On topological and measure entropies of semigroups

by

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The present paper contains a generalization of the theory of topological and measure entropies to the case of an action of an arbitrary subsemigroup of $\mathbb{Z}^n$. Some ideas were suggested to the author by M. Misiurewicz.

1. Definitions of the topological and measure entropies.

A subset $\bar{\Lambda} \subset \mathbb{R}^n$ will be called a cone in $\mathbb{R}^n$ if $\forall x \in \Lambda \quad \forall t > 0 \quad t \cdot x \in \Lambda$ and $\bar{\Lambda} \cap B(0,1)$ is of positive Jordan measure, where $B(0,1)$ is the unit-ball in $\mathbb{R}^n$.

The set $\Lambda$ of the form $\Lambda = \bar{\Lambda} \cap \mathbb{Z}^n$, where $\bar{\Lambda}$ is a cone in $\mathbb{R}^n$, will be called a cone in $\mathbb{Z}^n$.

If $G$ is a semigroup in $\mathbb{Z}^n$ then $G$ generates a subgroup of $\mathbb{Z}^n$ isomorphic to $\mathbb{Z}^{n'}$ for some $N' \in \mathbb{N}$ as usually denotes the set of positive integers / $\mathbb{N}$.

Thus without loss of generality, we can restrict ourselves to the study of these semigroups in $\mathbb{Z}^n$ which generate $\mathbb{Z}^n$. It is easy to prove the following.

Proposition 1. A semigroup $G \subset \mathbb{Z}^n$ generates $\mathbb{Z}^n$ iff $G$ contains a cone in $\mathbb{Z}^n$.

Commencing from now $G$ is a fixed semigroup in $\mathbb{Z}^n$ containing a cone $\bar{\Lambda}$ in $\mathbb{Z}^n$. 

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We introduce the following notations:

For \( r_1 = (r_1^1, \ldots, r_1^N), \ r_2 = (r_2^1, \ldots, r_2^N) \in \mathbb{R}^N \)
the relation \( r_1 < r_2 / r_1 < r_2 / \) means that \( r_1^i < r_2^i / \) for \( i = 1, \ldots, N. \)

\( \mathbb{R}_+^N = \{ \mathbb{R}^N : x \geq 0 \}. \)

For \( s \in \mathbb{R}_+^N \) we define \( J_s \) as \( \{ x \in \mathbb{R}_+^N : x < s \}. \)

\( z_+^N = \{ z \in \mathbb{Z}_+^N : z \geq 0 \}. \)

For \( w \in \mathbb{Z}_+^N \) we define \( I_w \) as \( \{ x \in \mathbb{Z}_+^N : x < w \}. \)

For \( \mathbb{N} \) we set \( \mathbb{N} = (\mathbb{Z}^+ \cup \{ 0 \}). \)

\( X \) is a non-empty, compact Hausdorff (probability) space.

\( T \) is an action of \( G \) in \( X \) (it is not assumed that \( T^0 = \text{id}_X \)).

\( \mathcal{A} \) denotes an open cover (a finite measurable partition) of \( X \).

For every subset \( B \) of \( G \) we set \( H_B = \bigvee_{s \in B} (T_s)^{-1} \mathcal{A} \).

\( H(\mathcal{A}, B) \) stands for the topological (measure) entropy of the cover (partition) \( \mathcal{A}_B \).

For \( n \in \mathbb{N} \) we set \( \Lambda_n \equiv \bigcap B(0, n) \), where \( B(0, n) \) is the ball with center 0 and radius \( n. \)

**Theorem 1.** \( \lim_{n} \frac{1}{\text{card} \lambda_n} H(\mathcal{A}, \lambda_n) \) exists and does not depend on the choice of \( \lambda = G. \)

**Lemma 1.** Let \( \delta \) be an arbitrary positive number. If \( \lambda \) is a cone in \( \mathbb{Z}_+^N \) and \( \{n_1\} \) is a sequence of positive
integers such that \( \lim_{n_1} \frac{1}{n_1} \) then there exist 

(i) \( n_1 = +\infty \) then there exist 

(1) positive integers \( l_1, \ldots, l_k, t_1, \ldots, t_k \)

(ii) \( w \in \mathbb{Z}_+^N \)

(iii) \( z_{i,j} \in I_w; \quad j=1, \ldots, t_i; \quad i=1, \ldots, k \)
such that 

\[
I_w = \bigcup_{j=1}^{t_k} (\Lambda^{n_{t,k}} + z_{1,t_1}) \cup \ldots \cup 
\]

\[
\ldots \cup \bigcup_{j=1}^{t_k} (\Lambda^{n_{t,k}} + z_{k,t_k}) \cup I_w' \quad \text{where all the sets in the}
\]

above sum are pairwise disjoint and \( \frac{\text{card } I_w'}{\text{card } I_w} < \delta \).

