribution of \( V_{N+1} \) is given by Lemma 4.2 c. The distributions of \( V_0 \) and \( V_2 \) are as before, except that \( p(x) \) is no longer necessarily one, so

\[
E[D_{x'y'} | D_{xy} = N] = \frac{p(x)(\hat{\gamma}' - \hat{\gamma})}{p(x')(\hat{y}' - \hat{y})} + (N-1) \frac{p(x)p(y)(\hat{\gamma} - \hat{\gamma})}{p(x')p(y')(\hat{y}' - \hat{y})} + \frac{p(x)(\hat{\gamma}' - \hat{\gamma})}{p(y')(\hat{y}' - \hat{y})}.
\]

Thus

\[
E[M_{x'y'} | \mathcal{U}_{xy}] = M_{xy} + p(x)p(y')(\hat{\gamma}' - \hat{\gamma}) + p(x')p(x)(\hat{\gamma}' - \hat{\gamma}) - p(x)p(y)(\hat{\gamma} - \hat{\gamma}).
\]

We can simplify this. Since \( p(x) = 1/1+c \bar{x} \),

\[
(4.11) \int_x^y p^2(u) du = p(y) p(x)(\hat{\gamma} - \hat{\gamma})
\]

so the above equals

\[
\int_x^{y'} + \int_x^{x'} - \int_x^y = \int_x^{y'} + \int_y^{y'} p^2(u) du,
\]

giving finally

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E(M', y | U_{xy}) = M_{xy} + \int_{x}^{x'} p^2(u)du.  

For fixed $x$, the processes of Theorem 4.4 are martingales in $y$, so that
\[
def \quad M_x = \lim_{y \to x} M_{xy},
\]
exists a.s. and in $L^p$ for all $p > 1$. As the processes are also martingales in $x$, so are their limits. We can unify them as follows. Set $U_x \triangleright y \triangleright x$ $U_{xy}$, and define
\[
\delta(x) = \begin{cases} 
0 & \text{if } x \geq a \\
1 & 0 \leq x < a \\
2 & x < 0
\end{cases}
\]
Then for any $x_o$, 
\[
(4.12) \quad \{M_x - \int_{x_o}^{x} \delta(u) p^2(u)du, U_{xy}, x \geq x_o\}
\]
is a martingale on the set $\{\inf X_t < x_o\} = \{M_{x_o} > 0\}$. Now $M_x$ is closely related to the local time $L_x$:
\[
L_x = \lim_{y \to x} (y-x)D_{xy} = \lim_{y \to x} \frac{y-x}{p(x)p(y)(y-x)} M_{xy}
\]
It follows that $L_x$ exists if $s(x)$ is differentiable, and
\[
(4.13) \quad L_x = (p^2(x) s'(x))^{-1} M_x.
\]
Now an elementary calculation with stochastic integrals gives us
\[
dL = (p^2 s')^{-1} dM - \frac{M}{x} (p^2 s') \frac{1}{(p^2 s')^2} dx = (p^2 s')^{-1} (dM - p^2 s' dx) + (\delta - \frac{p^2 s'}{p^2 s'} L_x)dx
\]
or
\[
L_x - L_{x_0} - \int_{x_0}^{x} (\delta(u) - \frac{p^2 s'}{p^2 s'} L_u)du = \int_{x_0}^{x} (p^2 s')^{-1} (dM - \delta p^2 s' dx).
\]
The last term is a martingale, which brings us to

**THEOREM 4.5**

Suppose \( s \in C^{(2)} \). Then for any \( x_0 \)

\[
\{ L_x - \int_{x_0}^x (\delta(u) - \frac{(p^2 s')'}{p s'}) L_u, x \geq x_0 \}
\]

is a martingale on the set \( \{ L_{x_0} > 0 \} \). Its increasing process is

\[
< L_x > = 2 \int_{x_0}^x L_u \, du.
\]

**Proof:** we have just shown that the process is a martingale. To identify its increasing process, we will identify the process \( < M_x > \) first. Notice that \( < M_x > \)
can be characterized as a continuous increasing process with the property that if \( x < x' \),

\[
(4.14) \quad \text{Var}(M_{x'} | L_x) = \mathbb{E} \{ < M_x > - < M_{x'} | L_x > \}.
\]

As in the proof of Theorem 4.4, let \( x < y < x' < y' \) and suppose all four lie in one of the intervals \([a, \infty)\), \([0, a]\), or \([\infty, 0]\). We have to treat all three cases separately. We will refer to them as cases (a), (b) and (c) respectively, and we will use the notation of the proof of Theorem 4.4.

