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A DIFFUSION WITH A DISCONTINUOUS LOCAL TIME

by

John B. WALSH

Let $X_t = M_t + V_t$ be a continuous semi-martingale, where M_t is a local martingale and V_t is a continuous process of locally bounded variation. In [2] Marc Yor has proved that X_t has a local time, $\phi(t,x)$, at each point x, which is continuous in the space variable except possibly at points x with the property that $d|V_t|$ charges the set $\{t : X_t = x\}$. The most obvious example of a process whose local time is discontinuous is reflecting Brownian motion ; there is clearly a discontinuity at zero, and, as the process V_t in this case is the local time at zero itself, $d|V_t|$ does charge $\{t : X_t = 0\}$. One might object that this discontinuity comes about because the natural state space of the process is $[0,\infty)$, rather than $(-\infty,\infty)$, so that it doesn't provide a satisfying illustration of Yor's theorem. We thought it would be worthwhile to give an example of a diffusion which has the whole line as its natural state space, and which has a discontinuity in its local time at the origin. Since the process has some interest in its own right, we will take the opportunity to bring to light a few properties which aren't relevant to local time, but which still seem to merit a showing.

Let R_t be a reflecting Brownian motion and fix $0 \leq \alpha \leq 1$. Ito and Mc Kean show how to construct what they call skew Brownian motion ([1], problem 1, section 4.2) from R_t. Consider the excursions of R_t from 0. Change the sign of each excursion independently with probability $1-\alpha$, so that a given excursion is positive with probability α, negative with probability $1-\alpha$. The resulting process is called a <u>skew Brownian motion with parameter</u> α . If $\alpha = 1/2$, this is an ordinary Brownian motion, and if α equals zero or one, it is a reflecting Brownian motion. If $\alpha \neq 1/2$, it is a diffusion which acts like a Brownian motion as long as it is away from the origin, and which passes the origin more easily in one direction that the other. It can be used to model a particle diffusing thru a semi-permeable membrane under osmotic pressure.

Ito and Mc Kean give its <u>scale function</u> s_α as

$$s_\alpha(x) = \begin{cases} (1-\alpha)x & \text{if } x \geq 0 \\ \alpha x & \text{if } x \leq 0. \end{cases}$$

Let r_α be the inverse of s_α :

$$r_\alpha(x) = \begin{cases} \frac{x}{1-\alpha} & \text{if } x \geq 0 \\ \frac{x}{\alpha} & \text{if } x \leq 0 \end{cases}$$

(with the convention that $1/0 = 0$ in case $\alpha = 0$ or 1). Then the <u>speed measure</u> ν_α is

$$\nu_\alpha(dx) = r_\alpha(x)dx.$$

It follows that its infinitesimal generator $\mathfrak{G}_\alpha$ is

$$\mathfrak{G}_\alpha f(x) = \frac{1}{2} \frac{d}{d\nu_\alpha} \left(\frac{d^+ f}{ds_\alpha}\right)(x) = \frac{1}{2} f''(x) \tag{1}$$

In order for f to be in the domain $D(\mathfrak{G}_\alpha)$ of $\mathfrak{G}_\alpha$ it must satisfy

(i) f'' is continuous on $\mathbb{R}-\{0\}$, and $f''(0+) = f''(0-)$;

(ii) $\alpha f'(0+) = (1-\alpha) f'(0-)$.

<u>A SECOND CHARACTERIZATION</u>

Skew Brownian motion can be characterized in several ways. For instance :

<u>Proposition 1</u>

<u>A diffusion X on $\mathbb{R}$ is a skew Brownian motion iff $\{|X_t|, t \geq 0\}$ is a reflecting Brownian motion.</u>

Proof : One can see from the construction that if X is a skew Brownian motion, then $|X|$ is a reflecting Brownian motion. Conversely, if X or $-X$ is itself a reflecting Brownian motion, it is already a skew Brownian motion with $\alpha = 1$ or 0 respectively. These are the extreme cases, and in all others, the origin will be a regular point.

