KRYSTYNA MAŁGORZATA ZIEMIAN

On topological and measure entropies of semigroups

Astérisque, tome 51 (1978), p. 457-472

<http://www.numdam.org/item?id=AST_1978__51__457_0>

Société Mathématique de France
Astérisque 51 (1978) p.457-472

On topological and measure entropies of semigroups

by

Krystyna Małgorzata Ziemian

The presentet paper contains a generalization of the theory of topological and measure entropies to the case of an action of an arbitrary subsemigroup of Z^N. Some ideas were suggested to the author by M. Misiurewicz.

1. Definitions of the topological and measure entropies.

A subset $\tilde{\Lambda} \subset R^N$ will be called a cone in R^N if $\forall x \in \Lambda \ \forall t > 0 \quad t \cdot x \in \Lambda$ and $\tilde{\Lambda} \cap B(0,1)$ is of positive Jordan measure, where $B(0,1)$ is the unit-ball in R^N. The set Λ of the form $\Lambda = \tilde{\Lambda} \cap Z^N$, where $\tilde{\Lambda}$ is a cone in R^N, will be called a cone in Z^N.

If G is a semigroup in Z^N then G generates a subgroup of Z^N isomorphic to $Z^{N'}$ for some $N' \in N$ / N as usually denotes the set of positive integers /. Thus without loss of generality, we can restrict ourselves to the study of these semigroups in Z^N which generate Z^N. It is easy to prove the following.

Proposition 1. A semigroup $G \subset Z^N$ generates Z^N iff G contains a cone in Z^N.

Commencing from now G is a fixed semigroup in Z^N containing a cone Λ in Z^N.

We introduce the following notations:

For $r_1 = (r_1^1, \ldots, r_1^N)$, $r_2 = (r_2^1, \ldots, r_2^N) \in \mathbb{R}^N$ the relation $r_1 < r_2$ / $r_1 \leqslant r_2$ / means that $r_1^i < r_2^i$ /$r_1^i \leqslant r_2^i$/ for $i = 1, \ldots, N$.

$\mathbb{R}_+^N \overset{df}{=} \{ x \in \mathbb{R}^N : x \geqslant 0 \}$.

For $\varrho \in \mathbb{R}_+^N$ $J_\varrho \overset{df}{=} \{ x \in \mathbb{R}_+^N : x < \varrho \}$. $J_\varrho + s$, where $s \in \mathbb{R}^N$, will be called a rectangle in $\mathbb{R}^N$.

$Z_+^N \overset{df}{=} \{ z \in Z^N : z \geqslant 0 \}$.

For $w \in Z_+^N$ $I_w \overset{df}{=} \{ x \in Z_+^N : x < w \}$. $I_w + z$, where $z \in Z^N$, will be called a rectangle in Z^N.

X is a non-empty, compact Hausdorff (probability) space.

T is an action of G in X (it is not assumed that $T^0 = id_X$).

$\mathcal{A}$ denotes an open cover (a finite measurable partition) of X.

For every subset B of G we set $\mathcal{A}_B \overset{df}{=} \bigvee_{g \in B} (T^g)^{-1} \mathcal{A}$.

$H(\mathcal{A}_B)$ stands for the topological (measure) entropy of the cover (partition) $\mathcal{A}_B$.

For $n \in N$ we set $\Lambda^n \overset{df}{=} \Lambda \cap B(0,n)$, where $B(0,n)$ is the ball with center 0 and radius n.

Theorem 1. $\lim_n \frac{1}{\operatorname{card} \Lambda^n} H(\mathcal{A}_{\Lambda^n})$ exists and does not depend on the choice of $\Lambda \subset G$.

Lemma 1. Let δ be an arbitrary positive number. If Λ is a cone in Z^N and (n_l) is a sequence of positive integers such that $\lim_{l} n_l = +\infty$ then there exist

(i) positive integers $l_1, \ldots, l_k, t_1, \ldots, t_k$

(ii) $w \in Z_+^N$

(iii) $z_{i,j} \in I_w$, $j=1, \ldots, t_i$, $i=1, \ldots, k$

such that $I_w = \bigcup_{j=1}^{t_1} (\Lambda^{n_{l_1}} + z_{1,j}) \cup \ldots \cup$

$\ldots \cup \bigcup_{j=1}^{t_k} (\Lambda^{n_{l_k}} + z_{k,j}) \cup I_w'$, where all the sets in the above sum are pairwise disjoint and $\frac{\text{card } I_w'}{\text{card } I_w} < \delta$.