Suppose \( D_{xy} = N \) and that \( y-x \) and \( y'-x' \) are small. Then Corollary 4.3e gives

\[
\text{Var}(V_o) = \frac{p(x)^2}{p(x') p(y')} \frac{(x'-\bar{x})(y'-\bar{x})}{(y'-\bar{x})^2}
\]

in cases (b) and (c), and in case (c), \( \text{Var}(V_{N+1}) = \text{Var}(V_o) \). If \( 1 \leq j \leq N \)

\[
\text{Var}(V_j) = \frac{2p(x)p(y) (\bar{x} - \bar{x})}{[p(x)p(y')(\bar{x}' - \bar{x})]^2} \left[ p(x)p(y')(\bar{x}' - \bar{x}) + o(1) \right].
\]

The \( V_i \) are conditionally independent given \( L_{xy} \), so that the variances add, hence in cases (a), (b) and (c) respectively, \( \text{Var}(D_{x'y'} | L_{xy}) \) is equal to
\[ D_{xy} \text{Var}\{V_i\}, \text{Var}\{V_o\} + (D_{xy} - 1) \text{Var}\{V_1\} \]

and

\[ \text{Var}\{V_o\} + (D - 1) (\text{Var}\{V\} + \text{Var}\{V^o\}) \]

This gives us \( \text{Var}\{D_{xy}x'y'|U_{xy}\} \) in all three cases. Now multiply by 
\[ \left[p(x')p(y')\left(y' - \bar{x}'\right)\right]^2 \]
and let \( y + x \) and \( y' + x' \) to see that in cases (a), (b), and (c) respectively, that \( \text{Var}\{M_x'|U_x\} \) equals

\begin{align*}
(4.15a) & \quad 2p(x')p(x)(\bar{x}' - \bar{x})M_x \\
(4.15b) & \quad 2p(x')p(x)(\bar{x}' - \bar{x})M_x + \left[p(x)p(x')\left(\bar{x}' - \bar{x}\right)\right]^2 \\
(4.15c) & \quad 2p(x')p(x)(\bar{x}' - \bar{x})M_x + 2\left[p(x)p(x')\left(\bar{x}' - \bar{x}\right)\right]^2. 
\end{align*}

We claim \( \langle M_x \rangle = 2 \int_{x_0}^x p^2(u)M_u ds(u) \) for all \( x \geq x_0 \).

Indeed,
\[
E\left\{2 \int_{x}^{x'} p^2(u)M_u ds(u) \mid U_x \right\} = 2 \int_{x}^{x'} E\{M_u \mid U_x\} p^2(u) ds(u) \\
= 2 \int_{x}^{x'} (M_x + \int_{x}^{u} \delta(v)p^2(v)ds(v))p^2(u)ds(u).
\]

Now \( \delta \) is constant on \( (x',x) \), so
\[
= 2 M_x \int_{x}^{x'} p^2(u)ds(u) + \delta(x) \left( \int_{x}^{x'} p^2(u)ds(u) \right)^2.
\]

by (4.11) this is
\[
= 2 M_x p(x')p(x)(\bar{x}' - \bar{x}) + \delta(x)\left[p(x')p(x)(\bar{x}' - \bar{x})\right]^2.
\]

Compare this with (4.15a-c) to verify (4.12).

But now that we know \( \langle M \rangle \), we get \( \langle L \rangle \) immediately since, by (4.13)
\[
d\langle L \rangle_x = (p^2 s')^{-2} d\langle M \rangle_x \\
= 2(p^2 s')^{-2} p s' M_x dx \\
= 2 L_x dx \]

qed
We can now prove Theorem 4.1. We proved the Markov property in a simpler setting in §3, and a similar argument can be used here, or we can just refer to [7, Theorem 2.3]. We are interested primarily in the infinitesimal generator. To find this, let \( f \in C^{(2)} \) be bounded. Then for any \( x \), on the set \( \{L_x > 0\} \)

\[
f(L_{x+h}) - f(L_x) = \int_x^{x+h} f'(L_y)dL_y + \frac{1}{2} \int_x^{x+h} f''(L_y)d<\Lambda>_y
\]

\[
= \int_x^{x+h} f'(L_y)(dL_y - (\delta(y) - \frac{(p^s)'(y)}{p^s}L_y)dy)
\]

\[
+ \int_x^{x+h} [f'(L_y)(\delta(y) - \frac{(p^s)'(y)}{p^s}L_y) + f''(L_y)L_y]dy,
\]

The first integral is with respect to a martingale, so it has expectation zero, and

\[
\mathbb{E}_x f(L_x) = \lim_{h \to 0} \frac{1}{h} \mathbb{E}[f(X_{x+h}) - f(L_x) | \mathcal{H}_x] = L_x f''(L_x) + \delta(x) - \frac{(p^s)'(y)}{p^s}L_x f''(L_x),
\]

which verifies (4.6), and we are done.

