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Astérisque, tome 52-53 (1978), p. 47-61
<http://www.numdam.org/item?id=AST_1978__52-53__47_0>

Société Mathématique de France
Astérisque 52-53 (1978) p.47-61

THE LOCAL TIME OF THE BROWNIAN SHEET

by

John B. WALSH

The Brownian sheet, $\{W_{st}, s \geq 0, t \geq 0\}$, is a Gaussian process with mean zero and covariance $E\{W_{uv} W_{st}\} = (u \wedge s)(v \wedge t)$. It has continuous paths, and it also has a local time which, as was shown in [1], is continuous in all variables. We will refer to this as the local time in the plane. We will also use a second local time, the local time on lines. Consider, for instance, the line $s=s_o$ in $\mathbb{R}^2_+$. $\{W_{s_o t}, t \geq 0\}$ is a one-parameter Brownian motion, and it therefore has a local time at any point x, which we call the local time along the line $s=s_o$.

Our approach is quite different from that of [1]. We will start with the local time on lines, and get the local time in the plane, $L(x;s,t)$, by integrating. $L(x;s,t)$ is not exactly the same as the local time of [1], but it is closely related to it. Our basic results, Theorems 1.2 and 2.1, state that $L(x;s,t)$ is continuous in x and is continuously differentiable in s and t, except at the boundary. Its partial derivatives are exactly the local times on lines, as one would expect. (This was mentioned in [1], but the delicate part, which is to show that the local time on lines changes continuously when the line is changed, eluded us).

We are also able to get some estimates on the moduli of continuity of $L(x;s,t)$ and its partials. It turns out that $L(x;s,t)$ is much smoother in all its variables than is the usual Brownian local time, while $t \to \frac{\partial}{\partial s} L(x;s,t)$ and $s \to \frac{\partial}{\partial t} L(x;s,t)$ are much rougher.

We wish to thank R. Cairoli : the theorems of this article were suggested by some of his earlier unpublished results, and we have greatly benefited from his ideas. We would also like to thank Marc Yor for a number of stimulating conversations on the subject.

1 - LOCAL TIMES ON LINES

Let $\{W_{st},\ s,\ t \geq 0\}$ be the Brownian sheet. If we fix t, $s \to W_{st}$ is a Brownian motion, and its local time at x up to s is given by Tanaka's formula :

$$(1.1)\qquad L_1(x;s,t) = \frac{2}{t}\left[(W_{st}-x)^+ - x^- - \int_0^s I_{\{W_{ut} > x\}}\partial_1 W_{ut}\right]$$

where the stochastic integral is an Ito integral with respect to $u \to W_{ut}$. We use "$\partial_1 W$" instead of "dW" to emphasize that this is a line integral, not an area integral. The particular normalization chosen above assures us that for a bounded Borel f,

$$(1.2)\qquad \int_0^s f(W_{ut})du = \int_{-\infty}^{\infty} L_1(x;s,t)\, f(x)dx \qquad \text{a.s.}$$

Similarly, the local time along the line s = constant is

$$(1.3)\qquad L_2(x;s,t) = \frac{2}{s}\left[(W_{st}-x)^+ - x^- - \int_0^t I_{\{W_{sv} > x\}}\,\partial_2 W_{sv}\right].$$

We can get the absolute distributions of L_1 and L_2 by a scaling argument. Both $\{s^{-1/2}\ W_{st},\ t \geq 0\}$ and $\{t^{-1/2}\ W_{st},\ s \geq 0\}$ are standard Brownian motions, so their local times at x are standard Brownian local times. If, say, $\phi(x,s)$ is the local time of $t^{-1/2}\ W_{st}$ at x, then, for a bounded Borel f

$$\int_0^s f(t^{-1/2}\ W_{ut})du = \int_{-\infty}^{\infty} \phi(x,s)dx.$$

Compare this with (1.2) to see that $\phi(x,s) = \sqrt{t}\ L_1(x\sqrt{t};s,t)$. Thus we have

PROPOSITION 1.1

For each $s,t > 0$, $\{\sqrt{t}\, L_1(x\sqrt{t}\,;\, s,t),\ s \geq 0\}$ *and* $\{\sqrt{s}\, L_2(x\sqrt{s}\,;\, s,t),\ t \geq 0\}$ *have the same distribution as standard Brownian local time at* x.