Proof: By assumption, $\Lambda = \tilde{\Lambda} \cap Z^N$, $\Lambda^{n_l} = \Lambda \cap B(0,n_l) = \tilde{\Lambda} \cap B(0,n_l) \cap Z^N$ for $l \in \mathbb{N}$. Let $\tilde{\Lambda}^{n_l} \overset{\text{df}}{=} \tilde{\Lambda} \cap B(0,n_l) \subset \mathbb{R}^N$. Fix $\varepsilon > 0$. If $|\cdot|$ denotes the Jordan measure on $\mathbb{R}^N$ then

(1) $$\lim_{l} \frac{\text{card}(\tilde{\Lambda}^{n_l} \cap Z^N)}{|\tilde{\Lambda}^{n_l}|} = 1,$$

by definition of Jordan measure.

Let $J \subset \mathbb{R}$ be a rectangle with vertices belonging to Z^N such that $\tilde{\Lambda}^1 \subset J$. Denote

(2) $$\beta \overset{\text{df}}{=} \frac{|\tilde{\Lambda}^1|}{|J|}$$

I_w can be constructed inductively. The idea is the following. We chose $l_1 \in \mathbb{N}$ such that $n_{l_1} \cdot J \setminus \tilde{\Lambda}^{n_{l_1}}$ can be covered by pairwise disjoint translates of $n_1 \cdot J$ by vectors with integer coordinates so precisely that if we denote the covered part of $n_{l_1} \cdot J$ by $(n_{l_1} \cdot J)_c$ then

$$(3) \qquad \frac{|(n_{l_1} \cdot J)_c|}{|n_{l_1} \cdot J \setminus \tilde{\Lambda}^{n_{l_1}}|} > 1 - \varepsilon .$$

Then, $n_{l_1} \cdot J$ contains both $\tilde{\Lambda}^{n_{l_1}}$ and the translates of $\tilde{\Lambda}^{n_1}$. Now, if $(n_{l_1} \cdot J)_{\tilde{\Lambda}}$ denotes the sum of and these translates then, in virtue of (2) and (3),

$$(4) \qquad \frac{|(n_{l_1} \cdot J)_{\tilde{\Lambda}}|}{|n_{l_1} \cdot J|} > \beta + (1 - \varepsilon)(1 - \beta) \cdot \beta .$$

Now, we chose $l_2 \in N$ such that $n_{l_2} \cdot J \setminus \tilde{\Lambda}^{n_{l_2}}$ can be convered pairwise disjoint translates of $n_{l_1} \cdot J$ by vectors with integer coordinates, so precisely that if we denote the covered part of $n_{l_2} \cdot J$ by $(n_{l_2} \cdot J)_c$ then

$$(5) \qquad \frac{|(n_{l_2} \cdot J)_2|}{|n_{l_2} \cdot J \setminus \tilde{\Lambda}^{n_{l_2}}|} > 1 - \varepsilon .$$

Then, $n_{l_2} \cdot J$ contains both $\tilde{\Lambda}^{n_{l_2}}$ and the translates of $\tilde{\Lambda}^{n_{l_1}}$ and $\tilde{\Lambda}^{n_1}$. Now, if $(n_{l_2} \cdot J)_{\tilde{\Lambda}}$ denotes the sum of $\tilde{\Lambda}^{n_{l_2}}$ and these translates then by (2), (4) and (5) we have

$$(6) \qquad \frac{|(n_{l_2} \cdot J)_{\tilde{\Lambda}}|}{|n_{l_2} \cdot J|} > \beta + (1 - \varepsilon)(1 - \beta) \cdot \frac{|(n_{l_1} \cdot J)_{\tilde{\Lambda}}|}{|n_{l_1} \cdot J|}$$

Continuing this procedure, after the k-th step we have $J_{n_{l_k}}$ which contains both $\tilde{\Lambda}^{n_{l_k}}$ and the translates of $\tilde{\Lambda}^{n_1}$, $\tilde{\Lambda}^{n_{l_1}}$, ..., $\tilde{\Lambda}^{n_{l_{k-1}}}$ by vectors with integer coordinates, and if $(n_{l_k} \cdot J)_{\tilde{\Lambda}}$ denotes the sum of $\tilde{\Lambda}^{n_{l_k}}$ and these translates then

$$(7) \quad \frac{|(n_{l_k} \cdot J)_{\tilde{\Lambda}}|}{|n_{l_k} \cdot J|} > \beta + (1-\varepsilon)(1-\beta) \cdot \frac{|(n_{l_{k-1}} \cdot J)_{\tilde{\Lambda}}|}{|n_{l_{k-1}} \cdot J|}$$

where $(n_{l_{k-1}} \cdot J)_{\tilde{\Lambda}}$ is the sum of $\tilde{\Lambda}^{n_{l_{k-1}}}$ and the translates of $\tilde{\Lambda}^{n_1}, \tilde{\Lambda}^{n_{l_1}}, \ldots, \tilde{\Lambda}^{n_{l_{k-2}}}$ covering $J_{n_{l_{k-1}}}$ after (k-1)-th step.

