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and diffemorphisms with infinitely many sinks

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ONE-DIMENSIONAL ATTRACTORS OF A-DIFFEOMORPHISMS ON $S^2$ AND DIFFEOMORPHISMS WITH INFINITELY MANY SINKS

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The first part of the present paper gives an account of series of structurally stable diffeomorphisms on $S^2$ having the spectral decomposition which consists of the connected one-dimensional attractor and four repulsive periodic points. Beginning with the example of such diffeomorphisms given in [10] we shall give a sequence of its modifications. With each diffeomorphism of that series we shall connect a geometric intersection matrix which enables us to calculate a topological entropy. Considering the sequence of values of entropy we shall prove that diffeomorphisms of given series are representatives of a countable set of class of topological conjugation which is not the result of any iteratonsof any finite series of diffeomorphisms (theorem of part 1).

In the second part of the paper the construction of one-dimensional attractor on $S^2$ is used for investigation of the question about $C^1$-typicalness of diffeomorphisms with

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* This paper is a part of collective report on the activity of the seminar on topology and dynamical systems in Obninsk Branch of Moscow Engineering-Physics Institute in 1976.
infinitely many sources (sinks) on smooth compact manifold
of dimension greater than two. The suggested theorem
result includes a specification of the of S. Newhouse in dimension
greater than two.

1. Diffeomorphisms with One-Dimensional Attractors

All the constructions will be carried out on the plane
$\mathbb{R}^2$ that will be turned afterwards into 2-sphere by adding a
point at infinity. Let us fix some Cartesian co-ordinate
system $(x, y)$ and introduce the following designations. For
any segment $A$ of axis Ox $\bar{A}$ and $\hat{A}$ are its left and right end
points accordingly. For an integer $i$, $\sigma(i)$ is the sign "+" if
$i$ is even and "-" if $i$ is odd. Let $L$ be a fixed smooth curve
on the plane with the following properties: $L$ is convex,
symmetrically with respect to axis Oy, close enough to semi-
circle $x^2 + y^2 = 1$, $y > 0$ and it has $(-1, 0), (1,0)$ as the end
points, moreover its tangents in these points are vertical.
For any segment $A$ of the axis Ox let $L^+(A)$ be the curve,
obtained from $L$ by similarity transformation centred at
origin followed by displacement along axis Ox. Moreover, the
end points of $L^+(A)$ are just the same as those of $A$. The
curve $L^-(A)$ is obtained from $L^+(A)$ by reflection in an axis
Ox. We say that curves $L^+(A)$ and $L^-(A)$ are based on segment
$A$. For nonintersecting closed segments $A, B$ on Ox (let $\hat{A} < \hat{B}$
for definiteness) we define
$\cap^\sigma(A, B)$ \hspace{1cm} ($\sigma = + , -$)
the closed region bounded by segments $A, B$ and curves
$L^\sigma(\hat{A}, \hat{B}), L^\sigma(\hat{A}, \hat{B})$ and let be
$\cap^\sigma(A, B) = \cap^\sigma(B, A)$
Now we are prepared to describe our construction. Let us lay on axis Ox nonintersecting segments

\[ A_i^1, A_i^2, A_i^3, A_i^4, A_i^5, A_i^6, A_i^7, A_i^8, A_i^9, A_i^{10}, A_i^{11}, A_i^{12}, A_i^{13}, A_i^{14}, A_i^{15}, A_i^{16}, A_i^{17}, A_i^{18}, A_i^{19}, A_i^{20} \]

ordered by growth of their x-co-ordinates. In addition to these segments \( A_1^1, A_1^2, A_1^3, A_1^4, A_1^5, A_1^6 \) that are nonintersecting and ordered by growth of x-co-ordinates too. Require

\[ A_i^1 < A_i^2 < A_i^3 < A_i^4 < A_i^5 \]