Notice that $L_1(0,s,t)$ has the same distribution as $\frac{1}{\sqrt{t}}\,\phi(0,s)$, so that as $t \to 0$, $L_1(0,s,t) \to \infty$ in probability, and $L_2(0,s,t) \to \infty$ in probability as $s \to 0$. Thus, L_1 and L_2 are badly-behaved at the points $x = t = 0$ and $x = s = 0$ respectively. Elsewhere, though, they are quite respectable, as the following theorem shows.

THEOREM 1.2

There exist versions of L_1 *and* L_2 *which are a.s. jointly continuous in* $(x;s,t)$ *in* $\mathbb{R} \times \mathbb{R}_+^2$ *except at* $x = t = 0$ *and* $x = s = 0$ *respectively.*

Before proving this, we need a lemma which is not new, but which we will prove for the sake of completeness.

LEMMA 1.3

Let $\{B_s,\ s \geq 0\}$ *be a standard Brownian motion from zero and let* $p \geq 1$. *For each* $x \in \mathbb{R}$ *and* $s > 0$

$$(1.3)\qquad \sup_{\substack{\varepsilon > 0\\ x \in \mathbb{R}}} E\{(\frac{1}{2\varepsilon}\int_0^s I_{\{|B_u - x| < \varepsilon\}}du)^p\} < \infty$$

Proof : the integral above is bounded by s, so if $\varepsilon \geq 1$ the whole expression is bounded by $(\frac{s}{2})^p$. Thus, suppose $0 < \varepsilon < 1$. Note first that, by a familiar argument, (1.3) is maximized over x by taking $x=0$. Let f_ε be a function whose second derivative is $(2\varepsilon)^{-1}\, I_{(-\varepsilon,\varepsilon)}$. By Ito's formula applied to f_ε:

$$(1.4)\qquad \frac{1}{2\varepsilon}\int_0^s I_{\{|B_u| < \varepsilon\}}du = f_\varepsilon(B_s) - f_\varepsilon(B_0) - \int_0^s f_\varepsilon'(B_u)dB_u.$$

Now $0 \leq f_\varepsilon' \leq 1$ and $0 \leq f_\varepsilon(t) \leq |t+\varepsilon| \leq |t| + 1$, so

$$(1.5) \quad E\{(f_\varepsilon(B_s) - f_\varepsilon(B_o))^p\} \leq E\{(|B_s| + 1)^p\} < \infty.$$

The stochastic integral on the right hand side of (1.4) is a continuous martingale with an increasing process

$$A_s = \int_o^s (f_\varepsilon'(B_u))^2 du \leq s.$$

By Burkholder's inequality, there is a constant $c_p \ni$:

$$(1.6) \quad E\{(\int_o^s f_\varepsilon'(B_u)dB_u)^p\} \leq c_p \; E\{(A_s)^{p/2}\} \leq c_p s^{p/2}.$$

The result follows from the fact that the bounds in (1.5) and (1.6) do not depend on ε. q e d

We can now prove Theorem 1.2.

Proof : let s, s', t and t' be positive, and let x and x' be real. Write

$$\begin{aligned} J(x';s',t') - J(x;s,t) &= (J(x';s',t') - J(x;s',t')) \\ &+ (J(x;s',t') - J(x,s,t')) + (J(x;s,t') - J(x;s,t)). \\ &\stackrel{\text{def}}{=} J_1 + J_2 + J_3. \end{aligned}$$

We will estimate $E\{|J_i|^p\}$ separately for each i. The plan is the same in all three cases:represent J_i as a stochastic integral, then use Burkholder's inequality to bound its L^p-norm by the $L^{p/2}$-norm of its increasing process. Once this is done, the theorem is a consequence of a general theorem of Kolmogorov.

We will treat J_1 and J_2 first since they are relatively straightforward to handle. J_3 is a bit tricky.

