KARL SIGMUND

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AFFINE TRANSFORMATIONS ON THE SPACE OF PROBABILITY MEASURES

Karl Sigmund

Introduction

Let X be a compact metric space, $M(X)$ the compact metrizable space of probability measures with the weak topology, T a homeomorphism from X onto itself and T_M the induced affine homeomorphism from $M(X)$ onto itself, defined by $\int f\, dT_M\mu = \int f\circ T\, d\mu$, $f \in C(X)$. The systems $(T_M , M(X))$ offer a wide variety of examples in topological dynamics. One may study, for example :

a) the structure of the set $M_T(X)$ of fixed points ;

b) the evolution of the time averages $N^{-1} \sum_{k<N} T_M^k\mu$;

c) inheritance properties : this includes properties which, whenever valid for T , are also valid for T_M , and properties which, whenever valid for $\mu \in M(X)$, are valid for μ-almost all $x \in X$.

Here we shall consider a few results on these questions.

The connectedness of ergodic measures

It is well known that the set $M_T(X)$ of T-invariant measures on X is a nonempty compact convex set whose extremal points are precisely the ergodic measures. Thus the convex combinations of ergodic measures are dense in $M_T(X)$. It happens quite frequently that the ergodic measures themselves are dense in $M_T(X)$. This is the case, for example, when T satisfies the specification property (see [4]).

In [9] it is shown that if T is a mixing subshift of finite type (or more generally a basic set for an Axiom A diffeomorphism) then the set of ergodic measures is not only dense, but also arcwise connected. The proof uses the fact that CO-measures, i.e. ergodic measures concentrated on a single periodic orbit, are dense in $M_T(X)$ and that any two such CO-measures can be joined by a continuous arc of strongly mixing Markov measures.

A corresponding result, incidentally, can be shown to be valid for the space of transformations on the unit interval preserving Lebesgue measure (with the weak topology). The set of ergodic transformations is dense and arcwise connected. Here the proof uses the approximation lemma of Rohlin and Keane's result on the contractibility of the group of measure preserving transformations.

Topological Transitivity

$T:X \to X$ is called topologically transitive if for any two nonempty open sets $U,V \subset X$ there exists an $N \in \mathbb{Z}$ such that $T^N U \cap V \neq \phi$, or, equivalently, if there exists a point $x \in X$ whose orbit is dense (such a point is also called topologically transitive). T is called weakly mixing if the product system $T \times T$ is topologically transitive.

Proposition :

If T_M is topologically transitive, then T_M is weakly mixing.

Proof.

We first show that if T_M is topologically transitive then T is weakly mixing. For this it is enough to show that for any two nonempty open sets $U,V \subset X$ there exists an $N \in \boldsymbol{Z}$ with $T^{-N}U \cap U \neq \phi$ and $T^{-N}U \cap V \neq \phi$ (see [8]). The sets $N_1 = \{\mu \in M(X) : \mu(U) > \frac{8}{10}\}$ and

$N_2 = \{\mu\varepsilon M(X) : \mu(U) > \frac{4}{10}$ and $\mu(V) > \frac{4}{10}\}$ are open . Thus there exists an $N \varepsilon Z$ with $T_M^N(N_2) \cap N_1 \neq \phi$ and hence a $\mu \varepsilon M(X)$ with $\mu(U) > \frac{8}{10}$, $\mu(T^{-N}U) > \frac{4}{10}$ and $\mu(T^{-N}V) > \frac{4}{10}$. This is obviously only possible if $U \cap T^{-N}U \neq \phi$ and $U \cap T^{-N}V \neq \phi$.

Now, we show that if T is weakly mixing then so is T_M . Let M_n be the (closed) set of probability measures consisting only of atoms whose weights are multiples of $1/n$. It is easy to see that the restriction of T_M to M_n is a factor of $T^{(n)} = TxTx...xT$. But $T^{(n)}$ is weakly mixing, see [6]. Hence $T_M|M_n$ is weakly mixing. Let U and V be nonempty open sets in $M(X)$. If n is large enough, M_n intersects these sets. The fact that $T_M|M_n$ is weakly mixing then implies that $T_M^N(U) \cap U \neq \phi$ and $T_M^N(U) \cap V \neq \phi$ for some $N \varepsilon \mathbb{Z}$, and hence that T_M is weakly mixing.