**Proof:** By assumption, \( \Lambda = \Lambda \cap \mathbb{Z}^N, \Lambda^{n_1} = \Lambda \cap B(0, n_1) = \Lambda \cap B(0, n_1) \cap \mathbb{Z}^N \) for \( 1 \leq n_1 \). Let \( \Lambda^{n_1} = \Lambda \cap B(0, n_1) \subset \mathbb{R}^N \).

Fix \( \varepsilon > 0 \). If \( | \cdot | \) denotes the Jordan measure on \( \mathbb{R}^N \) then

\[
(1) \quad \lim_{n_1} \frac{\text{card}(\Lambda^{n_1} \cap \mathbb{Z}^N)}{|\Lambda^{n_1}|} = 1,
\]

by definition of Jordan measure.

Let \( J \subset \mathbb{R} \) be a rectangle with vertices belonging to \( \mathbb{Z}^N \) such that \( \Lambda \subset J \). Denote

\[
(2) \quad \beta = \frac{|\Lambda^{n_1}|}{|J|}
\]

\( I_w \) can be constructed inductively. The idea is the following. We chose \( l_1 \in \mathbb{N} \) such that \( n_1 \cdot J \setminus \Lambda^{n_1} \) can be covered by pairwise disjoint translates of \( n_1 \cdot J \) by vectors with integer coordinates so precisely that if we denote the covered part of \( n_1 \cdot J \) by \( (n_1 \cdot J)_C \) then

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Then, \( n_1 \cdot J \) contains both \( \Lambda^{n_1} \) and the translates of \( \Lambda^{n_1} \). Now, if \( (n_1 \cdot J)^{\sim} \) denotes the sum of and these translates then, in virtue of (2) and (3),

\[
\frac{|(n_1 \cdot J)^{\sim}|}{|n_1 \cdot J|} > \beta + (1 - \varepsilon)(1 - \beta) \cdot \beta.
\]

Now, we chose \( l_2 \in \mathbb{N} \) such that \( n_1 \cdot J \setminus \Lambda^{n_1} \) can be covered pairwise disjoint translates of \( n_1 \cdot J \) by vectors with integer coordinates, so precisely that if we denote the covered part of \( n_1 \cdot J \) by \( (n_1 \cdot J)_c \) then

\[
\frac{|(n_1 \cdot J)_c|}{|n_1 \cdot J \setminus \Lambda^{n_1}|} > 1 - \varepsilon.
\]

Then, \( n_1 \cdot J \) contains both \( \Lambda^{n_1} \) and the translates of \( \Lambda^{n_1} \) and \( \Lambda^{n_2} \). Now, if \( (n_2 \cdot J)^{\sim} \) denotes the sum of \( \Lambda^{n_2} \) and these translates then by (2), (4) and (5) we have

\[
\frac{|(n_2 \cdot J)^{\sim}|}{|n_2 \cdot J \setminus \Lambda^{n_2}|} > \beta + (1 - \varepsilon)(1 - \beta) \cdot \frac{|n_1 \cdot J^{\sim}|}{|n_1 \cdot J|}.
\]

Continuing this procedure, after the \( k \)-th step we have \( J^{n_k} \) which contains both \( \Lambda^{n_k} \) and the translates of \( \Lambda^{n_k} \), \( \Lambda^{n_{k-1}} \), \ldots, \( \Lambda^{n_1} \) by vectors with integer coordinates,
and if \((n_k \cdot J)\Lambda\) denotes the sum of \(\Lambda^{n_k}\) and these translates then
\[
\frac{|(n_k \cdot J)\Lambda|}{|n_k \cdot J|} > \beta + (1 - \epsilon)(1 - \beta) \cdot \frac{|(n_k \cdot J)\Lambda|}{|n_k \cdot J|}
\]

where \((n_k \cdot J)\Lambda\) is the sum of \(\Lambda^{n_k}\) and the translates of \(\Lambda^n, \Lambda^{n_1}, ..., \Lambda^{n_{k-1}}\) covering \(J_n\) after \((k-1)\)-th step.