Thus, suppose $|X|$ is a reflecting Brownian motion and that the origin is regular for both $(-\infty,0)$ and $(0,\infty)$. Now X must be a Brownian motion on both $(-\infty,0)$ and $(0,\infty)$ so that if X has infinitesimal generator $\mathfrak{G}$ and $f \in D(\mathfrak{G})$,

$$\mathfrak{G}f(x) = \frac{1}{2} f''(x) \quad \text{if} \quad x \neq 0. \tag{2}$$

Let $s(x)$ be a strictly increasing solution of $\mathfrak{G}s = 0$, which must exist. Then s is continuous, and linear on both $(-\infty,0)$ and $(0,\infty)$. It may have a break in its slope at the origin. Now let

$$\alpha = s'(0+)\ (s'(0+) - s'(0-))^{-1},$$

and note that s_α is a linear function of s, so that $\mathfrak{G}s_\alpha = 0$. This means that s_α is a scale function of X. Using (2) and the relation

$$\mathfrak{G}f = \frac{1}{2} \frac{d}{d\nu} \left(\frac{d^+ f}{ds_\alpha}\right),$$

we see the speed measure ν of X equals ν_α on $\mathbb{R} - \{0\}$, while the fact that $P\{X_t = 0\} = P\{R_t = 0\} = 0$ for all $t > 0$ implies that $\nu\{0\} = 0$, so $\nu \equiv \nu_\alpha$. Since the scale and speed measure determine a diffusion, we are done.

Proposition 2

Let X be a skew Brownian motion. Then α is determined by either of

(a) $P^0\{X_t > 0\} = \alpha$ for some (hence all) $t > 0$;

or (b) $\{s_\alpha(X_t),\ t \geq 0\}$ is a martingale, and $0 < \alpha < 1$.

Proof : Part (a) follows from the original construction. If $0 < \alpha < 1$, $\mathfrak{G}_\alpha s_\alpha \equiv 0$, so $s_\alpha(X_t^\alpha)$ is a martingale. For any $\beta \neq \alpha$, $s_\beta(X_t^\alpha)$ is not a martingale. If it were, and if $\alpha < \beta < 1$, say, $f(X_t^\alpha) = \frac{1-\alpha}{1-\beta} s_\beta(X_t^\alpha) - s_\alpha(X_t^\alpha)$ would be one too. But $f \geq 0$ and $f(o) = 0$, so for any $t > 0$

$$0 = E^o\{f(X_o^\alpha)\} < E^o\{f(X_t^\alpha)\},$$

a contradiction.

THE TRANSITION FUNCTION

Proposition 2a implies a non-symmetric reflection principle around 0, and we can use this to determine the transition function in closed form. Indeed, if $p_t^\alpha(x,y)$ is the transition density of skew Brownian motion with parameter α, we have :

$$(3) \qquad \sqrt{2\pi t}\; p_t^\alpha(o,y) = \begin{cases} 2\alpha\, e^{-y^2/2t} & \text{if } y > 0 \\ 2(1-\alpha)\, e^{-y^2/2t} & \text{if } y > 0 \end{cases}$$

and for $x \neq 0$

$$(4) \qquad \sqrt{2\pi t}\; p_t^\alpha(x,y) = \begin{cases} e^{-\frac{(y-x)^2}{2t}} + (2\alpha-1)\, e^{-\frac{(y+x)^2}{2t}} & \text{if } x > 0,\ y > 0 \\ e^{-\frac{(y-x)^2}{2t}} + (1-2\alpha)\, e^{-\frac{(y+x)^2}{2t}} & \text{if } x < 0,\ y < 0 \\ 2\alpha\, e^{-\frac{(y-x)^2}{2t}} & \text{if } x < 0,\ y > 0 \\ 2(1-\alpha)\, e^{-\frac{(y-x)^2}{2t}} & \text{if } x > 0,\ y < 0. \end{cases}$$

Indeed, (3) is clear from the original construction and (4) can be seen as follows.

Let τ_o be the first hit of 0. If $x > 0$ and $y > 0$,

$$P^x\{X_t \in dy\} = P^x\{X_t \in dy,\ \tau_o > t\} + P^x\{X_t \in dy,\ \tau_o \leq t\}.$$

The first probability on the right equals that for ordinary Brownian motion, which is, by the reflection principle :

$$P^x\{B_t \in dy\} - P^x\{B_t \in -dy\} = \frac{1}{\sqrt{2\pi t}}\left(e^{-\frac{(y-x)^2}{2t}} - e^{-\frac{(y+x)^2}{2t}}\right)$$

where $\{B_t, t \geq 0\}$ is standard Brownian motion.

On the other hand, Proposition 2a implies

(5) $$P^x\{X_t \in dy, \tau_o \leq t\} = 2\alpha\, P^x\{B_t \in dy, \tau_o \leq t\} = \frac{2\alpha}{\sqrt{2\pi t}}\, e^{-\frac{(y+x)^2}{2t}}.$$

This gives us the first line of (4) ; the second follows by symmetry (replace α by $1-\alpha$), the third follows from (5) with x replaced by $-x$, and the last follows again by symmetry.