Let $p > 1$. Now either $x < x'$ or $x' \leq x$. Suppose the former, for instance. Write

$$J_1 = \frac{1}{t'} \int_o^{s'} I_{\{x < W_{ut'} \leq x'\}} \partial_1 W_{ut'}.$$

If we define $B_u = (t')^{-1/2} W_{ut'}$, B is a standard Brownian motion from zero, and

$$E\{|\frac{1}{x'-x} J_1|^p\} = (\frac{1}{t'})^p E\{|\frac{\sqrt{t'}}{x'-x} \int_0^{s'} I_{\{\frac{x}{\sqrt{t'}} < B_u \leq \frac{x'}{\sqrt{t'}}\}} dB_u|^p\}.$$

By Lemma 1.3, there is a constant $c_{s'p}$ such that this is

$$\leq (\frac{1}{t'})^p c_{s'p}.$$

Thus

$$(1.7) \qquad E\{|J_1|^p\} \leq c_{s'p}|\frac{x'-x}{t'}|^p$$

Moving on to J_2, suppose $s \leq s'$ and write

$$J_2 = \frac{1}{\sqrt{t'}} \int_s^{s'} I_{\{B_u > \frac{x}{\sqrt{t'}}\}} dB_u.$$

This is a martingale whose associated increasing process is

$$A_s = \frac{1}{t'} \int_s^{s'} I_{\{B_u > \frac{x}{\sqrt{t'}}\}} du \leq \frac{s'-s}{t'}.$$

By Burkholder's inequality

$$(1.8) \qquad E\{(J_2)^p\} \leq c_p(\frac{s'-s}{t'})^{p/2}.$$

This brings us to J_3. Assume that $t < t'$. We can write

$$J_3 = \frac{1}{t'} \int_0^s I_{\{W_{ut'} > x\}} \partial_1 W_{ut'} - \frac{1}{t} \int_0^s I_{\{W_{ut} > x\}} \partial_1 W_{ut}$$

$$= \frac{1}{t'} \int_0^s I_{\{W_{ut'} > x\}} \partial_1 (W_{ut'} - W_{ut}) - \frac{1}{t}\int_0^s I_{\{W_{ut'} < x < W_{ut}\}} \partial_1 W_{ut}$$

$$+ (\frac{1}{t'} - \frac{1}{t}) \int_0^s I_{\{W_{ut'} > x\}} \partial_1 W_{ut} + \frac{1}{t} \int_0^s I_{\{W_{ut} < x < W_{ut'}\}} \partial_1 W_{ut}$$

$$\stackrel{def}{=} K_1(s) + K'(s) + K_2(s) + K_3(s).$$

We will just consider K_1, K_2 and K_3, for K' is obtained from K_3 by simply interchanging t and t'. Now K_1, K_2 and K_3 are martingales with increasing processes A_1, A_2, and A_3 respectively

$$A_1(s) = \left(\frac{1}{t'}\right)^2 \int_0^s I_{\{W_{ut'} > x\}} (t'-t)\,du \le \frac{t'-t}{(t')^2}\, s$$

$$A_2(s) = \left(\frac{t'-t}{tt'}\right)^2 \int_0^s I_{\{W_{ut'} > x\}} t\,du \le \frac{(t'-t)^2}{t(t')^2}\, s$$

and

$$A_3(s) = \frac{1}{t^2} \int_0^s I_{\{W_{ut} < x < W_{ut'}\}}\, t\,du$$

Just as above, we apply Burkholder's inequality:

$$\text{(1.9)} \qquad E\{K_1^p\} \le c_p \left(\frac{t'-t}{t'^2}\, s\right)^{p/2} \; ; \; E\{K_2^p\} \le c_p \left(\frac{t'-t}{t'}\right)^p \left(\frac{s}{t}\right)^{p/2},$$

and

$$\text{(1.10)} \qquad E\{K_3^p\} \le \frac{c_p}{t^{p/2}}\, E\left\{\left(\int_0^s I_{\{W_{ut} < x \le W_{ut'}\}} du\right)^{p/2}\right\}$$

We must bound the expectation on the right hand side of (1.10). Write $t'=t+h$ and put

$$B_s^1 = \frac{1}{\sqrt{t}}\, W_{st} \quad \text{and} \quad B_s^2 = \frac{1}{\sqrt{h}}\, (W_{s,t+h} - W_{st}).$$