Denote \(r_0 \overset{df}{=} \beta\), \(r_1 \overset{df}{=} \frac{|(n_1 \cdot J)\Lambda|}{|n_1 \cdot J|}\), ..., \(r_k \overset{df}{=} \frac{|(n_k \cdot J)\Lambda|}{|n_k \cdot J|}\)

By (7) \(1 > r_k > \beta + (1 - \epsilon)(1 - \beta)r_{k-1}\) for \(k \in \mathbb{N}\).

It is easy to prove that the sequence \((r_k)\) satisfying the above condition tends to \(f(\epsilon)\) while \(k\) tends to infinity, where \(\lim_{\epsilon \to 0} f(\epsilon) = 1\). This fact together with (1) ends the proof.

Proof of Theorem 1: Suppose that \(\Lambda_1, \Lambda_2 \subset G\) are cones in \(\mathbb{Z}^N\). Denote \(\eta_1 \overset{df}{=} \lim \inf \frac{1}{\text{card } \Lambda_1^n} H(\mathcal{A}_{\Lambda_1^n})\), \(\eta_2 \overset{df}{=} \lim \sup \frac{1}{\text{card } \Lambda_2^n} H(\mathcal{A}_{\Lambda_2^n})\). Fix \(\epsilon > 0\).

There exist a sequence \((n_1)\) of positive integers such that
\[
\frac{1}{\text{card } \Lambda_1^{n_1}} H(\mathcal{A}_{\Lambda_1^{n_1}}) \leq \eta_1 + \epsilon \quad \text{for } l \in \mathbb{N}.
\]

If \(I_w\) is a rectangle from Lemma 1 constructed for \((n_1)\) and \(\epsilon\), then for sufficiently large \(n \in \mathbb{N}\)
(9) \[ \Lambda_2^n = \bigcup_{i=1}^{t} (I_i + \gamma) \cup (\Lambda_2^n)'
\]

where \( \gamma \in G, \ i=1, \ldots, t, \) the sets in the above sum are pairwise disjoint and \( \frac{\text{card}(\Lambda_2^n)}{\text{card} \Lambda_2^n} < \varepsilon \).

By (8), (9) and Lemma 1 we have
\[ \frac{1}{\text{card} \Lambda_2^n} H(\Lambda_2^n) \leq \eta_1 + \varepsilon + 2 \varepsilon H(\mathcal{A}), \text{ so } \eta_2 \leq \eta_1. \]

Definition 1. (a) The topological (measure) entropy of a cover (partition) \( \mathcal{A} \) with respect to an action \( T \) of the semigroup \( G \) is the number
\[ h(T, \mathcal{A}) \overset{df}{=} \lim_{n \to \infty} \frac{1}{\text{card} \Lambda_2^n} H(\Lambda_2^n). \]

(b) The topological (measure) entropy of an action \( T \) of the semigroup \( G \) is the number \( h(T) \overset{df}{=} \sup_{\mathcal{A}} h(T, \mathcal{A}). \)

Example. Let \( H \not\cong \mathbb{Z}^N \) be a semigroup in \( \mathbb{Z}^N \) containing 0 and a cone in \( \mathbb{Z}^N \). Equip the set \( \{0,1\} \) with the discrete topology and put \( X \overset{df}{=} \{0,1\}^H \) with the product topology. We define an action \( T \) of \( H \) as a shift on \( X : (T^h(x))_g = x_{h+g} \) for \( x \in X, h, g \in H \). It is easy to prove that \( T \) cannot be extended to an action of a semigroup \( H', H \not\subseteq H' \subseteq \mathbb{Z}^N \).

This example shows that the above definition is a substantial generalisation of classical one.

It can be easily proved that the above defined notions of entropy possess all the basic properties of entropy which can be found e.g. in [7] and [3].
2. The relation between the entropy of a semigroup and the entropy of its subsemigroup.

For \( A \subseteq \mathbb{Z}^N \), \( \langle A \rangle \) will denote the additive group generated by \( A \).

Let \( P \) be a subsemigroup of \( G \). We know that for some \( K \in \mathbb{N} \) there exists an isomorphism \( \varphi : \mathbb{Z}^K \rightarrow \langle P \rangle \). \( \varphi \) induces a linear mapping \( \overline{\varphi} : \mathbb{R}^K \rightarrow \mathbb{R}^N \). Let

\[ V \overset{df}{=} \varphi(\langle \mathbf{1}, \ldots, \mathbf{1} \rangle) \cap \mathbb{Z}^N. \]

\( G \) contains a cone in \( \mathbb{Z}^N \), thus there exists \( h \in G \) such that \( V + h \subseteq G \).