Note that a weakly mixing T_M need not be strongly mixing. It is not known whether any affine homeomorphism of a compact convex space which is topologically transitive is also weakly mixing.

Proposition :

Let T_r be the set of topologically transitive points in X . Then $\mu(T_r) = 1$ whenever $\mu \varepsilon M(X)$ is topologically transitive.

Proof.

Since $T_r = \{x \varepsilon X$: for all open $U \neq \phi$, there is an $n \varepsilon \mathbb{Z}$ with $T^n x \varepsilon U\}$, and since there exists a countable base of open sets in X , it is enough to show that $\mu(\bigcup_{n\varepsilon\mathbb{Z}} T^{-n}U) = 1$ whenenver μ is topologically transitive and $U \neq \phi$ is open. If $\mu(\bigcup_{n\varepsilon\mathbb{Z}} T^{-n}U) = a < 1$, then $\mu(T^nU) \leq a$ for all $n \varepsilon Z$. Let $\nu \varepsilon M(X)$ be such that

$\nu(U) = 1$. Since μ is topologically transitive, $T^{n_k}\mu \to \nu$ for some subsequence n_k and hence $\liminf \mu(T^{-n_k} U) \geq \nu(U) = 1$, a contradiction.

Nonwandering points

A point $x \in X$ is called nonwandering if for every neighbourhood U of x there is an $n \neq 0$ such that $T^n U \cap U \neq \phi$. Let Ω denote the closed nonempty set of nonwandering points.

Proposition :

If $\mu \in M(X)$ is nonwandering for T_M , then $\mu(\Omega) \geq \frac{1}{2}$.

Proof.

Let μ be nonwandering and suppose $\mu(\Omega) < \frac{1}{2}$. By the regularity of μ , there exists an open set $U \supset \Omega$ with $\mu(U) < \frac{1}{2}$ and $\mu(U) = \mu(\bar{U})$. Since the compact set $X \setminus U$ consists of wandering points, one can cover it by finitely many -say N- open balls O such that $T^n O \cap O = \phi$ for all $n \neq 0$. It is clear then that for any $n > N$ and any $x \in X\setminus U$, one has $T^n x \in U$. The set $V = \{\rho \in M(X) : \rho(\bar{U}) < \frac{1}{2}\}$ is open, and contains μ . Since μ is nonwandering, there exists a $\nu \in V$ with $T^n \nu \in V$ for some $n > N$. By a density argument we may assume that ν is of the form $k^{-1}(\delta(x_1)+\ldots+\delta(x_k))$, where the $x_k \in X$ need not be distinct. $\nu \in V$ implies $\nu(X\setminus U) > \frac{1}{2}$, thus at least "half" of the x_j belong to $X\setminus U$. The corresponding $T^n x_j$ are in U, thus $T^n \nu \notin V$, a contradiction .

It is not true that if μ is nonwandering, then $\mu(\Omega) = 1$. In fact the bound $\frac{1}{2}$ is sharp, as can be shown by an example due to Grillenberger. Let $\underline{x}_n$ be the sequence $(x_i) \in \{0,1\}^{\mathbf{Z}}$ with $x_n = 1$, $x_i = 0$ for $i \neq n$, and let $\underline{x}_\infty$ be the sequence $(\ldots 000 \ldots)$. Let $X = \{\underline{x}_n , n \in \mathbf{Z} \cup \infty\}$ and let T be the corresponding subshift. Its only nonwandering point is $\underline{x}_\infty$. But any measure $\mu \in M(X)$ with $\mu(\{\underline{x}_\infty\}) \geq \frac{1}{2}$ is nonwandering. For example, if μ has mass $\frac{1}{2}$ at $\underline{x}_\infty$ and $\frac{1}{2}$ at $\underline{x}_0$, and if U is an arbitrarily small neighbourhood of μ, then U contains a measure ν with mass $\frac{1}{2}$ at $\underline{x}_0$ and $\frac{1}{2}$ at $\underline{x}_{(-n)}$, for some large n. Then $T^n\nu$ has mass $\frac{1}{2}$ at $\underline{x}_0$ and $\frac{1}{2}$ at $\underline{x}_n$, and thus belongs to U if n is large enough. Hence $T^n U \cap U \neq \phi$.

On the other hand, $\mu(\Omega) = 1$ does not imply that μ is nonwandering. This follows from an example in the next paragraph.