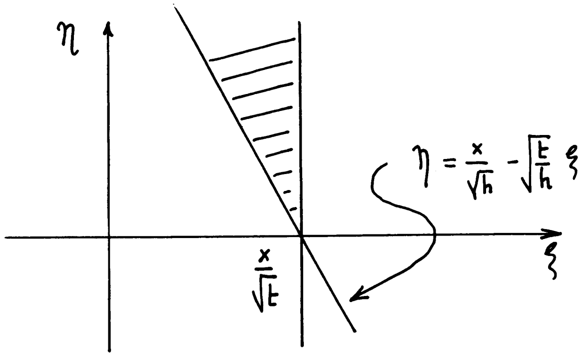
Then B^1 and B^2 are independent standard Brownian motions from zero, and

$$W_{st} \le x < W_{s,t+h} \quad \text{iff} \quad B_s^1 \le x/\sqrt{t}\,,\; B_s^1 + \sqrt{\frac{h}{t}}\, B_s^2 > x/\sqrt{t}$$

or, iff $(B_s^1, B_s^2) \in R_{xth} \overset{\text{def}}{=} \{(\xi,\eta) : \xi \le x/\sqrt{t},\ \eta > \frac{x}{\sqrt{h}} - \sqrt{\frac{t}{h}}\,\xi\}$

Thus, put $\bar{B}_s = (B_s^1, B_s^2)$. This is a standard planar Brownian motion, and we are finally led to look at

$$\frac{1}{\sqrt{h}} \int_0^s I_{R_{xth}}(\bar{B}_u)\,du.$$



R_{xth}

Note that as $h \to 0$ this tends to a local time along the half-line $\{\eta > 0,\ \xi = \frac{x}{\sqrt{t}}\}$. We don't need to show quite that much ; the following rather crude estimate is enough for our purposes. We claim that

$$E\{(\frac{1}{\sqrt{h}}\int_o^s I_{R_{xth}}(\bar{B}_u)du)^{p/2}\}$$

is bounded in x,t and h. It is clearly bounded for $h \geq 1$, so we may assume $h < 1$. Let $M = \max_{0\leq u\leq s} |B_u^2|$. Then for $u \leq s$, $I_{R_{xth}}(\bar{B}_u) \leq I_{\{\frac{x}{\sqrt{t}} - M\sqrt{h} < B_u^1 < \frac{x}{\sqrt{t}}\}}$,

so that

$$E\Big\{(\frac{1}{\sqrt{h}}\int_o^s I_{R_{xth}}(B_u)du)^{p/2}\Big\} \leq E\Big\{(M\frac{1}{M\sqrt{h}}\int_o^s I_{\{\frac{x}{\sqrt{t}} - M\sqrt{h} < B_u^1 < \frac{x}{\sqrt{t}}\}} du)^{p/2}\Big\}$$

But $M\sqrt{h}$ is independent of B_u^1, so this is

$$\leq E\{M^{p/2}\}\sup_{\varepsilon>0}E\{(\frac{1}{\varepsilon}\int_o^s I_{\{\frac{x}{\sqrt{t}} - \varepsilon < B_u^1 < \frac{x}{\sqrt{t}}\}} du)^{p/2}\}$$

Now $E\{M^{p/2}\}<\infty$, while thê expectation is bounded by Lemma 1.3. If the bound for this expression is $D(t,s,p)$, we have shown

$$(1.11) \qquad E\{K_3^p\} \leq C_p \frac{D(t,s,p)}{t^{p/2}} (t'-t)^{p/4}.$$

Denote $\Delta x = x'-x$, $\Delta s = s'-s$, $\Delta t = t'-t$. Collecting (1.7),(1.8), (1.9) and (1.11), we have

$$E\{|J_1|^p\} = O(|\Delta x|^p),\ E\{|J_2|^p\} = O(|\Delta s|^{p/2}),\ E\{|J_3|^p\} = O(|\Delta t|^{p/4}),$$

and this is uniform in $x,s,$ and t for $t \geq \varepsilon > 0$ and $s \leq s_o$. It follows that if $\Delta(x,s,t) = (\Delta x^2+\Delta s^2+\Delta t^2)^{1/2}$, that there is a constant α such that for $t \geq \varepsilon$, $s \leq s_o$, and $x \in \mathbb{R}$,

$$E\{|J(x';s',t') - J(x;s,t)|^p\} \leq \alpha\, \Delta(x,s,t)^{p/4}.$$

If $p = 12+4\varepsilon$, this is

$$= \alpha\, \Delta(x,s,t)^{3+\varepsilon}$$

Thus Kolmogorov's theorem applies, proving that there exists a version of $J(x;s,t)$ which is a.s. jointly continuous in the three variables on $t > 0$. In fact, if $x\neq 0$, we can extend this continuity to $t=0$, since as $W_{so} \equiv 0$,