Denote $r_0 \overset{df}{=} \beta$, $r_1 \overset{df}{=} \frac{|(n_{l_1} \cdot J)_{\tilde{\Lambda}}|}{|n_{l_1} \cdot J|}$, ...

..., $r_k \overset{df}{=} \frac{|(n_{l_k} \cdot J)_{\tilde{\Lambda}}|}{|n_{l_k} \cdot J|}$

By (7) $1 \geq r_k \geq \beta + (1-\varepsilon)(1-\beta) r_{k-1}$ for $k \in N$.

It is easy to prove that the sequence (r_k) satisfying the above condition tends to $f(\varepsilon)$ while k tends to infinity, where $\lim_{\varepsilon \to 0} f(\varepsilon) = 1$. This fact together with (1) ends the proof.

Proof of Theorem 1: Suppose that $\Lambda_1, \Lambda_2 \subset G$ are cones in Z^N. Denote $\eta_1 \overset{df}{=} \liminf_n \frac{1}{\operatorname{card} \Lambda_1^n} H(\mathcal{A}_{\Lambda_1^n})$,

$\eta_2 \overset{df}{=} \limsup_n \frac{1}{\operatorname{card} \Lambda_2^n} H(\mathcal{A}_{\Lambda_2^n})$. Fix $\varepsilon > 0$.

There exist a sequence $(n_l)_{l=1}$ of positive integers such that

$$(8) \quad \frac{1}{\operatorname{card} \Lambda_1^{n_l}} H(\mathcal{A}_{\Lambda_1^{n_l}}) \leq \eta_1 + \varepsilon \qquad \text{for } l \in N.$$

If I_w is a rectangle from Lemma 1 constructed for (n_l) and ε, then for sufficiently large $n \in N$

$$(9) \quad \Lambda_2^n = \bigcup_{i=1}^{t} (I_w + \gamma_i) \cup (\Lambda_2^n)'$$

where $\gamma_i \in G$, $i=1,\ldots,t$, the sets in the above sum are pairwise disjoint and $\dfrac{\operatorname{card}(\Lambda_2^n)'}{\operatorname{card}\Lambda_2^n} < \varepsilon$.

By (8), (9) and Lemma 1 we have

$$\frac{1}{\operatorname{card}\Lambda_2^n} H(\mathcal{A}_{\Lambda_2^n}) \leq \eta_1 + \varepsilon + 2\varepsilon H(\mathcal{A}), \text{ so } \eta_2 \leq \eta_1.$$

Definition 1. (a) The topological (measure) entropy of a cover (partition) $\mathcal{A}$ with respect to an action T of the semigroup G is the number

$$h(T,\mathcal{A}) \stackrel{df}{=} \lim_{n} \frac{1}{\operatorname{card}\Lambda^n} H(\mathcal{A}_{\Lambda^n}).$$

(b) The topological (measure) entropy of an action T of the semigroup G is the number $h(T) \stackrel{df}{=} \sup_{\mathcal{A}} h(T,\mathcal{A})$.

Example. Let $H \neq Z^N$ be a semigroup in Z^N containing 0 and a cone in Z^N. Equip the set $\{0,1\}$ with the diserete topology and put $X \stackrel{df}{=} \{0,1\}^H$ with the product topology. We define an action T of H as a shift on X : $(T^h(x))_g = x_{h+g}$ for $x \in X$, $h,g \in H$. It is easy to prove that T cannot be extended to an action of a semigroup H', $H \subsetneq H' \subset Z^N$.

This example shows that the above definition is a substantial generalisation of classical one.

It can be easily proved that the above defined notions of entropy possess all the basic properties of entropy which can be found e.g. in [1] and [3].

2. The relation between the entropy of a semigroup and the entropy of its subsemigroup.

For $A \subset Z^N$, $\langle A \rangle$ will denote the additive group generated by A.