Poisson stability and central points

Let C be the set of central points for $T:X \to X$, i.e. the largest invariant set such that the corresponding restriction of T has only nonwandering points. C is the closure of the set P of Poisson stable points, i.e. of the points x such that $T^{n_k} x \to x$ for some sequence $n_k \to \infty$ (see [1]).

Proposition :

If μ is Poisson stable, then $\mu(P) = 1$. If μ is central, then $\mu(C) = 1$.

Proof.

The first part of the proposition is due to Jacobs [7]. Let $G \subset X$ be a nonempty open set and write $U = G \cup T^{-1}G \cup T^{-2}G \cup \ldots$. Then $U \supset T^{-1}U \supset \ldots$ and therefore

$$\mu(U) \geq T_M\, \mu(U) \geq T_M^2 \mu(U) \geq \ldots \qquad (*)$$

If μ is Poisson stable, i.e. if $T_M^{n_k} \mu \to \mu$ for some subsequence n_k, one has $\liminf T_M^{n_k} \mu(U) \geq \mu(U)$, and therefore equalities in (*). Thus $\mu(U \setminus T^{-1}U) = 0$, i.e. $\mu(G \setminus (T^{-1} G \cup T^{-2} G \cup \ldots) = 0$. Taking for G the sets of a countable base for the open sets, one sees that μ-almost all x are Poisson stable.

If μ is central, there exists a sequence ν_n of Poisson stable measures with $\nu_n \to \mu$. By the first part of the proposition, $\nu_n(P) = 1$, hence $\nu_n(C) = 1$. Since C is closed, it follows that $\mu(C) = 1$.

The converse statements are not valid, as can be shown by the following example. Let X be a subshift of $\{0,1,2\}^Z$, consisting of orbit closures of $\underline{x} = (x_i)$ and $\underline{y} = (y_i)$, where $x_i = 1$ for $i=k_n$, $x_i = 0$ otherwise, $y_i = 2$ for $i = k_n'$, $y_i = 0$ otherwise. k_n and k_n' are two monotonically increasing sequences with $k_0 = k_0' = 0$. They may be chosen in such a way that x and y are both Poisson stable, but that they recur at different times. Let μ be the measure with mass $\frac{1}{2}$ at $\underline{x}$ and $\frac{1}{2}$ at $\underline{y}$. Then $\mu(C) = \mu(P) = 1$, but μ is not Poisson stable, and not even central. Indeed, every measure ν near μ has mass approximately $\frac{1}{2}$ on both cylinder sets $\{\underline{z} \in X : z_0 = 1\}$ and $\{\underline{z} \in X : z_0 = 2\}$. But at most one of these two sets can have a measure near $\frac{1}{2}$ for $T^n\nu$, $n > 0$.

Note, incidentally, that if R is the set of recurrent points (points whose orbit closure is minimal), then $\mu(R) = 1$ does not imply that μ is recurrent, and μ recurrent does not imply $\mu(R) = 1$. The same is true for pseudo-recurrence.

Ljapunoff stability

A point $x \in X$ is said to be (positively) Ljapunoff stable for $T:X \to X$ if for every $\varepsilon > 0$, there is a $\delta > 0$ such that $d(x,y) < \delta$ implies $d(T^n x, T^n y) < \varepsilon$ for all $n \in N$. T is said to be Ljapunoff stable if every $x \in X$ is Ljapunoff stable for T.

Proposition :

If T is Ljapunoff stable, then so is T_M.

Proof.

As a metric on $M(X)$, we shall use the metric $\bar{d}$ defined by $\bar{d}(\mu,\nu) = \inf\{\varepsilon : \mu(A) \leq \nu(A^{\varepsilon}) + \varepsilon$ for all Borel sets $A \subset X\}$. (See [2]). Here A^{ε} denotes the ε-ball around A. Let T be Ljapunoff stable. Since X is compact, there exists for every $\varepsilon > 0$ a $\delta < \varepsilon$ such that $d(x,y) < \delta$ implies $d(T^n x, T^n y) < \varepsilon$ for all $n \in N$. For every $A \subset X$ and $n \in N$ one has $(T^{-n}A)^{\delta} \subset T^{-n}(A^{\varepsilon})$. Suppose $\bar{d}(\mu,\nu) < \delta$. Then $\mu(T^{-n}A) \leq \nu(T^{-n}A)^{\delta} + \delta$ for every Borel set A, hence $\mu(T^{-n}A) \leq \nu(T^{-n}(A^{\varepsilon})) + \varepsilon$, i.e. $T_M{}^n\mu(A) \leq T_M{}^n\nu(A^{\varepsilon}) + \varepsilon$. This shows that $\bar{d}(T_M{}^n\mu, T_M{}^n\nu) \leq \varepsilon$ for all $n \in N$.