$L_1(x,s,t) = 0$ for all small enough t if $x \neq 0$. q e d

Remark : the behavior of $L_1(x;s,t)$ for fixed t has been well-studied, since it is then a Brownian local time. It is interesting to consider its behavior as t varies for fixed x and s. We can show that it has a modulus of continuity which is $O(|t|^{1/4-\varepsilon})$ for any $\varepsilon > 0$. Indeed, (1.9) and (1.11) imply that for $\delta > 0$, there is a constant K depending on s and p such that if $t,t' \geq \delta$, $|t'-t| \leq 1$, then

$$E\{|L_1(x;s,t') - L_1(x;s,t)|^p\} \leq K|t'-t|^{1/4}$$

If we set $\Psi(t) = |t|^p$ and $\rho(t) = t^{1/4}$ in the theorem of Garsia, Rodemich, and Rumsey [2], we see that

$$E\{\int_\delta^{1+\delta} \int_\delta^{1+\delta} \left| \frac{L_1(x;s,t') - L_1(x;s,t)}{|t'-t|^{1/4}} \right|^p dt\, dt'\} \leq K.$$

If B is the value of the double integral above, it is a.s. finite, and the theorem tells us that for all t, t' in $(\delta, 1+\delta)$:

$$|L_1(x;s,t') - L_1(x;s,t)| \leq 2\int_o^{|t'-t|} \left(\frac{B}{v^2}\right)^{1/p} \frac{dv}{v^{3/4}}$$

This is finite if $p > 8$. If we set $\varepsilon = 2/p$ (which can be made as small as we please since p is arbitrary) this is

$$= \frac{8\ B^{1/p}}{1-4\ \varepsilon}\ |t'-t|^{1/4-\varepsilon}$$

While the above does not prove it, it suggests that $t \to L_1(x,s,t)$ is much rougher than, say, a Brownian path which would have a modulus of continuity of $O(\sqrt{x \log x})$. We conjecture, for instance, that its quadratic variation is infinite but that its fourth-power variation is finite. (This might be interesting to check. It could give us a process which is not a semi-martingale).

2 - LOCAL TIME IN THE PLANE

Let us define local time in the plane by

$$L(x;s,t) = \int_o^t L_1(x;s,v)\,dv \tag{2.1}$$

The integral exists, for if $x \neq 0$, L_1 is continuous in v, and if $x \neq 0$,

$$E\{\int_o^t L_1(0;s,v)\,dv\} = \int_o^t E\{L_1(0;s,v)\}\,dv,$$

where the application of Fubini is justified since the integrand is positive. By Proposition 1.1, $L_1(0;s,v)$ has the same distribution as $v^{-1/2}\,L_1(0;s,1)$ so this is

$$= E\{L_1(0;s,1)\}\int_o^t \frac{dv}{\sqrt{v}} < \infty.$$

It follows that $L(0;s,t)$ exists and is integrable (but not square-integrable !).

This gives us a local time in the following sense : if we integrate (1.2) we get

$$\int_o^s \int_o^t f(W_{uv})\,du\,dv = \int_{-\infty}^{\infty} L(x;s,t)\,dx \tag{2.2}$$

for each bounded Borel f. We can do the same with L_1 replaced by L_2. Since (2.2) remains true, we conclude that for each s,t

$$L(x;s,t) = \int_o^t L_1(x;s,v)\,dv = \int_o^s L_2(x;u,t)\,du, \tag{2.3}$$

for a.e. x, hence, by continuity, for all x. Since L_1 and L_2 are continuous, L is continuously differentiable in s and t. In fact, we have the following.