Let P be a subsemigroup of G. We know that for some $K \in N$ there exists an isomorphism $\varphi : Z^K \longrightarrow \langle P \rangle$. φ induces a linear mapping $\tilde{\varphi} : R^K \longrightarrow R^N$. Let

$V \overset{\text{df}}{=} \tilde{\varphi}\,(J_{(1,\dots,1)}) \cap Z^N$. G contains a cone in Z^N, thus there exists $h \in G$ such that $V + h \subset G$.

We set $\mathcal{A}^V \overset{\text{df}}{=} \mathcal{A}_{V+h}$ and $p \overset{\text{df}}{=} \text{card } V$.

T_p denotes an action of P on X defined by $P \ni g \longmapsto T^g$.

Theorem 2 / cf [3] 2.1 /. If $K = N$ then $h(T_p, \mathcal{A}^V) = p \cdot h(T, \mathcal{A})$.

Proof : I. $h(T_p, \mathcal{A}^V) \geqslant p \cdot h(T, \mathcal{A})$.

By assumption $\varphi^{-1}(P)$ generates Z^N, thus there is a cone Λ_p in Z^N, $\varphi(\Lambda_p) \subset P$.

Fix $\varepsilon > 0$. We set $\eta \overset{\text{df}}{=} h(T, \mathcal{A})$, $\eta_p \overset{\text{df}}{=} h(T_p, \mathcal{A}^V)$.

For some $n_o \in N$ we have

$$(10) \qquad \frac{1}{\text{card}\,\Lambda_p^n} H(\mathcal{A}_{\varphi(\Lambda_p^n)}) \leqslant \eta_p + \varepsilon \quad \text{for} \quad n \geqslant n_o$$

Let I_w be a rectangle in Z^N from Lemma 1, constructed for the sequence $(\Lambda^n)_{n=n_0}^{\infty}$ and ε. For some $k \in G$, $\varphi(I_w)+V+k \subset G$, because G contains a cone in Z^N. For sufficiently large n we can find $s \in N$, $\lambda_j \in G, j=1,\dots,s$ such that

$$(11) \qquad \Lambda^n = \bigcup_{j=1}^{s} (\varphi(I_w) + V + h + k + \lambda_j) \cup (\Lambda^n)',$$

where the sets appearing in this sum are pairwise disjoint and $\dfrac{\operatorname{card}(\Lambda^n)'}{\operatorname{card}\Lambda^n} < \varepsilon$.

From (12), (13) and Lemma 1 we get

$$\frac{1}{\operatorname{card}\Lambda^n} H(\mathcal{A}_{\Lambda^n}) \leqslant \frac{1}{\operatorname{card}\Lambda^n} \sum_{j=1}^{s} H(\mathcal{A}_{\varphi(I_w)+V+h+k+\lambda_j}) + \varepsilon\cdot H(\mathcal{A}) \leqslant \varepsilon\cdot H(\mathcal{A}) + \frac{1}{\operatorname{card}(\varphi(I_w)+V)}\cdot H(\mathcal{A}^V_{\varphi(I_w)+k})$$ but

$\operatorname{card}(\varphi(I_w)+V) = p \operatorname{card} I_w$ and in virtue of (12) and Lemma 1, and $\dfrac{1}{\operatorname{card} I_w} H(\mathcal{A}^V_{\varphi(I_w)+k}) \leqslant \eta_p + \varepsilon + \varepsilon\cdot H(\mathcal{A}^V)$.

Hence $\dfrac{1}{\operatorname{card}\Lambda^n} H(\mathcal{A}_{\Lambda^n}) \leqslant \dfrac{1}{p}\cdot\eta_p + \varepsilon\cdot H(\mathcal{A}) + \dfrac{1}{p} + \dfrac{1}{p}H(\mathcal{A}^V))$

which implies $p\cdot\eta \leqslant \eta_p$.

II. $p\cdot h(T, \mathcal{A}) \geqslant h(T_p, \mathcal{A}^V)$.

Fix $\varepsilon > 0$. There exists $n_0 \in \mathbb{N}$ such that

$$(12)\qquad \frac{1}{\operatorname{card}\Lambda^n} H(\mathcal{A}_{\Lambda^n}) \leqslant \eta + \varepsilon \qquad \text{for} \qquad n \geqslant n_0 .$$

Let I_w be a rectangle in Z^N from Lemma 1, constructed for $(\Lambda^n)_{n=n_0}^{\infty}$ and ε. There exists $t \in \mathbb{N}$, $z_0, z_i \in Z^N$, $i = 1,\ldots,t$, such that

$$(13)\qquad \varphi(I_{z_0}) + V = \bigcup_{i=1}^{t} (I_w + z_i) \cup (\varphi(I_{z_0}) + V)',$$

the sets appearing in this sum are pairwise disjoint and

$$\frac{\operatorname{card}(\varphi(I_{z_0})+V)'}{\operatorname{card}(\varphi(I_{z_0})+V} < \varepsilon .$$