We set \( \mathcal{A}^V = \mathcal{A} \upharpoonright V + h \) and \( p \overset{df}{=} \text{card } V \).

\( T_p \) denotes an action of \( P \) on \( X \) defined by \( P \ni g \mapsto T_g \).

**Theorem 2** / cf [3] 2.1/. If \( K = \mathbb{N} \) then

\[ h(T_p, \mathcal{A}^V) = p \cdot h(T, \mathcal{A}). \]

**Proof:**

1. \( h(T_p, \mathcal{A}^V) \geq p \cdot h(T, \mathcal{A}). \)

By assumption \( \varphi^{-1}(P) \) generates \( \mathbb{Z}^N \), thus there is a cone \( \Lambda_p \) in \( \mathbb{Z}^N \), \( \varphi(\Lambda_p) \subseteq P \).

Fix \( \varepsilon > 0 \). We set \( \eta \overset{df}{=} h(T, \mathcal{A}) \), \( \eta_p \overset{df}{=} h(T_p, \mathcal{A}^V) \).

For some \( n_0 \in \mathbb{N} \) we have

\[
\frac{1}{\text{card } \Lambda_p^n} H(\mathcal{A} \varphi(\Lambda_p^n)) \leq \eta_p + \varepsilon \quad \text{for } n \geq n_0.
\]

Let \( I_w \) be a rectangle in \( \mathbb{Z}^N \) from Lemma 1, constructed for the sequence \( (\Lambda_p^n)_{n \geq n_0} \), and \( \varepsilon \). For some \( k \in G \), \( \varphi(I_w) + V + k \subseteq G \), because \( G \) contains a cone in \( \mathbb{Z}^N \). For sufficiently large \( n \) we can find \( s \in \mathbb{N} \), \( \lambda_j \in G, j = 1, \ldots, s \) such that

\[
\Lambda_p^n = \bigcup_{j=1}^s (\varphi(I_w) + V + h + k + \lambda_j) \cup (\Lambda_p^n)^c.
\]
where the sets appearing in this sum are pairwise disjoint and
\[
\frac{\text{card}(\Lambda^n)}{\text{card} \Lambda^n} < \varepsilon.
\]

From (12), (13) and Lemma 1 we get
\[
\frac{1}{\text{card} \Lambda^n} H(\mathcal{A}_n) \leq \frac{1}{\text{card} \Lambda^n} \sum_{j=1}^{s} H(\mathcal{A}_j \phi(I_w) + V + h + k + \chi_j) + \varepsilon \cdot H(\mathcal{A}_r) + \frac{1}{\text{card}(\phi(I_w) + V)} H(\mathcal{A}_r \phi(I_w) + k)
\]
but
\[
\text{card} (\phi(I_w) + V) = p \cdot \text{card} I_w
\]
and in virtue of (12) and Lemma 1, and
\[
\frac{1}{\text{card} \Lambda^n} H(\mathcal{A}_n) \leq \frac{1}{p} \cdot \eta_p + \varepsilon \cdot H(\mathcal{A}_r) + \frac{1}{p} + \frac{1}{p} H(\mathcal{A}_r)
\]
which implies
\[
p \cdot \eta_p \leq \eta_p.
\]

II. \( p, h (T, \mathcal{A}) \geq h (T_p, \mathcal{A}) \).

Fix \( \varepsilon > 0 \). There exists \( n_0 \in \mathbb{N} \) such that
\[
(12) \quad \frac{1}{\text{card} \Lambda^n} H(\mathcal{A}_n) \leq \eta + \varepsilon \quad \text{for} \quad n \geq n_0.
\]

Let \( I_w \) be a rectangle in \( \mathbb{Z}^N \) from Lemma 1, constructed for \( (\Lambda^n)_{n=n_0}^\infty \) and \( \varepsilon \). There exists \( t \in \mathbb{N} \), \( z_0, z_1 \in \mathbb{Z}^N \), \( i = 1, \ldots, t \), such that
\[
(13) \quad \phi(I_{z_0}) + V = \bigcup_{i=1}^{t} (I_w + z_i) \cup (\phi(I_{z_0}) + V),
\]
the sets appearing in this sum are pairwise disjoint and
\[
\frac{\text{card}(\phi(I_{z_0}) + V)}{\text{card}(\phi(I_{z_0}) + V)} < \varepsilon.
\]