<u>Proposition</u> :

If $\mu \in M(X)$ is Ljapunoff stable for T_M, then every x in the support of μ is Ljapunoff stable for T.

Proof.

Suppose some x in the support of μ is not Ljapunoff stable. There exists an $\varepsilon < 1$ and a sequence $x_n \to x$ such that for every $n \in N$ there is a $j = j(n)$ with $d(T^j x, T^j x_n) > \varepsilon$. We may approximate μ by a measure ν of the form $N^{-1} \sum_{i<N} \delta(y_i)$, where $y_1 = x$, such that ν is Ljapunoff stable. Let ν_n be the measure $N^{-1} \sum_{i<N} \delta(y_i^n)$, where $y_1^n = x_n$ and $y_i^n = y_i$ for $i=2,\dots,N$. Clearly $\nu_n \to \nu$. We shall show that

$$d(T_M{}^{j(n)}\nu, T_M{}^{j(n)}\nu_n) \geq \frac{\varepsilon}{N} \quad \text{for all} \quad n \in N, \qquad (*)$$

and so obtain a contradiction. For given n (and $j = j(n)$) let A^k be the closed ball with radius $k\frac{\varepsilon}{N}$ and center $T^j x$, $k=0,1,\dots,N$. There is at least one k with $1 \leq k \leq N$ such that $A^k \setminus A^{k-1}$ contains no point of the form $T^j x_i$, $2 \leq i \leq N$. The atoms of $T^j \nu_n$ lying in A^{k-1} are exactly the same as those of $T^j \nu$, except for

$T^j x$. Thus $T_M{}^j \nu(A^{k-1}) = T_M{}^j \nu_n(A^{k-1}) + \frac{1}{N} = T_M{}^j \nu_n(A^k) + \frac{1}{N}$. Setting $A = A^{k-1}$ one sees that $T_M{}^j \nu(A) \geq T_M{}^j \nu_n(A^{\varepsilon/N}) + \frac{\varepsilon}{N}$, which implies(*)

Entropy

Let $h(T)$ denote the topological entropy of a transformation T. Clearly $h(T) > 0$ implies $h(T_M) = \infty$. Ledrappier and Walters (oral communication) proved that for the map $T:(x,y) \to (x,x+y) \pmod 1$ on $\mathbf{T}^{\Omega}$, for which $h(T) = 0$, one has $h(T_M) = \infty$.

Proposition :

Either $h(T_M) = 0$ or $h(T_M) = \infty$.

Proof.

Let $f_1, f_2, \ldots,$ be a dense sequence in the unit ball of $C(X)$. We shall use the metric $\bar{\bar{d}}$ on the space of finite signed measures on X given by $\bar{\bar{d}}(\mu,\nu) = \sum_{i \in N^*} 2^{-i} \mid \int f_j d\mu - \int f_j d\nu \mid$

This metric induces the weak topology in $M(X)$. Note that $\bar{\bar{d}}$ is invariant under translations. $\bar{\bar{d}}(\alpha\mu,\alpha\nu) = \alpha\bar{\bar{d}}(\mu,\nu)$ for $\alpha > 0$ and $\bar{\bar{d}}(\mu,\nu) \leq 1$ for $\mu,\nu \in M(X)$.

Now suppose, $h(T_M) > 0$. Then there exists an $\varepsilon > 0$ such that

$$\limsup \frac{1}{n} \log s_n(\varepsilon) = a > 0$$

where $s_n(\varepsilon)$ is the maximal cardinality of an (n,ε)-separated subset of $M(X)$. Let E be such a subset (i.e. for $\mu,\nu \in E$ with $\mu \neq \nu$, there is a j, $0 \leq j < n$, with $d(T_M{}^j \mu, T_M{}^j \nu) \geq \varepsilon$). Choose N such that $2(10^N - 1)^{-1} < \varepsilon$. Let $\varepsilon' = \varepsilon .\ 10^{-N}$ and define

$$E' = \{\mu \in M(X) : \mu = (10^N - 1)10^{-N}\mu' + 10^{-N}\mu'' , \text{ for } \mu',\mu'' \in E\}$$