For $n \in \mathbb{N}$ sufficiently large we can find $l \in N$, $\lambda_i \in \Lambda_p^n$, $i = 1,\ldots,l$, such that

$$(14)\qquad \Lambda_p^n = \bigcup_{i=1}^{l} (I_{z_0} + \lambda_i) \cup (\Lambda_p^n)',$$

all the sets in the above sum are pairwise disjoint and

$$\frac{\operatorname{card}(\Lambda_p^n)}{\operatorname{card}\ \Lambda_p^n} < \varepsilon .$$

By (14), (15) and (16) we have $\frac{1}{\operatorname{card}\varphi(\Lambda_p^n)} H(\mathcal{A}^V_{\varphi(\Lambda_p^n)}) \leqslant$

$$\leqslant \varepsilon \cdot H(\mathcal{A}^V) + \frac{1}{\operatorname{card}\ \Lambda_p^n} \sum_{i=1}^{l} H(\mathcal{A}_{\varphi(I_{z_0})+V+h+\varphi(\lambda_i)}) \leqslant \varepsilon \cdot H(\mathcal{A}^V)$$

$$+ \frac{1}{\operatorname{card}\ \Lambda_p^n} \sum_{i=1}^{l}\Big(\sum_{j=1}^{t} H(\mathcal{A}_{I_w+z_j+h+\varphi(\lambda_i)}) + H(\mathcal{A}_{(\varphi(I_{z_0})+V)+h+\varphi(\lambda_i)})\Big)$$

$\leqslant p\cdot\eta + \varepsilon\big(p\cdot H(\mathcal{A}) + p + H(\mathcal{A}^V)\big)$ which gives the inequality $\eta_p \leqslant p\cdot\eta$.

Corollary 1 (cf [3] 2.3), If $K = N$ then $h(T_p) = p\cdot h(T)$.

Theorem 3 (cf [3] 2.5) . If $K < N$ and $h(T) > 0$ then $h(T_p) = +\infty$.

Proof: Recall that $\langle P\rangle \simeq Z^K$, $\varphi : Z^K \to \langle P\rangle$ is an isomorphism , $K < N$. We extend φ to an isomorphism of Z^N into Z^N . In the sequel this extention is denoted also by φ . Let p^* denotes the index of subsemigroup $\varphi(Z^N)$ in Z^N and $P^* \overset{df}{=} \varphi(Z^N) \cap G$. By Theorem 1, $h(T_{P^*}) = p^*\cdot h(T)$. The extension φ can be chosen in such a way that p^* is arbitrarily large. Thus it suffices to prove that $h(T_{P^*}) \leqslant h(T_P)$.

$\varphi^{-1}(P)$ contains a cone Λ_p in Z^K. $\varphi^{-1}(P^*)$ contains a cone Λ_* in Z^N . Fix $\varepsilon > 0$. There exists $n_0 \in N$ such that for $n \geqslant n_0$

$$(15) \qquad \frac{1}{\operatorname{card} \Lambda_p^n} H(\mathcal{A}_{\varphi(\Lambda_p^n)}) \leq h(T_p, \mathcal{A}) + \varepsilon .$$

Let I_W be a rectangle from Lemma 1, constructed for $(\Lambda_p^n)_{n=n_0}^{\infty}$ and ε. For $n \in \mathbb{N}$ sufficiently large we can cover Λ_*^n by pairwise disjoint translates of I_W so precisely, that by a standard estimation we obtain the desired inequality.

Corollary 2. /of [3] 2.6./. If $K < N$, $h(T_p) < +\infty$, then $h(T) = 0$.

Note that everything that was proved in part 2 is also valid for measure entropy (proofs without modifications).

3. Theorem of Dinaburg - Goodwyn - Goodman.

We introduce the following notations :

$\mathfrak{M}(X)$ - the space of all Borel, normalised measures on X with weak * - topology.

$\mathfrak{M}(X,T)$ - the subspace of all T-invariant measures in $\mathfrak{M}(X)$.

W - the set of all neighbourhoods of the diagonal in $X \times X$ directed by the inclusion.

Let $\delta \in W$. $\delta_C \overset{\text{df}}{=} \bigcap_{g \in C} (T^g \times T^g)^{-1} \delta$ for arbitrary $C \subset G$.