For \( n \in \mathbb{N} \) sufficiently large we can find \( l \in \mathbb{N} \), \( \chi_i \in \Lambda^n_p \), \( i = 1, \ldots, l \), such that
\[
(14) \quad \Lambda^n_p = \bigcup_{i=1}^{l} (I_{z_0} + \chi_i) \cup (\Lambda^n_p)',
\]

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all the sets in the above sum are pairwise disjoint and
\[
\frac{1}{\text{card} \Lambda_p^n} < \varepsilon.
\]

By (14), (15) and (16) we have
\[
\frac{1}{\text{card} \varphi(\Lambda_p^n)} H(\Lambda\times \varphi(\Lambda_p^n)) \leq
\]
\[
\leq \varepsilon \cdot H(\mathcal{A}^V) + \frac{1}{\text{card} \Lambda_p^n} \sum_{i=m}^t H(\mathcal{A}^\varphi(\mathcal{I}_{z_0}^+ \cup V) + \varphi(\mathcal{A})) \leq \varepsilon \cdot H(\mathcal{A}^V)
\]
\[
+ \frac{1}{\text{card} \Lambda_p^n} \sum_{i=m}^t \left( \sum_{j=m}^t H(\mathcal{A}^\varphi(\mathcal{I}_{z_0}^+ z_j \cup \varphi(\mathcal{A})) + H(\mathcal{A}^\varphi(\mathcal{I}_{z_0}^+) \cup \mathcal{V}) + \varphi(\mathcal{A})) \right)
\]
\[
\leq p \cdot \eta + \varepsilon (p \cdot H(\mathcal{A}) + p + H(\mathcal{A}^V)) \quad \text{which gives the inequality}
\]
\[
\eta_p \leq p \cdot \eta.
\]

Corollary 1 (cf. [3] 2.3). If $K = N$ then
\[
\eta(T_p) = p \cdot \eta(T).
\]

Theorem 3 (cf. [3] 2.5). If $K < N$ and $h(T) > 0$ then
\[
h(T_p) = + \infty.
\]

Proof: Recall that $\langle \mathcal{P} \rangle \sim \mathbb{Z}^K$, $\varphi: \mathbb{Z}^K \rightarrow \langle \mathcal{P} \rangle$
is an isomorphism, $K \leq N$. We extend $\varphi$ to an isomorphism of
$\mathbb{Z}^N$ into $\mathbb{Z}^N$. In the sequel this extension is denoted
also by $\varphi$. Let $p^\mathbb{Z}$ denotes the index of subsemigroup
$\varphi(\mathbb{Z}^N)$ in $\mathbb{Z}^N$ and $p^\mathbb{Z} \stackrel{df}{=} \varphi(\mathbb{Z}^N) \cap G$. By Theo-
rem 1, $h(T_p^\mathbb{Z}) = p^\mathbb{Z} \cdot h(T)$. The extension $\varphi$ can be cho-
sen in such a way that $p^\mathbb{Z}$ is arbitrarily large. Thus it suf-
fices to prove that $h(T_p^\mathbb{Z}) \leq h(T_p)$.

$\varphi^{-1}(\mathcal{P})$ contains a cone $\Lambda_p$ in $\mathbb{Z}^K$. $\varphi^{-1}(p^\mathbb{Z})$
contains a cone $\Lambda_+$ in $\mathbb{Z}^N$. Fix $\varepsilon > 0$. There exists
$n_0 \in \mathbb{N}$ such that for $n \geq n_0$
(15) \( \frac{1}{\text{card } \Lambda_p^n} H(\mathcal{Q}(\Lambda_p^n)) \leq h(T_p, \mathcal{Y}) + \varepsilon \).

Let \( \mathcal{W} \) be a rectangle from Lemma 1, constructed for 
\((\Lambda_p^n)_{n=1}^{\infty} \) and \( \varepsilon \). For \( n \in \mathbb{N} \) sufficiently large
we can cover \( \Lambda_p^n \) by pairwise disjoint translates of \( \mathcal{W} \)
so precisely, that by a standard estimation we obtain the de-
sired inequality.