Of course, rather than deriving these formulae, one can show by direct calculation that p_t^α is the transition function of a diffusion and that its infinitesimal generator is $\mathfrak{G}_\alpha$.

A SECOND PATH CONSTRUCTION

To avoid trivialities, we will suppose that $0 < \alpha < 1$ from now on. Note that $\{r_\alpha(B_t), t \geq 0\}$ will be a diffusion with scale function s_α. The scale function won't change if we time-change the process, so let

$$A_t = \int_o^t r_\alpha^2(B_s)ds$$

and define

$$T_t = \inf\{s > 0 : A_s > t\}.$$

We then define

(6) $$X_t^\alpha = r_\alpha(B_{T_t}),\ t \geq 0.$$

<u>Proposition 3</u>

X^α <u>is a skew Brownian motion with parameter</u> α.

<u>Proof</u> : if we notice that $s_\alpha(X_t^\alpha)$ is a martingale, and that we have chosen

the time-change in such a way as to make $|X_t^\alpha|$ into a reflecting Brownian motion, then Proposition 3 follows from Propositions one and two.

SKEW BROWNIAN MOTION AS A SEMI_MARTINGALE

The representation (6) makes it easy to decompose X_t^α into a martingale and a process of finite variation. Let L_t^o be the Brownian local time and apply the extended form of Ito's formula to $r_\alpha(B_t)$:

$$(7) \qquad r_\alpha(B_t) = r_\alpha(B_o) + \int_o^t r'_\alpha(B_s)dB_s + \frac{1}{2}(2\alpha-1)L_t^o.$$

Since $X_t^\alpha = r_\alpha(B_{T_t})$, it follows that we can write $X_t^\alpha = M_t+V_t$, where M_t, the local martingale, comes from the stochastic integral, and $V_t = (\alpha-\frac{1}{2})L_{T_t}^o$.

Notice that dV_t is carried by the set $\{t : X_t^\alpha = 0\}$, so that, by Yor's theorem, X^α has a local time which will be continuous except possibly at the origin.

THE DISCONTINUITY OF THE LOCAL TIME

$|X^\alpha|$ is a reflecting Brownian motion, so that all reasonable definitions of its local time at x will agree, at least if $x\neq 0$. To keep to the spirit of (2), we will use Meyer's decomposition to define the local time. Fix $x\neq 0$. Then the local time at x, $\phi_\alpha(t,x)$, of X^α is the unique continuous increasing process for which $\phi_\alpha(x,o) = 0$ and

- if $x > 0$, then $\{(X_t^\alpha-x)^+ - \frac{1}{2}\phi_\alpha(x,t), t \geq 0\}$ is a martingale(*) ;
- if $x < 0$, then $\{(X_t^\alpha-x)^- - \frac{1}{2}\phi_\alpha(x,t), t \geq 0\}$ is a martingale.

The use of different expressions for $x > 0$ and $x < 0$ above is not significant. We are merely trying to avoid discussing $x=o$, which is somewhat singular for X^α.

(*) That is, $\phi_\alpha(x,t)$ is the increasing process in Meyer's decomposition of the sub-martingale $\{(X_t^\alpha-x)^+, t \geq 0\}$.

Proposition 4

$x \to \phi_\alpha(x,t)$ is discontinuous at zero unless $\alpha = 1/2$.

Proof : let L_t^x be the standard Brownian localtime at x. If $x > 0$, by (6):

$$(X_t^\alpha - x)^+ = \frac{1}{1-\alpha}(B_{T_t} - (1-\alpha)x)^+$$

$$= \frac{1}{1-\alpha}(B_o - (1-\alpha)x)^+ + \frac{1}{1-\alpha}\int_o^{T_t} I_{\{B_s > (1-\alpha)x\}} dB_s$$

$$+ \frac{1}{2(1-\alpha)} L_{T_t}^{(1-\alpha)x} ,$$

this last by Tanaka's formula. The stochastic integral is a martingale, so $\phi(x,t)$ must equal $\frac{1}{1-\alpha} L_{T_t}^{(1-\alpha)x}$ if $x > 0$. On the other hand, if $x < 0$,

$$(X_t^\alpha - x)^- = \frac{1}{\alpha}(B_{T_t} - \alpha x)^-$$

$$= \frac{1}{\alpha}(B_o - \alpha x)^- + \frac{1}{\alpha}\int_o^{T_t} I_{\{B_s < \alpha x\}} dB_s + \frac{1}{2\alpha} L_{T_t}^{\alpha x}.$$