Let regions \( \bigcap_i \rho \quad (i = 1, 2, 3; \quad 1 \leq \rho \leq 5) \) be defined by

\[ \bigcap_i \rho = \bigcap_i^{(i+\rho-1)} (A_i^\rho, A_i^{\rho+1}) \]

and

\[ \bigcup_i \rho \quad (i = 1, 2, 3) \quad \bigcap' \quad \text{by} \]

\[ \bigcup_{\rho=1}^{5} \bigcap_i \rho \quad (i = 1, 2, 3) \]

(see region \( \bigcap' \) on the fig. 1a). Subsequently \( \bigcap' \) will perform the function of image of region \( \Phi' \) (see fig. 1b) under diffeomorphism \( f \), that will be defined among others below.

Now we proceed to describe some construction that will be used to modify region \( \bigcap' \) into regions \( \bigcap'' \) for any natural \( n \).

Then we shall define regions \( \Phi'' \) and diffeomorphisms \( f_n : \Phi'' \rightarrow \bigcap'' \) for any \( n \). The construction is inductive and looks "on the figure" as follows. Let us consider a part of \( \bigcap' \) lying in the region \( \Sigma' \) bounded in bold outline on fig. 1a. The set \( Q^1 = \bigcap' \cap \Sigma' \) is drawn on fig. 2a and we shall transform it into the region \( R^1 \) showed on fig. 2b, so that

\[ R^2 \cap \partial \Sigma^1 = \bigcap' \cap \partial \Sigma' \]
Let $\Pi^2 = (\Pi \setminus \Sigma^1) \cup R^2$ (see fig. 3). A part $Q^2$ of $\Pi^2$ lying in the region $\Sigma^2$ bounded in bold outline on fig. 2b is exactly the same as $Q^1$ on fig. 2a. Consequently we may subject $Q^2$ to the same transformation to produce region $\Pi^3 = (\Pi^2 \setminus \Sigma^2) \cup R^3$ and so on.

To be more formal let us consider closed regions $Q^1_1$, $Q^2_1$, $Q^3_1$, $Q^4_1$ defined below by the sequence:

$$a'^n, a'^2, a'^3, a'^n, a'^n, b'^n, b'^3, b'^n, b'^n, b'^n$$

of nonintersecting segments on $Ox$, ordered by growth of $x$-co-ordinate.

$$Q^1_1 = \Pi^+(a'^3, a'^n), \quad Q^2_1 = \Pi^+(a'^3, b'^3) \cup \Pi^-(b'^n, b'^n),$$
$$Q^3_1 = \Pi^+(a'^n, a'^n) \cup \Pi^-(a'^n, b'^n), \quad Q^4_1 = \Pi^-(b'^n, b'^n) \quad (2)$$

The closed region $\Sigma^n$ bounded by curves $L^+(a'^i, b'^i)$, $L^-(a'^n, b'^n)$ and segments $[a'^n, a'^n]$, $[b'^n, b'^n]$ contains the set $Q^n = \bigcup Q^n_i$ (see fig. 2a). Let us lay on $[a'^n, a'^n]$ nonintersecting segments $a'^n, a'^n, a'^n, a'^n, a'^n, a'^n, b'^n, b'^n, b'^n, b'^n, b'^n, b'^n, b'^n, b'^n, b'^n$ so that $a'^n = a'^n$, $b'^n = b'^n$ and let

$$R^n_1 = \Pi^-(b'^1, a'^1) \cup Q^n_1 \cup \Pi^+(a'^3, b'^3)$$
$$R^n_2 = \Pi^+(a'^2, b'^2) \cup Q^n_2 \cup \Pi^-(a'^1, b'^1)$$
$$R^n_3 = \Pi^+(a'^1, b'^1) \cup Q^n_3 \cup \Pi^-(a'^2, b'^2)$$
$$R^n_4 = \Pi^+(a'^3, b'^3) \cup Q^n_4 \cup \Pi^+(a'^n, b'^n)$$

(see fig. 2b), where $Q^n_i$ are defined by (2) with $n + 1$
instead of \( n \).