Corollary 2. /of [3] 2.6./. If \( K < N \), \( h(T_p) < +\infty \),
then \( h(T) = 0 \).

Note that everything that was proved in part 2 is also
valid for measure entropy (proofs without modifications).


We introduce the following notations:
\( \mathcal{M}(X) \) - the space of all Borel, normalised measures on \( X 
\) with weak \( \ast \) - topology.
\( \mathcal{M}(X,T) \) - the subspace of all \( T \)-invariant measures
in \( \mathcal{M}(X) \).

\( W \) - the set of all neighbourhoods of the diagonal in \( X \times X 
directed by the inclusion.

Let \( \mathcal{C} \in W \). \( \mathcal{C} \triangleq \cap \{ (T^g \times T^{g})^{-1} \} \) for arbitrary
\( g \in G \).

A finite subset \( e \) of \( X \) is called a/ \( (\mathcal{C}, \mathcal{S}) \) - sepa-
rated, if for all \( x, y \in e \), \( x \neq y \) we have \( (x,y) \notin \mathcal{S}_C \); b/ \( (\mathcal{C}, \mathcal{S}) \) - spanning, if for all \( x \in X \) there exists
\( y \in e \) such that \( (x,y) \in \mathcal{S}_C \).
Let $\tau(C,\mathcal{E}) \overset{df.}{=} \min \{ |\text{card} e : e \text{ is } (C,\mathcal{E})\text{-spanning} \}$. We define

$$\overline{\tau}_T(A,\mathcal{E}) \overset{df.}{=} \limsup_n \frac{1}{\text{card } A^n} \log \tau(A^n,\mathcal{E}),$$

$$\underline{\tau}_T(A,\mathcal{E}) \overset{df.}{=} \limsup_n \frac{1}{\text{card } A^n} \log \tau(A^n,\mathcal{E}).$$

By an argument analogous to the one applied in [3] the following definition makes sense,

Definition 3. $h_T(A) = \lim \overline{\tau}_T(A,\mathcal{E}) = \lim \underline{\tau}_T(A,\mathcal{E}) = \sup_{\mathcal{E}} \overline{s}_T(A,\mathcal{E}) = \sup_{\mathcal{E}} \underline{s}_T(A,\mathcal{E}).$

Theorem 4. For all $A \subset G$ we have $h_T(A) = h(T).$

The proof of this theorem is a translation of the proof [3] 4.8 to the language of the form structure $\mathcal{W}$ on $X$.

The following lemma will be used in the proof of Dinaburg-Goodwyn-Goodman theorem.

Lemma 2. Assume that $\mu \in \mathcal{M}(X,T)$ and $\mathcal{J}$ is a $\mu$-measurable finite partition of $X$. Let $p_i \in \mathbb{Z}_+^N$ for $i \in \mathbb{N}$ and $\lim p_i = +\infty$. Chose $g_i \in G$ such that $I_p + g_i \subset G$ for $i \in \mathbb{N}$. Then

$$h_\mu(T,\mathcal{J}) = \lim_i \frac{1}{\text{card } I_p} H_\mu(I_p,\mathcal{J} + g_i).$$

Proof: $I \limsup_i \frac{1}{\text{card } I_p} H_\mu(I_p,\mathcal{J} + g_i) \leq h_\mu(T,\mathcal{J}).$

There exists a sequence of positive integers $(n_i)$ such that $\frac{1}{\text{card } A^n} H_\mu(A^n) \leq h_\mu(T,\mathcal{J}) + \epsilon.$
For $i$ sufficiently large we cover $I_{p_i} + g_i$ by pairwise disjoint translates of a rectangle $I_w$ from Lemma 1, constructed for $(\Lambda^n)$ and $\varepsilon$.

A standard estimation yealds the desired inequality.

$$\limsup_{i} \frac{1}{\text{card } I_{p_i}} H_{\mu}(\bigcup_{i} (I_{p_i} + g_i))$$

If $i \in \mathbb{N}$ then for sufficiently large $n \in \mathbb{N}$ we can find $k \in \mathbb{N}, \lambda_i \in \Lambda^n, l = 1, \ldots, k$, such that

$$(\bigcup_{i=1}^{k} (I_{p_i} + \lambda_i) \cup (\Lambda^n)),$$

where the sets appearing in this sum are pairwise disjoint and

$$\frac{1}{\text{card } \Lambda^n} < \varepsilon.$$

Since $\mu(A_{I_{p_i} + \lambda_i}) = \mu(A_{I_{p_i} + \lambda_i + g_i}) = \mu(A_{I_{p_i} + \lambda_i + g_i})$, the following inequality holds:

$$\frac{1}{\text{card } \Lambda^n} H_{\mu}(\bigcup_{i} A_{I_{p_i} + \lambda_i}) \leq \varepsilon \cdot H_{\mu}(\bigcup_{i} A_{I_{p_i} + \lambda_i}).$$

This inequality implies II.