THEOREM 2.1

$L(x;s,t)$ *is a.s. continuous in the three variables* (x,s,t) *and, for fixed* x, *continuously differentiable in* s *and* t, *with*

$$\frac{\partial}{\partial s} L(x;s,t) = L_2(x;s,t) \quad \text{and} \quad \frac{\partial}{\partial t} L(x;s,t) = L_1(x;s,t).$$

Moreover, for each $s,t > 0$, $\varepsilon > 0$ *and* $N < \infty$, *there is a finite random* variable B such that

$$(2.4)\qquad |L(y;s,t) - L(x;s,t)| \le B|y-x|^{1-\varepsilon} \text{ if } \varepsilon \le |x|\,,\ |y| \le N.$$

We should point out that $L(x;s,t)$ is not the same as the local time $\phi(x;s,t)$ introduced in [1,§6], although they are closely related. Indeed, ϕ satisfied

$$(2.5)\qquad \int_0^s \int_0^t f(W_{uv})uv\,du\,dv = \int_{-\infty}^{\infty} \phi(x;s,t)\,dx.$$

If we compare this with (2.2), we see that

$$\phi(x;s,t) = \int_0^s \int_0^t uv\,L(x;du,dv)$$

The factor uv makes ϕ better-behaved at the origin than L, while L is perhaps a more natural quantity, since it is the occupation-time density.

We have proved all but (2.4) of this theorem, but we can't prove that yet. It will follow from theorem 2.4 below.

We will need to establish a number of results before we are ready to prove this theorem. Because of the singular behavior of L near the origin, we will first consider the local time in a region bounded away from the axes, which we may as well take to be $s \ge 1$, $t \ge 1$. Let

$$\hat{L}_1(x;s,t) = L_1(x;s,t) - L_1(x;1,t)$$
$$\hat{L}_2(x;s,t) = L_2(x;s,t) - L_2(x;s,1),$$

and put

$$\hat{L}(x;s,t) = \int_1^t \hat{L}_1(x;s,v)\,dv$$
$$= \int_1^s \hat{L}_2(x;u,t)\,du.$$

Note that $s \to \hat{L}_1(x;s,t)$ is increasing ; instead of writing $d_s\hat{L}_1(x;s,t)$ we will write $\hat{L}_1(x;ds,t)$. Equation (1.2) can be generalized easily to include functions varying with time : if $f(x,t)$ is bounded and Borel measurable :

$$(2.6)\qquad \int_0^s f(W_{ut},u)\,du = \int_{-\infty}^{\infty}\int_0^s f(x,u)\; L_1(x;du,t)\,dx$$

(To see this, note that is is clearly true if $f(x,u) = g(x)\; I_{[a,b]}(u)$, and apply a monotone class argument).

Let $R_{st} = [0,s] \times [0,t]$, with the obvious convention in case either s or t is infinite. We recall a few facts about stochastic integrals.

If $\{\phi_{st},\ s \geq 0,\ t \geq 0\}$ is measurable and adapted to the fields $\mathcal{F}_{st} = \sigma\{W_{uv},\ u \leq s,\ v \leq t\}$, and if $E\{\int_{R_{st}} \phi^2(u,v)\,du\,dv\} < \infty$, one can define the stochastic area integral

$$\int_{R_{st}} \phi_{uv}\,d\,W_{uv}.$$

We can also do this if ϕ is only weakly adapted, i.e adapted to the fields $\mathcal{F}_{s\infty}$ (1). Furthermore, one can write line integrals in this form, e.g. if $\{\rho_s,\ s \geq 0\}$ is adapted to $\mathcal{F}_{st}$, then

$$\int_0^s \rho_u\,\partial_1\,W_{ut} = \int_{R_{s\infty}} \rho_u\, I_{\{v \leq t\}} dW_{u,v}$$

This brings us to another formula for $\hat{L}$.