For \( R^{n+1} = \cup_{i=1}^{n} R_i \), we have \( Q^n \cap \partial \Sigma^n = R^n \cap \partial \Sigma^n \). Thus, given set \( Q^1 = \cup_{i=1}^{n} \Sigma_{c_i}^n \) as in (2) an induction defines sets \( Q^n, \Sigma^n (n \geq 1) \) and \( R^n (n \geq 2) \), moreover, \( Q^n_{c_i} = R^n_{c_i} \cap \Sigma^n \) (\( n \geq 2 \)).

Now let the sequence \( a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3, \ldots, b_n \) be a part of (1) from \( A^1 \) to \( A^4 \) inclusive. Then \( Q_1 = \Pi_2, Q_2 = \Pi_3 \cup \Pi_4, Q_3 = \Pi_2 \cup \Pi_4, Q_4 = \Pi_3 \cup \Pi_4 \) and boundary of \( \Sigma^i \) is the union of \( L^+ (A_3^1, A_5^1), L^- (A_3^1, A_5^1), [A_3^1, A_5^1], [A_3^1, A_5^1] \) (a bond outline on fig. 1a). We construct sets \( Q^n, \Sigma^n \) and \( R^n (n \geq 2) \) and define \( \Pi_1^{n+1} = (\Pi_1^n \cup \Sigma^n) \cup U R^n \cup R^n_{1}, \Pi_2^{n+1} = (\Pi_2^n \cup \Sigma^n) \cup U R^n \cup R^n_{2}, \Pi_3^{n+1} = (\Pi_3^n \cup \Sigma^n) \cup U R^n \cup R^n_{3}, \Pi_4^{n+1} = (\Pi_4^n \cup \Sigma^n) \cup U R^n \cup R^n_{4} \) by induction, then \( \Pi_{1}^{n+1} = U \Pi_{i}^{n+1} = (\Pi_{1}^{n} \cup \Sigma^{n}) \cup U R^{n} \).

Now let us define sets \( \phi_i^{n+1} = \cup_{i=1}^{n} \phi_i^{n} \) and diffeomorphisms \( f_n : \phi^n \to \Pi^n \) (\( n \geq 1 \)). Let \( L_1 = L^- (0, A_2^1), L_3 = L^+ (q_1, 0), L_2 = L^- (A_5^3, 1), L_4 = L^+ (A_5^3, 1), L_3 = L^- (A_5^3, 1), L_1 = L^+ (A_5^3, 1) \) and \( \phi_i \) be a closed region bounded by a pair of \( x \)-segments and curves \( l_i, l_i^n \). It is easy to see that \( \phi_i^n = \cup_{i=1}^{n} \phi_i^n \) contains \( \Pi^n \).

Before constructing the diffeomorphisms \( f_n : \phi^n \to \Pi^n \) let us define some partition \( \Gamma \) of \( \phi^n \) consisting of line segments. Let the line \( l_i^n \) be based on the segment \( \Delta_i^n \) and choose some point \( M_i^n \) in interior of \( \Delta_i^n \). For any interior point \( h \) of \( l_i \) an intersection of a line passing through points \( M_i^n \) and \( h \) with \( \phi_i^n \). If \( h \) is an end point of some \( l_i \) then let \( \gamma_i^n = [0, 1] \). Thus \( \Gamma = \{ \gamma_i^n \} \). Thus \( \Gamma = \{ \gamma_i^n \} \) is a partition of \( \phi^n \).