Theorem 5. /Dinaburg-Goodwyn-Goodman/.

$$h(T) = \sup_{\mu \in M} h_{\mu}(T).$$

Proof: I. $\sup_{\mu \in M} h_{\mu}(T) \leq h(T) /$Goodwyn/.

The proof is analogous to the proof of Theorem 4.1 in [4].

II. $h(T) \leq \sup_{\mu \in M} h_{\mu}(T) /$cf [5] /.

Fix $\varepsilon > 0$ and $\mathcal{C} \in W$. Let for all $n \in \mathbb{N}$ $e_n$ be a set $(\Lambda^n, \mathcal{C})$ - separated of maximal cardinality.
For some sequence \( (n_k) \) of positive integers there exists
\[
\lim_{k} \frac{1}{\text{card } \Lambda^{n_k}} \log \text{card } e_{n_k} = h_T(\Lambda, \mathcal{E}).
\]

We construct a measure \( \mu \in \mathcal{M}(X, \mathcal{T}) \) in the way indicated in [5]:
\[
\mathcal{S}_n = \frac{1}{\text{card } \Lambda^n} \sum_{g \in \Lambda^n} T^{g_n} \delta_{\mathcal{S}_n}
\]
/definition of \( T^{\mathcal{S}_n} \) is given in [5]. In virtue of the theorem of Alaoglu there exists a cluster point \( \mu \in \mathcal{M}(X) \) of the sequence \( (\mu_{n_k}) \). As in [5] one proves that \( \mu \in \mathcal{M}(X, \mathcal{T}) \).

Let \( \mathcal{A} \) be a finite Borel partition of \( X \) such that \( a \times a \subset \mathcal{E} \) for \( a \in \mathcal{A} \). Then for \( a \in \mathcal{A}_n \) \( a \times a \subset \mathcal{E}_n \) thus \( \forall a \in \mathcal{A}_n \) \( \text{card } (e_n \cap a) \leq 1 \), so
\[
\mathcal{H}_{\mathcal{S}_n}(\mathcal{A}_n) = - \sum_{\gamma \in \mathcal{E}_n} \mathcal{S}_n(\{y_\gamma\}) \log \mathcal{S}_n(\{y_\gamma\}) = \log \text{card } e_n.
\]

Let \( (p_1^{m} + g_1) \) be a sequence from Lemma 2.

We can assume that \( g_i \in \mathbb{Z}_+^N \) for \( i \in \mathbb{N} \).

Fix \( m \in \mathbb{N} \) and \( \varepsilon \), \( 0 < \varepsilon < \frac{1}{2 \log \text{card } \mathcal{A}} \). There exists \( l_0 \in \mathbb{N} \) such that for \( l \geq l_0 \) \( p_1 - g_m - p_m \in \mathbb{Z}_+^N \) and
\[
\frac{\text{card } I_{p_1 - g_m - p_m}}{\text{card } I_{p_1}} \Rightarrow 1 - \varepsilon.
\]

If \( l \geq l_0 \), \( l \in \mathbb{N} \), then for \( n \) sufficiently large we can find \( t \in \mathbb{N} \), \( \Lambda_t \in \Lambda^n \), \( i = 1, \ldots, t \), such that
\[
\Lambda^n = \bigcup_{i=1}^{t} (I_{p_1} + \Lambda_1) \cup (\Lambda^n)^{1};
\]
the sets appearing
in this sum are pairwise disjoint and \( \frac{\text{card}(A^n)}{\text{card}(A^n)} \leq \varepsilon \).