A finite subset e of X is called a/ (C, δ) - separated, if for all $x, y \in e$, $x \neq y$ we have $(x,y) \notin \delta_C$;
b/ (C, δ) - spanning, if for all $x \in X$ there exists $y \in e$ such that $(x,y) \in \delta_C$.

Let $r(C, \delta) \overset{df.}{=} \min \{ \text{card } e : e \text{ is } (C, \delta)\text{-spanning} \}$, $s(C, \delta) = \max \{ \text{card } e : e \text{ is } (C, \delta)\text{-separated} \}$. We define

$$\bar{r}_T(\Lambda, \delta) \overset{df.}{=} \limsup_n \frac{1}{\text{card } \Lambda^n} \log r(\Lambda^n, \delta) ,$$

$$\bar{s}_T(\Lambda, \delta) \overset{df.}{=} \limsup_n \frac{1}{\text{card } \Lambda^n} \log s(\Lambda^n, \delta) .$$

By an argument analogous to the one applied in [3] the following definition makes sense,

Definition 3. $h_T(\Lambda) = \lim_{\delta} \bar{s}_T(\Lambda, \delta) = \lim_{\delta} \bar{r}_T(\Lambda, \delta) = \sup_{\delta} \bar{s}_T(\Lambda, \delta) = \sup_{\delta} \bar{r}_T(\Lambda, \delta).$

Theorem 4. For all $\Lambda \subset G$ $h_T(\Lambda) = h(T)$.

The proof of this theorem is a translation of the proof [3] 4.8 to the language of the form structure W on X.

The following lemma will be used in the proof of Dinaburg-Goodwyn-Goodman theorem.

Lemma 2. Assume that $\mu \in \mathfrak{M}(X, T)$ and $\mathcal{A}$ is a μ - measurable finite partition of X. Let $p_i \in Z_+^N$ for $i \in \mathbb{N}$ and $\lim p_i = +\infty$. Chose $g_i \in G$ such that $I_p + g_i \subset G$ for $i \in \mathbb{N}$. Then

$$h_\mu(T, \mathcal{A}) = \lim_i \frac{1}{\text{card } I_{p_i}} H_\mu(\mathcal{A}_{I_p + g_i})$$

Proof : I $\limsup_i \frac{1}{\text{card } I_{p_i}} H_\mu(\mathcal{A}_{I_{p_i} + g_i}) \leq h_\mu(T, \mathcal{A}).$

There exists a sequence of positive integers (n_l) such that $\frac{1}{\text{card } \Lambda^{n_l}} H_\mu(\mathcal{A}_{\Lambda^{n_l}}) \leq h_\mu(T, \mathcal{A}) + \varepsilon$.

For i sufficiently large we cover $I_{p_i} + g_i$ by pairwise disjoint translates of a rectangle I_W from Lemma 1 , constructed for (Λ^{n_l}) and ε .
A standard estimation yealds the desired inequality.

$$\text{II.} \quad h_\mu(T, \mathcal{A}) \leq \liminf_i \frac{1}{\operatorname{card} I_{p_i}} H_\mu(\mathcal{A}_{I_{p_i}+g_i}) .$$

If $i \in \mathbb{N}$ then for sufficiently large $n \in \mathbb{N}$ we can find $k \in \mathbb{N}$, $\lambda_l \in \Lambda^n$, $l = 1,\dots,k$, such that
$\Lambda^n = \bigcup_{l=1}^{k} (I_{p_i} + \lambda_l) \cup (\Lambda^n)'$, where the sets appearing in this sum are pairwise disjoint and $\dfrac{\operatorname{card} (\Lambda^n)'}{\operatorname{card} \Lambda^n} < \varepsilon$. Since

for $l = 1,\dots,k$, $H_\mu(\mathcal{A}_{I_{p_i}+\lambda_l}) = H_\mu(\mathcal{A}_{I_{p_i}+\lambda_l+g_i}) = H_\mu(\mathcal{A}_{I_{p_i}+g_i})$, the following inequality holds :
$\dfrac{1}{\operatorname{card} \Lambda^n} H_\mu(\mathcal{A}_{\Lambda^n}) \leq \varepsilon \cdot H_\mu(\mathcal{A}) + \dfrac{1}{\operatorname{card} I_{p_i}} H_\mu(\mathcal{A}_{I_{p_i}+g_i})$. This inequality implies II .

Theorem 5. /Dinaburg-Goodwyn-Goodman/.

$$h(T) = \sup_{\mu \in \mathfrak{M}(X,T)} h_\mu(T) .$$

Proof: I . $\sup_{\mu \in \mathfrak{M}(X,T)} h_\mu(T) \leq h(T)$ /Goodwyn/.