Now define \( f_n : \phi^n \to \Pi^n \) so that \( \phi_n (\phi_i^n) = \Pi_i \).
and for any $y_h \in \Gamma$, $f^n_h(y_h) = y_h'$ for some $y_h' \in \Gamma$. First define $f^n$ on $\bigcup_i l_i$. The boundary of each $l_i^n$ consists of the pair of line segments and pair of curves $m_i$ and $m_i'$. We can choose designations of them so that $\bigcup l_i$ is connected and contains the point $(0,0)$. Let $\bigcup_i l_i$ be an orientation preserving diffeomorphism uniformly expanding each $l_i$ and mapping it onto $m_i$ with fixed point $(0,0)$. Define $f_h^n$ for interior points of $\bigcup_i l_i$. Let $f_h^n$ map any $y_h$, $h \in \bigcup l_i$ contracting it onto the connected component of $\gamma^h$ contains $f_h^n(h)$. The boundary of each $\bigcup_i l_i$ consists of the pair of line segments and pair of curves $m_i$ and $m_i'$. We can choose designations of them so that $\bigcup l_i$ is connected and contains the point $(0,0)$. Let $\bigcup_i l_i$ be an orientation preserving diffeomorphism uniformly expanding each $l_i$ and mapping it onto $m_i$ with fixed point $(0,0)$. Define $f_h^n$ for interior points of $\bigcup_i l_i$. Let $f_h^n$ map any $y_h$, $h \in \bigcup l_i$ contracting it onto the connected component of $\gamma^h$ contains $f_h^n(h)$. Note that if we make our construction more accurately as in [10] assigning the length to all segments and curves and defining contraction and expansion coefficients we can obtain that the product of those in $(0,0)$ is greater than $1$. We shall need that in the second part of the paper.

Any $f$ may be extended using lemma 1 of part 2 to be diffeomorphism of $S^2$. We can produce the extention so that $f^n_h : S^2 \to S^2$ will have exactly four repulsive periodic points with one in each connected component of $S^2 - \Phi^n$ and won't have nonwandering points in $S^2 - \Phi^n$ any more. All these repulsive points are fixed if $n$ is odd. If $n$ is even then two of these are fixed while two others form a periodic orbit. These points are in regions bounded by $l_2^n$, $l_3^n$ and respective segments of axis $0x$. One can prove as in [10] the hyperbolicity of the invariant set $\Lambda_h = \bigcap_{k>0} f_h^k(\Phi^n)$ using criterion of hyperbolicity by Hirsch and C. Pugh [4]. Robbins theorem implies structural
stability of $f_n$. One-dimensional compact $\Lambda_n$ is common frontier of four domains thus being well known Wada's continuum.

Define nonnegative integer $3 \times 3$ matrix $G(f_n) = G_n$ which will be called (following M. Shub and D. Sullivan [5]) geometric intersection matrix of $f_n$ in respect to partition $(\Phi^1, \Phi^2, \Phi^3)$ of $\Phi$. Element $g_{ij}^n$ of $G_n$ is equal to the number of connected components of $f_n(\Phi^1_i) \cap \Phi^2_j$ that is

$$g_{ij}^n = \text{card } \Pi_0 (\Pi^1_i \cap \Phi^2_j)$$

since $f_n(\Phi^2_i) = \Pi^2_j$. This permits to find $G_n$. It is easy to see that

$$G_1 = \begin{pmatrix} 3 & 0 & 2 \\ 4 & 3 & 2 \\ 2 & 2 & 1 \end{pmatrix}$$

(see fig. 1a) and

$$G_{n+1} = G_n + \begin{pmatrix} 0 & 0 & 0 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}$$

(see fig. 2a and 2b) implies

$$G_n = \begin{pmatrix} 3 & 0 & 2 \\ 2n + 2 & n + 2 & n + 1 \\ 2n & n + 1 & n \end{pmatrix}$$