We claim that E' is (ε',n)-separated. Indeed, let μ and ν be two distinct points of E' , with $\mu = (10^N - 1)10^N\mu' + 10^{-N}\mu''$,

$\nu = (10^N - 1)10^{-N}\nu' + 10^{-N}\nu''$ and $\mu', \mu'', \nu', \nu'' \in E$.

a) if $\mu' \neq \nu'$, then
$\bar{\bar{d}}(T_M{}^j\mu, T_M{}^j\nu) \geq \bar{\bar{d}}(T_M{}^j\mu', T_M{}^j\nu') - \bar{\bar{d}}(T_M{}^j\mu, T_M{}^j\mu') - \bar{\bar{d}}(T_M{}^j\nu, T_M{}^j\nu')$.

But $\bar{\bar{d}}(T_M{}^j\mu, T_M{}^j\mu') \leq \bar{\bar{d}}(0, 10^{-N}\mu'') \leq 10^{-N}$ and similarly $\bar{\bar{d}}(T_M{}^j\nu, T_M{}^j\nu') \leq 10^{-N}$. On the other hand, there is $j, 0 \leq j < n$, with $d(T_M{}^j\mu', T_M{}^j\nu') \geq \varepsilon$. For this j , then ,

$\bar{\bar{d}}(T_M{}^j\mu, T_M{}^j\nu) \geq \varepsilon - 2.10^{-N} \geq \varepsilon \,.\, 10^{-N} = \varepsilon'$.

b) if $\mu' = \nu'$, then $\mu'' \neq \nu''$. There is a j, $0 \leq j \leq n$, with $\bar{\bar{d}}(T_M{}^j\mu'', T_M{}^j\nu'') \geq \varepsilon$. For this j ,

$\bar{\bar{d}}(T_M{}^j\mu, T_M{}^j\nu) = 10^{-N}\, \bar{\bar{d}}(T_M{}^j\mu'', T_M{}^j\nu'') > \varepsilon \,.\, 10^{-N} = \varepsilon'$.

It follows that $s_n(\varepsilon') \geq$ card E' . Since card $E' = [s_n(\varepsilon)]^2$ this leads to

$h(\varepsilon') = \limsup \frac{1}{n} \log s_n(\varepsilon') \geq 2 \limsup \frac{1}{n} \log s_n(\varepsilon) = 2a$.

Iterating this procedure, one sees that $h(\varepsilon) \to \infty$ for $\varepsilon \to 0$. Hence $h(T_M) = \infty$.

Minimal centers of attraction and transformations which are tracing in the mean

For $x \in X$ the ω-limit is the nonempty closed invariant set $\omega(x)$ of accumulation points of $\{T^n x : n \in \mathbb{N}\}$. In [5] and [3], Dowker and Bowen characterized the dynamical systems $S:Y \to Y$ which occur as restrictions of some system $T:X \to X$ such that $\omega(x) = Y$ for some $x \in X$. Such so-called abstract ω-limits are exactly the systems $S:Y \to Y$ which are S-connected, i.e. such that there exists no nonempty open set $U \subset Y$ with $S(\bar{U}) \subset U$.

A similar result can be obtained for minimal centers of attraction. For $x \in X$ the minimal center of attraction $I(x)$ is the nonempty closed invariant set

$\{y \varepsilon X : \limsup N^{-1} \sum_{n<N} 1_U(T^n x) > 0$ for every open $U \ni y\}$. A dynamical system $S:Y \to Y$ is called an abstract minimal center of attraction if it occurs as restriction of some system $T:X \to X$ such that $I(x) = Y$ for some $x \varepsilon X$. Such systems are characterised by the fact that they support an invariant measure which is positive on all nonempty open sets. This is a consequence of the following results (see [10] and [11]).

For $\mu \varepsilon M(X)$ let $V(\mu)$ denote the V-limit of μ, i.e. the set of accumulation points of $N^{-1} \sum_{n<N} T_M{}^n \mu$. This is a nonempty closed and connected subset of $M_T(X)$. It is easy to see that $I(x)$ is just the support of $V(\delta(x))$, i.e. the smallest closed set C with $\nu(C) = 1$ for all $\nu \varepsilon V(\delta(x))$. It can be shown that if $V \subset M_T(X)$ is a closed connected nonempty set, then there exists a dynamical system $S:Y \to Y$ having $T:X \to X$ as restriction and such that $V = V(\mu)$ for some measure μ (and even some point measure) on Y.