Now, let \( q \in I_{P_m} \). We define

\[
s(q) = \left( \left[ \frac{p_1 - g_m - q_1}{p_m} \right], \ldots, \left[ \frac{p_N - q_m - q_N}{p_m} \right] \right).
\]

Observe that \( I_{P_l} = \bigcup_{r \in I_{X(q)}} (I_{P_l} + g_m + q + r \cdot p_m) \cup (I_{P_l})' \),
where the sets appearing in this sum are pairwise disjoint and

\[
\text{card} (I_{P_l}') \leq \text{card} I_{P_l} - \text{card} I_{P_l} + g_m - p_m \leq \varepsilon \cdot \text{card} I_{P_l}
\]

(by (16)). So, finally we can represent \( A^n \) as a sum of pairwise disjoint sets as follows

\[
A^n = \bigcup_{i \in A} \left( \bigcup_{r \in I_{X(q)}} (I_{P_l} + \gamma_i + g_m + q + r \cdot p_m) \cup (I_{P_l} + \gamma_i) \cup (A^n) \right).
\]

Thus, for all \( q \in I_{P_m} \)

\[
(17q) \quad H_{\Sigma_n}(A^n) \leq \text{card}(A^n)' \cdot \log \text{card} A + \sum_{i \in A} \frac{\text{card} I_{P_l}}{\log \text{card}} \sum_{i \in A} \sum_{r \in I_{X(q)}} H_{\sigma_n}(T^{\gamma_i + q + r \cdot p_m} A_{I_{P_m} + g_m}) + \sum_{q \in A^n} H_{\sigma_n}(T^q)^{-1} A_{I_{P_m} + g_m}.
\]

Adding the inequalities (17q), \( q \in I_{P_m} \), by sides we obtain

\[
(18) \quad \text{card} I_{P_m} \cdot \log \text{card} e_n \leq \text{card} I_{P} \cdot \log \text{card} A' + \left( \text{card}(A^n) + t \cdot \text{card} I_{P_l} \right) + \sum_{i \in A} \sum_{r \in I_{X(q)}} H_{\sigma_n}(T^{\gamma_i + q + r \cdot p_m} A_{I_{P_m} + g_m}) \leq \text{card} I_{P_m} \cdot \log \text{card} A \left( \text{card}(A^n) + t \cdot \text{card} I_{P_l} \right) + \sum_{q \in A^n} H_{\sigma_n}(T^q)^{-1} A_{I_{P_m} + g_m}.
\]

Dividing the inequality (18) by \( \text{card} I_{P} \cdot \text{card} A^n \) and applying the inequalities
\[ \frac{1}{\text{card } \Lambda^n} \sum_{g \in \Lambda^n} H_n \left( (T^g)^{-1} A_{I_{p_m} + q_m} \right) \leq H_\mu \left( A_{I_{p_m} + q_m} \right) \]

and

\[ \frac{t \cdot \text{card } I_{p_k} \cdot \mathcal{E}}{\text{card } \Lambda^n} \leq \frac{t \cdot \text{card } I_{p_k} \cdot \mathcal{E}}{\text{card } \Lambda^n} \leq \mathcal{E}, \] we obtain

\[ (19) \quad \frac{1}{\text{card } \Lambda^n} \log \text{card } \epsilon_n \leq 2 \cdot \epsilon \log \text{card } \mathcal{A} + \]

\[ + \frac{1}{\text{card } I_{p_m}} H_\mu \left( A_{I_{p_m} + q_m} \right). \]

Inequality (19) is true for all \( n \in \mathbb{N} \) sufficiently large and \( \mathcal{A} \) can be chosen in such a way that the boundaries of the elements of \( \mathcal{A} \) have measure \( \mu \) zero, hence taking the limit with respect to \( n \) or with respect to a subsequence \( (n_k) \) if necessary / we get \( h_T(\Lambda, \mathcal{D}) \leq 2 \cdot \epsilon \log \text{card } \mathcal{A} + \)

\[ + \frac{1}{\text{card } I_{p_m}} H_\mu \left( A_{I_{p_m} + q_m} \right) \leq \mathcal{E} + \frac{1}{\text{card } I_{p_m}} H_\mu \left( A_{I_{p_m} + q_m} \right) \]

for all \( \mathcal{D} \in \mathcal{W} \) and \( m \in \mathbb{N} \). Passing to the limit with \( \mathcal{D} \) and \( m \), owing to the arbitrariness of \( \mathcal{D} \), we obtain finally \( h(T) \leq h_\mu (T) \).

**Corollary 3.** If \( T_\Omega \) denotes an action of \( G \) on the set of nonwandering points \( \Omega \) defined by \( T^g_\Omega (x) = T^g (x) \) for \( x \in \Omega \), then \( h(T_\Omega) = h(T) \).
Bibliography


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