Matrix $G_n$ has somewhat another geometric meaning if is considered as a solenoid that is to say as an inverse limit of branched manifold and its expanding map (see R. Williams [7]). This construction in our case is obtained
as follows. Let us consider an equivalence relation \( \sim \) on \( \mathcal{P}^n \) which is generated by partition \( \Gamma \) of \( \mathcal{P}^n \) and a factor space \( K = \mathcal{P}^n / \sim \). It is easy to see that \( K \) is homeomorphic to bunch of cicles \( K = \bigvee_{i=1}^{s} S'_i \) and we enumerate \( S'_i \) so that \( \pi(\mathcal{P}^n_i) = S'_i \) where \( \pi: \mathcal{P}^n \to K \) is a projection. Since \( \Gamma \) is invariant under a map \( \varphi_n = \pi \circ f_n \circ \pi^{-1}: K \to K \) is well defined. R. Williams' axioms [7] are valid for \( \varphi_n \) in particular \( \varphi_n \) is local homeomorphism in a neighbourhood of any point \( x \in K \) except branch point \( x' \). Thus we may introduce \( 3 \times 3 \)-matrix \( G_n \) setting \( G_{ij}^n = \text{card}(\varphi_n^{-1}(x) \cap S'_i) \) for any point \( x \neq x' \) of \( S'_i \). Evidently, \( G_n \) is just the same matrix as introduced above.

Let \( X^n \) denote an inverse limit of spectrum
\[ K \leftarrow \mathcal{P}^n \leftarrow \mathcal{P}^{n-1} \leftarrow \ldots \quad \text{and} \quad \varphi_n : X^n \to X^n \]
its shift automorphism defined by
\[ \varphi_n(x, x_2, \ldots) = (\varphi_n x, x_2, \ldots) \]
We claim that for homeomorphism \( S: \Lambda_n \to X^n \) defined by
\[ S(\rho) = (\pi \rho, \pi \varphi^{-1} f_n(\rho), \ldots, \pi \varphi^{-k} f_n(\rho), \ldots) \], \( \rho \in \Lambda_n \)
the diagram
\[
\begin{array}{ccc}
\Lambda_n & \xrightarrow{f_n} & \Lambda_n \\
\downarrow{S} & & \downarrow{S'} \\
X^n & \xrightarrow{\varphi_n} & X^n \\
\end{array}
\]
commutes.

The fact that segments of invariant partition \( \Gamma \) contracts with coefficients which are less than 1 uniformly implies
injectivity of $S$. To prove that $S$ is onto let us consider for any $(x_1, x_2, \ldots) \in X^n$ a family of segments
$$\alpha_k = f_n \circ \pi^{-1}(x_{k+1}) \quad (k \geq 0)$$
containing in elements of $f$. Since $S(n+1) \subset \alpha_k$ and the length of $\alpha_k$ tends to zero as $k \to \infty$, $\bigcap_{k \geq 0} \alpha_k$ consists of unique point $\rho \in \Lambda_n$.

Evidently $S(\rho) = (x_1, x_2, \ldots)$ Commutativity of the diagram (2) is a consequence of the commutative diagram

$$
\begin{array}{ccc}
\phi^n & \xrightarrow{f_n} & \phi^n \\
\downarrow \Pi & & \downarrow \Pi \\
K & \xrightarrow{\psi_k} & K
\end{array}
$$

Topological entropy of $\psi_k$ is equal to logarithm of the maximal eigenvalue of the matrix $G_n$ (see [12]). Since the spectral decomposition of $\Omega(f_n)$ consists of four periodical points and one attractor $\Lambda_n$ and $f_n | \Lambda_n$ is conjugate to $\psi_k$

$$h(f_n | \Lambda_n) = h(f_n | \Lambda_n) = h(\psi_k | X^n)$$
implies

$$h(f_n) = \log (n+2+\sqrt{(n+1)(n+3)})$$

**THEOREM.** Diffeomorphisms of the series $\{f_n, n \in N\}$ are representatives of countable many topological conjugacy classes, these classes are not results by iterations of any finite series of diffeomorphisms.

**PROOF.** The first assertion is a consequence of the fact that values of the entropy of $f_n$ form infinitely sequence.