It is easy to see that any system $(T_M, M(X))$ is an abstract ω-limit. But there exist such systems which are not abstract minimal centers of attraction. (Consider the example in §4 : since every invariant measure is concentrated on the nonwandering set, the transformation $T_M:M(X) \to M(X)$ admits no invariant measure positive on all open sets, and therefore is no abstract minimal center of attraction).

A transformation $T:X \to X$ is called tracing (resp. tracing in the mean) if for every sequence x_n in X with $d(Tx_n, x_{n+1}) \to 0$ (resp. $N^{-1} \sum_{n<N} d(Tn_n, x_{n+1}) \to 0$), there exists a $z \varepsilon X$ with $d(T^n z, x_n) \to 0$ (resp. $N^{-1} \sum_{n<N} d(T^n Z, x_n) \to 0$). In [3] Bowen shows that if T is tracing and $Z \subset X$ is an abstract ω-limit, then Z is actually an ω-limit. Similarly, it can be shown that if T is tracing in the mean and $Z \subset X$ is an abstract minimal center of attraction, then Z is a minimal center of attraction. Subshifts of finite type

and basic sets for Axiom A diffeomorphisms are tracing. If T is tracing and strongly mixing, then T satisfies the specification property. If T satisfies the specification property, then T is tracing in the mean. The β-shifts provide examples of transformation which are tracing in the mean, but without the specification property.

Lagrange stability

Suppose now that X is a complete separable metric space and K_i an increasing sequence of compact sets such that every compact set $K \subset X$ lies in some K_i. Let L denote the set of points which are (positively) Lagrange stable, i.e. such that their positive orbit is relatively compact.

Proposition :

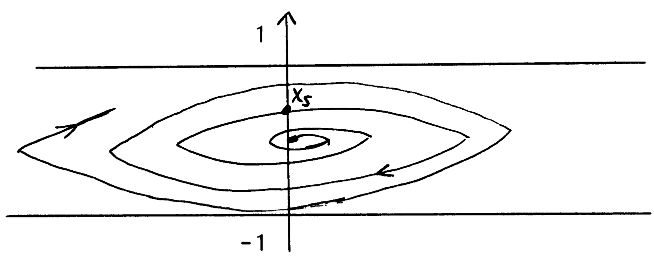
If $\mu(L) = 1$, then μ is Lagrange stable for T_M.

Proof.

Suppose that $\{T_M{}^n\mu\}$ is not relatively compact. By the portmanteau theorem (see [2]), there exists an $a > 0$ such that for any compact K, $T_M{}^n\mu(K) < 1-a$ for some $n \in \mathbb{N}$. For each $i \in \mathbb{N}$ there exists an n_i with $T_M{}^{n_i}\mu(K_i) < 1-a$, and hence a set L_i with $T^{n_i}L_i \cap K_i = \phi$ and $\mu(L_i) \geq a$. Setting $L_\infty = \bigcap_{m\in N} \bigcup_{i\geq m} L_i$, one has $\mu(L_\infty) \geq a$. But each $x \in L_\infty$ belongs to infinitely many L_i, i.e. the orbit escapes from infinitely many K_i and hence is not Lagrange stable, a contradiction.

The converse is not valid, there exist measures μ which are Lagrange stable but such that $\mu(L) = 0$. Consider the following (continuous time) example. Let X be the space $[0,1] \times [-1,1] \times \mathbb{R}$. For each $s \in [0,1]$ the set $X_s = \{s\} \times [-1,1] \times \mathbb{R}$ is globally invariant and the phase portrait looks as in the figure below. For each s let

x_s be $(s,\frac{1}{2},0)$, which is obviously not Lagrange stable, and let μ be Lebesgue measure on the segment $\{x_s : s \in |0,1|\}$. Let C_i be the compact set $[0,1] \times [-1,1] \times [-i,i]$. By suitable parameterizing the flows on X_s, one can ensure that $\mu(\{s \in [0,1] : T^t(x_s) \notin C_i\} < \frac{1}{i}$ for all $t \geq 0$, $i = 1,2,\ldots$. Hence the family $T_M^t\mu$ is uniformly tight, i.e. the orbit of μ has compact closure. But $\mu(L) = 0$.



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K. Sigmund
Mathematisches Institut
Strudlhofgasse 4
A-1090 Vienne
Autriche