<u>PROPOSITION 2.2</u>

<u>If</u> $s \geq 1,\ t \geq 1$

$$(2.7)\qquad \hat{L}(x;s,t) = 2\int_1^t ((W_{s\tau}-x)^+ - (W_{1\tau}-x)^+)\,\frac{d\tau}{\tau}$$

$$- 2\int_{R_{st}-R_{1t}} \left(\int_x^{\infty}\int_{v\vee 1}^{t} \hat{L}_2(y;s,d\tau)\,\frac{1}{\tau}\,dy\right) dW_{uv}$$

<u>Proof</u> : we begin with Tanaka's formula (1.1), writing the line integral as an area integral with respect to a weakly adapted integrand.

$$\hat{L}_1(x;s,t) = \frac{2}{t}\left[(W_{st}-x)^+ - (W_{1t}-x)^+ - \int_{R_{st}-R_{1t}} I_{\{W_{ut}>\ x\}}\, I_{\{0 \leq v \leq t\}} d\,W_{uv}\right].$$

By the definition of $\hat{L}$:

$$\hat{L}(x;s,t) = 2\int_1^t \left[(W_{s\tau}-x)^+ - (W_{1\tau}-x)^+\right]\frac{d\tau}{\tau} - 2\int_1^t \int_{R_{st}-R_{1t}} I_{\{W_{u\tau}>x\}} I_{\{0\le v\le\tau\}}\, dW_{uv}\,\frac{d\tau}{\tau}$$

Interchange the order

$$= 2\int_1^t \left[(W_{s\tau}-x)^+ - (W_{1\tau}-x)^+\right]\frac{d\tau}{\tau} - 2\int_{R_{st}-R_{1t}} \left[\int_{v\vee 1}^t I_{\{W_{u\tau}>x\}}\frac{d\tau}{\tau}\right] dW_{uv}$$

By (2.6) with L_1 replaced by $\hat{L}_2$:

$$\int_{v\vee 1}^t I_{\{W_{u\tau}>x\}}\frac{d\tau}{\tau} = \int_x^\infty \int_{v\vee 1}^t \hat{L}_2(y;u,d\tau)\,\frac{1}{\tau}\,dy.$$

Substituting this above gives (2.6) q e d

Let us see how $\hat{L}(x;s,t)$ varies with x. From (2.7) we can write

$$\hat{L}(x;s,t) = M(x) + N(x).$$

Now $|M(y) - M(x)| \le 2(t-1)|y-x|$, so M is Lipschitz continuous, and it remains to investigate N. Let $x < y$ and define :

$$X_s^{xy} = N(y)-N(x) = -2\int_{R_{st}-R_{1t}} \left(\int_x^y \int_{v\vee 1}^t \hat{L}_2(z;s,d\tau)\,\frac{1}{\tau}\,dz\right) dW_{uv}$$

This expression will be more transparent if we set

$$\phi(u,v) = \int_x^y \int_{v\vee 1}^t \frac{1}{\tau}\,\hat{L}_2(z;u,d\tau)\,dz.$$

Now $\hat{L}_2$ has moments of all orders, hence so does ϕ , and $\phi(s,v)$ is $\mathcal{F}_{st}$-measurable, so that

$$X_s^{xy} = -2\int_{R_{st}-R_{1t}} \phi(u,v)\,dW_{uv}$$

is an integral with respect to a weakly adapted integrand. Moreover, $\{X_s^{xy}, \mathcal{F}_{st}, s \ge 0\}$ is a continuous martingale whose increasing process is

$$\langle X^{xy}\rangle_s = 4\int_{R_{st}-R_{1t}} \phi^2(u,v)\,du\,dv$$

$$= 4 \int_0^t \int_1^s \left(\int_x^y \int_{v\vee 1}^t \frac{1}{\tau} \hat{L}_2(z;u;d\tau)dz \right)^2 du\,dv$$

$$\leq 4 \int_o^t \int_1^s \left(\int_x^y \hat{L}_2(z;u,t)^2\, dz \right)^2 du\,dv$$

By Hölder, if $p > 1$

$$< X^{xy} >_s^p \leq 4^p\,(t\,(s-1))^{p-1} \int_o^t \int_1^s \left(\int_x^y \hat{L}_2(z;u,t)dz \right)^{2p} du\,dv.$$

Applying Hölder again

$$(2.8) \qquad < X^{xy} >_s^p \leq 4^p\,(\,t\,(s-1)\,)^{p-1}\,(y-x)^{2p-1} \int_1^t \int_1^s \int_x^y \hat{L}_2(z;u;t)^{2p}\,dz\,du\,dv$$