We conclude that

$$(8) \qquad \phi(x,t) = \begin{cases} \frac{1}{1-\alpha} L_{T_t}^{(1-\alpha)x} & \text{if } x > 0 \\ \frac{1}{\alpha} L_{T_t}^{\alpha x} & \text{if } x < 0 \end{cases}$$

Since $x \to L_{T_t}^x$ is continuous, it follows that $x \to \phi(x,t)$ has a jump at the origin of $\frac{2\alpha-1}{\alpha(1-\alpha)} L_{T_t}^o$. If $\alpha \neq 1/2$, this is non-zero as soon as $T_t > \tau_o$, i.e. as soon as $L_{T_t}^o > 0$.

Note : the local time calculated above is exactly the density of the occupation time, and we could have calculated (8) directly, using elementary calculus rather than Tanaka's formula.

EPILOGUE

Ito and Mc Kean do not prove that the path construction of skew Brownian motion - by changing the signs of the excursions of reflecting Brownian motion - actually does lead to a diffusion. They outline an argument which, if we have read it correctly, assumes the process is a diffusion and then computes its infinitesimal generator on that basis. This is not a serious omission, for it is possible to prove the process is a diffusion in several ways, and we had originally intended to show here how this can be neatly deduced from Ito's theory of Poisson point processes. However, an early draft of this article convinced us that the machinery of the Poisson point process itself took so much space to set up that, no matter how easy the argument, the reader would still finish with the feeling that he had just watched a butterfly being assassinated with an elephant gun.

We decided to leave this out and, trusting the reader's faith that such constructions do indeed lead to diffusions, to remark instead that a similar procedure leads to a diffusion in the plane - or in $\mathbb{R}^n$, if one wishes - with an interesting singularity at the origin.

The idea is to take each excursion of R_t and, instead of giving it a random sign, to assign it a random variable Θ with a given distribution in $[0,2\pi)$, and to do this independently for each excursion. That is, if the excursion occurs during the interval (u,v), we replace R_t by the pair (R_t,Θ) for $u \leq t < v$, Θ being a random variable with values in $[0,2\pi)$. This provides a process $\{(R_t,\Theta_t),\ t \geq 0\}$, where Θ_t is constant during each excursion from 0, has the same distribution as Θ, and is independent for different excursions. We then consider $X_t = (R_t,\Theta_t)$ as a process in the plane, expressed in polar coordinates. It is a diffusion which, when away from the origin, is a Brownian motion along a ray, but which has what might be called a <u>roundhouse singularity</u> at the origin : when the process enters it, it, like Stephen Leacock's hero, immediately rides off in all directions at once.

As it stands, X is even singular away from the origin, since it only travels along rays, but it is possible to perturb the angular coordinate slightly in order to get a genuinely two-dimensional diffusion which still has the same type of roundhouse singularity at the origin, and for which all other points are polar. For instance, we could replace X by

$$\hat{X}_t = (R_t, \theta_t + \eta_t),$$

where η_t is a process which satisfies a stochastic differential equation during each excursion : if there is an excursion starting at time u, ending at v, η satisfies

$$\begin{cases} d\eta_t = f(\hat{X}_t)dB_t + g(\hat{X}_t)dt & \text{if } u \leq t < v \\ \eta_u = 0 \end{cases}$$

where B_t is a Brownian motion independent of X and f and g are bounded and continuous on $\mathbb{R}^2$.

We have no intention of trying to classify all possible behaviors at a roundhouse singularity ; the situation is already complicated enough so that even the most indulgent reader must be uneasily clearing his throat, shuffling his feet, and hinting that he would like to see a proof before going any further. But we can't resist pointing out that it should be possible to choose g - not necessarily continuous - to get a process which always leaves the origin along one of the rays $\theta = \theta_i$, i=1,...,n - this is true of the process $\hat{X}$ above if the distribution of Θ is concentrated on $\{\theta_1,\ldots,\theta_n\}$ - and which in addition can only return to the origin along a different set of rays $\theta = \theta_j$, j=n+1,...,N. This would represent a charged particle diffusing in a dipole or multipole field.

REFERENCES

(1) K. ITO and H.P. McKEAN : Diffusion processes and their sample paths. Springer-Verlag, 1974.

(2) M. YOR : Sur la continuité des temps locaux associés à certaines semi-martingales. Dans le même volume.