For the second let us consider any diffeomorphisms $F_1, \ldots, F_k$
on $S$. Let $\mu_i = \exp h(F_i)$ $(i = 1, \ldots, \kappa)$. If for any $f_n$ there exist some $F_i$ such that $f_n$ is conjugate to $F_i^p$ for some integer $p$, then $\lambda_n = \mu_i^p$ were $\lambda_n = \exp h(f_n)$. It is easy to see that $\lambda_n = 2n + 4 - \alpha_n$ for some $\phi < \alpha_n < 1$. Thus our supposition implies that for any integer number of the form $2n + 4$ there exists some integer $p$ such that $|2n + 4 - \mu_i^p| < 1$ for some $\mu_i$ belonged to some fixed finite set. This means that any element of the arithmetical progression $2n + 4$ is approximated by elements of finite family of geometrical progressions which is impossible.

2. On Diffeomorphisms with Infinitely Many Sources (Sinks)

Let $\text{Diff}_2(M)$ denote the space of $C^\infty$-diffeomorphisms of $C^\infty$ manifold $M$ without boundary $\text{Diff}_2^\kappa(M)$ ($K \leq 2$) denote the same set with the uniform $C^\kappa$-topology.

**Definition 1**

Some property of the elements of the set $\text{Diff}_2(M)$ is called $C^\kappa$-typical for $f \in \text{Diff}_2(M)$ if there is a residual subset $\mathcal{B}$ of an open neighbourhood $N(f)$ of $f$ in $\text{Diff}_2^\kappa(M)$ with this property for each element of $\mathcal{B}$.

Perhaps it is worthwhile saying that the notion of the $C^\kappa$-typical property is the generalization of the $C^k$-stability. Throughout this paper the notion of $C^\kappa$-stability is used in its ordinary meaning.

Our aim is to prove the following result.
For any manifold $\mathbb{M}$ of dimension greater than two, the set of diffeomorphisms in $\text{Diff}_r^2(\mathbb{M})$ ($r \geq 2$), for which the property of having infinitely many sources is $C^k$-typical ($k > 1$), is $C^0$-dense in $\text{Diff}_r(\mathbb{M})$.

Our theorem will be proved as follows. First, sufficient conditions for the appearance of infinitely many sources will be given. This result is due to S.E. Newhouse [1]. Infinitely many sources appearance is based on the fact that tangency points existence for stable and unstable manifolds is $C^k$-stable, for some basic set. Secondly, it will be shown that the diffeomorphisms with this property are dense in $\text{Diff}_r^0(\mathbb{M})$, $\dim \mathbb{M} \geq 3$. Before proving the theorem it is worth saying a few words about notations.

Let us recall (see [1]) that a compact $f$-invariant set $\Lambda$ is a basic set for $f$ if it is hyperbolic, topologically transitive, the periodic points of $f$ are dense in $\Lambda$, and $\Lambda$ has a local product structure. The basic set is non-trivial if it contains more than one orbit. In this case it must be infinite. Given a basic set $\Lambda_f$ for $f$, there is a neighbourhood $N$ of $f$ in $\text{Diff}_r^k(\mathbb{M})$ where each $g \in N$ has a unique basic set $\Lambda_g$ near $\Lambda_f$ and there is a homeomorphism $h : \Lambda_f \rightarrow \Lambda_g$ such that $gh = hf$. For $x \in \Lambda_f$ and $g \in N$, we denote $h(x)$ by $x_g$.

Let $p$ be a hyperbolic periodic point of $f$ with period $\nu$. Let $\mu_1, \mu_2, \ldots, \mu_s, \lambda_1, \ldots, \lambda_t$ be the eigenvalues of the derivative $D_p^f$ with $|\mu_1| \leq \ldots \leq |\mu_s| < 1 < |\lambda_1| \leq \ldots \leq |\lambda_t|$ and let be $\mu(p) = |\mu_s|$, $\lambda(p) = |\lambda_t|$. Now we can formulate the propositions discussed above.
THEOREM A. (S.E. Newhouse [1]) Suppose $\mathcal{A}$ is a non-trivial basic set for $f \in \text{Diff}^k(\mathbb{M})$ which contains a periodic point $p_f$ such that $\dim W^s(p_f) = \dim \mathbb{M} - 1$ ($\dim W^u(p_f) = \dim \mathbb{M} - 1$).