Let us estimate this. Since $\sqrt{s}\,L_2(x\sqrt{s},s,.)$ has the same distribution as $L_2(x,1,.)$, (proposition 1.1) then if $u \geq 1$, certainly

$$E\{\hat{L}_2(z;u,t)^{2p}\} \leq E\{L_2(z;u,t)^{2p}\} \leq E\{L_2(0;1,t)^{2p}\}.$$

But standard Brownian local time at zero has the same distribution as the Brownian maximum process, so, by the reflection principle

$$E\{L_2(0,1,t)^{2p}\} = E\{W_{1t}^{2p}\} = \frac{(2p)!}{p!}\left(\frac{t}{2}\right)^p$$

(This last can be easily calculated from the characteristic function, for instance). It follows from (2.8) that

$$(2.9) \qquad E\{ < X^{xy} >_s^p\} \leq 4^p \left(\frac{t^2}{2}\,(s-1)\,\right)^p \frac{(2p)!}{p!}\,(y-x)^{2p}.$$

By Burkholder's inequality

$$(2.10) \qquad E\{(X_s^{xy})^p\} \leq (p+1)^p\, E\{ < X^{xy} >_s^{p/2}\},$$

where we have used Klincsek's bound on the constant (3).

If we combine (2.9) and (2.10), we have proved

THEOREM 2.3

For each $p > 1$, $s \geq 1$ and $t \geq 1$ there is a constant K_{pst} such that for each x,y

$$E\{|\hat{L}(y;s,t) - \hat{L}(x;s,t)|^p\} \leq K_{pst} \, |y-x|^p$$

This theorem immediately gives us a modulus of continuity for $\hat{L}$, thanks to the theorem of Garsia, Rodemich, and Rumsey [2].

THEOREM 2.4

Fix $s \geq 1$, $t \geq 1$. For all $\varepsilon > 0$, and any N, there exists a random variable B such that for all $|x|$, $|y| \leq N$,

$$(2.11) \qquad |\hat{L}(y;s,t) - L(x,s,t)| \leq B|y-x|^{1-\varepsilon}.$$

Proof : let $\Psi(x) = |x|^p$ and $\rho(x) = x$ in the Garsia-Rodemich-Rumsey theorem. Now

$$E\{\int_o^1 \int_o^1 \left| \frac{\hat{L}(y;s,t) - \hat{L}(x;s,t)}{y-x} \right|^p dy\, dx\} \leq K_{pst}$$

by theorem 2.3. If $D(\omega)$ is the double integral above, $D(\omega) < \infty$ a.s. so that for all $0 < x,\ y < 1$

$$|\hat{L}_t(y) - \hat{L}_t(x)| \leq 8 \int_o^{|y-x|} \left(\frac{D(\omega)}{t^2}\right)^{1/p} dt = \frac{8p\, D^{1/p}}{p-2} \, |y-x|^{1-2/p}$$

This can be extended from $[0,1]$ to any finite interval. Since p can be taken as large as we please, we can assume $\frac{2}{p} < \varepsilon$, and we are done. q e d

Theorem 2.1 now follows easily. We have already proved all but (2.4). To prove this, note first that if $\hat{L}^\varepsilon(x;s,t)$ is the local time in $s \geq \varepsilon$, $t \geq \varepsilon$ (so that $\hat{L}^1 = \hat{L}$) then theorem 2.4 remains true with $\hat{L}$ replaced by $\hat{L}^\varepsilon$. But now if $|x|$, $|y| \geq \delta > 0$ there exists some $\varepsilon > 0$, depending on ω, such that $|W_{st}| < \delta$ if either $s \leq \varepsilon$, $t \leq N$ or $t \leq \varepsilon$, $s \leq N$. But $L(x;s,t,\omega) = \hat{L}^\varepsilon(x;s,t,\omega)$ and $L(y;s,t,\omega) = \hat{L}^\varepsilon(y;s,t,\omega)$ for all x,y such that $|x|$, $|y| \geq \delta$, and we are done, since $\hat{L}^\varepsilon$ satisfies (2.11).

The modulus of continuity in theorem 2.1 is tantalizingly close to $O(|y-x|)$, which leads us to end the article with a question : is $x \to L(x;s,t)$ Lipschitz continuous ?.

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