The proof is analogous to the proof of Theorem 4.1 in [4] .

$$\text{II.} \quad h(T) \leq \sup_{\mu \in \mathfrak{M}(X,T)} h_\mu(T) \qquad \text{/cf [5] /.}$$

Fix $\sigma > 0$ and $\delta \in W$. Let for all $n \in \mathbb{N}$ e_n be a set (Λ^n, δ) - separated of maximal cardinality.

For some sequence (n_k) of positive integers there exists

$$\lim_k \frac{1}{\operatorname{card} \Lambda^{n_k}} \log \operatorname{card} e_{n_k} = h_T(\Lambda, \delta) .$$

We construct a measure $\mu \in \mathfrak{M}(X,T)$ in the way indicated in [5] : $\sigma_n(\{y\}) = \frac{1}{\operatorname{card} e_n}$ for $y \in e_n$;

$\mu_n \overset{df}{=} \frac{1}{\operatorname{card} \Lambda^n} \sum_{g \in \Lambda^n} T^{**} g\sigma_n$ /definition of T^{**} is given in [5] /. In virtue of the theorem of Alaoglu there exists a cluster point $\mu \in \mathfrak{M}(X)$ of the sequence (μ_{n_k}) . As in [5] one proves that $\mu \in \mathfrak{M}(X,T)$.

Let $\mathcal{A}$ be a finite Borel partition of X such that $a \times a \subset \delta$ for $a \in \mathcal{A}$. Then for $a \in \mathcal{A}_{\Lambda^n}$ $a \times a \subset \delta_{\Lambda^n}$ thus $\forall a \in \mathcal{A}_{\Lambda^n}$ $\operatorname{card}(e_n \cap a) \leq 1$, so

$$H_{\sigma_n}(\mathcal{A}_{\Lambda^n}) = - \sum_{y \in e_n} \sigma_n(\{y\}) \log \sigma_n(\{y\}) = \log \operatorname{card} e_n .$$

Let $(I_{p_i} + g_i)$ be a sequence from Lemma 2.

We can assume that $g_i \in Z_+^N$ for $i \in N$.

Fix $m \in \mathbb{N}$ and ε , $0 < \varepsilon < \frac{\sigma}{2\log\operatorname{card} \mathcal{A}}$. There exists $l_0 \in \mathbb{N}$ such that for $l \geqslant l_0$ $p_l - g_m - p_m \in Z_+^N$ and

$$(16) \qquad \frac{\operatorname{card} I_{p_l - g_m - p_m}}{\operatorname{card} I_{p_l}} \geqslant 1 - \varepsilon .$$

If $l \geqslant l_0$, $l \in \mathbb{N}$, then for n sufficiently large we can find $t \in \mathbb{N}$, $\lambda_i \in \Lambda^n$, $i = 1,\dots,t$, such that $\Lambda^n = \bigcup_{i=1}^{t} (I_{p_l} + \lambda_i) \cup (\Lambda^n)'$; the sets appearing in this sum are pairwise disjoint and $\dfrac{\text{card}(\Lambda^n)'}{\text{card}\ \Lambda^n} \leq \varepsilon$.

Now, let $q \in I_{p_m}$. We define

$$s(q) = \left(\left[\frac{p_l^1 - g_m^1 - q^1}{p_m^1}\right], \dots, \left[\frac{p_l^N - g_m^N - q^N}{p_m^N}\right]\right).$$

Observe that $I_{p_l} = \bigcup_{r \in I_{s(q)}} \left(I_p + g_m + q + r \cdot p_m\right) \cup \left(I_{p_l}\right)'$, where the sets appearing in this sum are pairwise disjoint and card $(I_{p_l})' \leq$ card I_{p_l} - card $I_{p_l - g_m - p_m} \leq \varepsilon \cdot$ card I_{p_l} /by (16) /. So, finally we can represent Λ^n as a sum of pairwise disjoint sets as follows $\Lambda^n = \bigcup_{i=1}^{t} \Big(\bigcup_{r \in I_{s(q)}} (I_{p_m} + \lambda_i + g_m + q + r \cdot p_m) \cup (I'_{p_l} + \lambda_i)\Big) \cup (\Lambda^n)'$. Thus, for all $q \in I_{p_m}$

$$(17_q) \quad H_{\sigma_n}(\mathcal{A}_{\Lambda^n}) \leq \text{card}(\Lambda^n)' \cdot \log \text{card}\, \mathcal{A} + \sum_{i=1}^{t} \text{card}\, I'_{p_l} \log \text{card}\, \mathcal{A} + \sum_{i=1}^{t} \sum_{r \in I_{s(q)}} H_{\sigma_n}\left(\left(T^{\lambda_i + q + r \cdot p_m}\right)^{-1} \mathcal{A}_{I_{p_m} + g_m}\right).$$