Let $N$ be a small neighbourhood of $f$ as above such that each $g \in N$ has a unique basic set $\mathcal{A}_g$ near $\mathcal{A}$. Assume there is a neighbourhood $N_1 \subset N$ of $f$ in $\text{Diff}^k(\mathbb{M})$ such that if $g \in N_1$, then $W^u(\mathcal{A}_g)$ and $W^s(\mathcal{A}_g)$ have a point of tangency and 

$$\mu(p_g) \lambda(p_g) < 1 \quad (> 1)$$

Then there is a residual subset $B$ of $N$, such that each $g \in B$ has infinitely many sinks (sources). Using our notation the property of having infinitely many sinks (sources) is $C^k$-typical for $f$.

THEOREM A'. (S.E. Newhouse [1]) Let $p$ be a hyperbolic periodic point of $f$. $\dim W^u(o(p)) = 1$, $\mu(p) \lambda(p) > 1$ ($\dim W^s(o(p)) = 1$, $\mu(p) \lambda(p) < 1$). Assume $x$ is a point of tangency of $W^s(o(p))$ and $W^u(o(p))$. Then given any neighbourhood $U$ of $x$ in $\mathbb{M}$ and $N$ of $f$ in $\text{Diff}^k(\mathbb{M})$, $k \geq 1$, there is a $g \in N$ which has a sink in $U$.

THEOREM B. Given any manifold $\mathbb{M}$, $\dim \mathbb{M} \geq 3$, the set of $C^r$-diffeomorphisms, $r \geq 2$, for which condition of theorem A holds with $k \geq 1$ is dense in $\text{Diff}^r(\mathbb{M})$.

Remarks.

1. S.E. Newhouse [1] has shown that on any manifold $\mathbb{M}$, $\dim \mathbb{M} \geq 2$, there is a diffeomorphism for which conditions of theorem A are satisfied with $k \geq 2$.

2. In [9] N.K. Gavrilov has adduced conditions when diffeomorphism $f$ has infinitely many sinks, $f \in \text{Diff}^r(\mathbb{M}^2)$, $r \geq 2$. 

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Recall some definitions from general topology and differentiable dynamics.

Let $X$, $Y$ be metric spaces and $X$ be compact. Let $2^X$ denote the set of all closed subsets of $X$.

1. $\{A_n \mid A_n \in 2^X, n \in \mathbb{N}\}$ is a sequence of closed subsets of $X$.

$$\lim_{n \to \infty} (A_n) = \{ \rho \mid \forall n \exists \rho_n \in A_n : \rho = \lim_{m \to \infty} \rho_m \}$$

(see [1])

2. The map $F : Y \to 2^X$ is lower semi continuous, if for any point $y \in Y$ and for any sequence $\{y_n \in Y, n \in \mathbb{N}\}$, $y_n \to y$

$$F(y) \subseteq \lim_{n \to \infty} (F(y_n))$$

3. The Hausdorff metric on $2^X$ is defined by the formula

$$\text{dist} (A, B) = \max \left\{ \sup_{a \in A} \rho(a, B), \sup_{b \in B} \rho(A, b) \right\}$$

where $\rho$ is metric on $X$ (see [8]).

4. Diffeomorphism $f \in \text{Diff}_r(M)$ is a Kupka–Smale diffeomorphism (KS-diffeomorphism) if the following conditions are satisfied

   (1) All periodic points of $f$ are hyperbolic.

   (2) For any periodic point $p$ $W^s(p)$ and $W^u(p)$ are immersed copies of Euclidean space.

   (3) For any periodic points $p, q$ of $f$ $W^s(p)$ and $W^u(q)$ are transversal.