Example: it is clear from (4.6) that \( \{L_x, x \geq m\} \) is not time-homogeneous, for \( \delta(x) \) is not constant. It will, however, be homogeneous on each of the intervals \((-\infty,0)\), \((0,a)\), and \((a,\infty)\) if \( \frac{(p^s)'(y)}{p^s} \) is constant there. This was the case in Theorem 3.3 since \( p^s = 1 \) and \( s(x) \equiv x \) there. It is true of any diffusion on natural scale for \( x > 0 \) for the same reason. Another celebrated case, due to D. Ray, is the following. Let \( B_t \) be a Brownian motion and \( S \) an exponential random variable with parameter \( \alpha \), independent of \( B_t \). Let \( \overline{B}_t \) be \( B_t \) killed at \( S \). Finally, let \( X_t \) be the diffusion \( \overline{B}_t \), conditioned on \( B_S = 0 \). (This conditioning can be made rigorous via Doob's \( h \)-path processes. The semi-group of \( \overline{B}_t \) is \( \overline{P}_t = e^{-\alpha t} P_t \), where \( P_t \) is the Brownian semi-group, and the semi-group of \( X \) is the \( h \)-path transform of \( \overline{P}_t \) for \( h(x) = e^{2\alpha x} \sqrt{2\alpha x} \). This is a \( \overline{P} \)-excessive function which has a pole at the origin, and which is invariant away from the origin). The infinitesimal generator \( G \) of \( X \) is the \( h \)-transform of the generator of \( \overline{B} \).
\[ \mathcal{C}f(x) = \frac{1}{2h(x)} (hf)'' - \alpha f \]

\[ = \frac{1}{2} f''(x) - 2\sqrt{2\alpha} \ \text{sgn} x \ f'(x) \]

The scale function \( s \) satisfies \( \mathcal{C}s = 0 \); such a function which vanishes at 0 is

\[ s(x) = \begin{cases} 
    e^{2\sqrt{2\alpha} \ x} - 1 & \text{if } x \geq 0 \\
    1 - e^{-2\sqrt{2\alpha} \ x} & \text{if } x \leq 0 
\end{cases} \]

and it follows that for some \( c > 0 \),

\[ p(x) = \begin{cases} 
    1 & x \geq 0 \\
    \frac{1}{1 - cs(x)} & x \leq 0 
\end{cases} \]

To identify \( c \), it is easily seen from Lemma 4.2 that

\[ \mathbb{E}^{o}\{L_o\} = \lim_{x \to 0} \frac{x}{s(x)} \mathbb{E}^{x}\{D_{\infty}\} = \frac{1}{cs'(0)} \frac{1}{\sqrt{2\alpha}} \]

Next calculate \( \mathbb{E}^{o}\{L_o\} \) from another standpoint. Let \( \overline{E} \) be the expectation operator for \( \overline{B} \), and \( E \) the expectation operator for \( X \). Now \( X \) is identical to the process \( \overline{B} \), killed at its last exit from 0 \( \{5\} \), so

\[ \overline{E}(L_o(S)) = E(L_o(c)) = E(L_o) \]

But as \( \frac{1}{2} |\overline{B}_t| - L_o(t) \) is a martingale,

\[ \overline{E}(L_o(S)) = \frac{1}{2} \overline{E}(|\overline{B}_S|) \]

\[ = \int_0^\infty \int_0^\infty \frac{x}{2\pi t} e^{-\frac{x^2}{2t}} a e^{-at} \ dx \ dt \]

\[ = \frac{1}{2\sqrt{2\alpha}} \]

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Comparing this with the above, we see $c=1$, hence
\[
p(x) = \begin{cases} 
1 & \text{if } x \geq 0 \\
\text{if } x \leq 0 
\end{cases}
\]
so that $p^2 = 2\sqrt{2\alpha}$, and the generator of $L_x$ is
\[
g_x f(u) = u \frac{d^2}{du^2} + (\delta(x) - 2\sqrt{2\alpha} u) \frac{d}{du}
\]
For the interpretation of this in terms of Bessel processes, see (6) or (8).

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