Adding the inequalities $(17q)$, $q \in I_p$, by sides we obtain

$$(18) \quad \text{card}\, I_{p_m} \cdot \log \text{card}\, e_n \leq \text{card}\, I_p \cdot \log \text{card}\, \mathcal{A} \cdot \left(\text{card}(\Lambda^n)' + t\ \text{card}\, I'_{p_l}\right) + \sum_{i=1}^{t}\Big(\sum_{q \in I_{p_m}} \sum_{r \in I_{s(q)}} H_{\sigma_n}\left(\left(T^{\lambda_i + q + r \cdot p_m}\right)^{-1} \mathcal{A}_{I_{p_m} + g_m}\right)\Big) \leq$$
$$\leq \text{card}\, I_{p_m} \cdot \log \text{card}\, \mathcal{A}\left(\text{card}\,(\Lambda^n)' + t \cdot \text{card}\, I'_{p_l}\right) + \sum_{g \in \Lambda^n} H_{\sigma_n}\left((T^g)^{-1} \mathcal{A}_{I_{p_m} + g_m}\right).$$

Dividing the inequality (18) by card I_p · card Λ^n and applying the inequalities

$$\frac{1}{\operatorname{card} \Lambda^n} \sum_{g \in \Lambda^n} H_{\sigma_n}\left((T^g)^{-1} \mathcal{A}_{I_{p_m}+g_m} \right) \leq H_\mu(\mathcal{A}_{I_{p_m}+g_m})$$

and $\dfrac{t \cdot \operatorname{card} I'_{p_l}}{\operatorname{card} \Lambda^n} \leq \dfrac{t \cdot \operatorname{card} I_{p_l} \cdot \varepsilon}{\operatorname{card} \Lambda^n} \leq \varepsilon$, we obtain

$$(19) \qquad \frac{1}{\operatorname{card} \Lambda^n} \log \operatorname{card} e_n \leq 2 \cdot \varepsilon \log \operatorname{card} \mathcal{A} +$$

$$+ \frac{1}{\operatorname{card} I_{p_m}} \cdot H_{\mu_n}(\mathcal{A}_{I_{p_m}+g_m}).$$

Inequality (19) is true for all $n \in \mathbb{N}$ sufficiently large and $\mathcal{A}$ can be chosen in such a way that the boundaries of the elements of $\mathcal{A}$ have measure μ zero, hence taking the limit with respect to n/or with respect to a subsequence (n_k) if necessary / we get $h_T(\Lambda, \delta) \leq 2 \cdot \varepsilon \log \operatorname{card} \mathcal{A}$ +

$$+ \frac{1}{\operatorname{card} I_{p_m}} H_\mu(\mathcal{A}_{I_{p_m}+g_m}) \leq \sigma + \frac{1}{\operatorname{card} I_{p_m}} \cdot H_\mu(\mathcal{A}_{I_{p_m}+g_m})$$

for all $\delta \in W$ and $m \in \mathbb{N}$. Passing to the limit with δ and m , owing to the arbitraryness of σ , we obtain finally $h(T) \leq h_\mu(T)$.

Corollary 3. If T_Ω denotes on action of G on the set of nonwandering points Ω defined by $T^g_\Omega(x) = T^g(x)$ for $x \in \Omega$, then $h(T_\Omega) = h(T)$.

Bibliography

[1] Adler R.L., Konheim A.G., Mc Andrew M.H. : Topological entropy. Trans. Amer. Math. Soc., 114 /1965/, 309-319.

[2] Conze J.P. : Entropie d'un group abélien de transformations. Zeitschr . Wahr. verw. Geb., 25/1973/, 11-30.

[3] Eberlein E. : On topological entropy of semigroups of commuting transformations. Asterisque, 40 /1976/, 17-62.

[4] Misiurewicz M. : Topological conditional entropy. Stud. Math., 55/1976/, 177-200.

[5] Misiurewicz M. : A short proof of the variational principle for a Z_+^N action on a compact space. Bull. Acad. Polon. Sci., Série des sciences math., astr. et phys., 24/1976/, 1069-1075.

Krystyna Małgorzata Ziemian

Wydz.Mat. i Mech.
Instytut Matematyki
Uniwersytet Warszawski
PKiN
00-901 Warszawa
Poland