(see [6])

Let $\text{KS}(N)$ denote the set of Kupka–Smale diffeomorphisms in $N \subset \text{Diff}_r(M)$.

The lower semicontinuous map is continuous on a residual subset $B$ of $Y$. The set $\text{KS}(M)$ is residual subset of $\text{Diff}_r(M)$. 

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**Lemma 1.** Let \( J_n : Y \to 2^X \) be lower semicontinuous map for any \( n \in \mathbb{N} \), then the map \( J : Y \to 2^X \) given by the formula

\[
J(x) = \text{cl} \left( \bigcup_{n \in \mathbb{N}} J_n(x) \right)
\]

is semicontinuous.

**Proof.** Recall two elementary properties of \( \text{Li} \) (see [8])

1. \( \text{cl}(\text{Li} A_n) = \text{Li}(\text{cl} A_n) \)
2. \( \bigcup_{\kappa \in \mathcal{G}} [\text{Li} A_n(\kappa)] \subset \text{Li} \left( \bigcup_{\kappa \in \mathcal{G}} A_n(\kappa) \right) \)

where \( \kappa \in \mathcal{G} \) and \( \mathcal{G} \) is arbitrary.

Let the sequence \( x_\kappa \in Y \), \( \kappa \in \mathbb{N} \) converge to \( x \in Y \).

From the properties (1)-(2) and the fact that \( J_n \) is lower semicontinuous one obtains

\[
\text{Li} \left( \bigcup_{\kappa} J_n(x_\kappa) \right) = \text{Li} \left( \bigcup_{\kappa} J_n(x) \right) \supset
\bigcup_{\kappa} \text{Li} J_n(x_\kappa) \supset \bigcup_{\kappa} J_n(x)
\]

and hence

\[
J(x) = \text{cl} \left( \bigcup_{n} J_n(x) \right) \subset \bigcup_{\kappa} J(x_\kappa)
\]

This completes the proof of lemma 1.

Let \( \Lambda_f \) be non-trivial basic set of \( f \in \text{Diff}_k(M) \), \( \rho \in \Lambda_f \) be hyperbolic periodic point. Let \( \mathcal{H} \) be neighbourhood of \( f \) in \( \text{Diff}_k(M) \) described above. For any point \( g \in \mathcal{H} \), denote the homoclinic class of \( p_g \) that is \( H_p \) consists of all periodic points \( q \) of \( g \) such that \( W^u(o(q)) \) has a point of transversal intersection with \( W^s(o(p)) \) and \( W^u(o(p)) \) has a point of transversal intersection with \( W^s(o(q)) \). Then \( H_{p_g} \) contains the periodic points of \( \Lambda_g \) (see [1]). The formula \( H(g) = \text{cl} H_{p_g} \) defines the map \( H : \text{KS}(\mathcal{H}) \to 2^M \). \( S(g) \) denotes the closure of the set of all sources for \( g \).
and thus it defines the map $S: \text{KS}(\mathcal{N}) \rightarrow 2^M$.

**Lemma 2** (see [3]) $H$ and $S$ are continuous on a residual subset $\mathcal{J}_S \subset \mathcal{N}$.

**Proof.** For any $n$ let's define maps $H_n, S_n : \text{KS}(\mathcal{N}) \rightarrow 2^M$ by the formulas

$$H_n(g) = H_{\mathcal{P}} \cap \text{Per}_n(g)$$
$$S_n(g) = S(g) \cap \text{Per}_n(g)$$

where $\text{Per}_n(g)$ is the set of fixed points of $g^n$. $g$ is $\text{KS}$-diffeomorphism hence the set $\text{Per}_n(g)$ is finite and thus the sets $H_n(g)$ and $S_n(g)$ have only finite number of elements. We shall show that $H_n$ and $S_n$ are lower semicontinuous. Then $H$ and $S$ are lower semicontinuous because $H(g) = \text{cl}(\cup H_n(g))$ and $S(g) = \text{cl}(\cup S_n(g))$.

There are residual subsets $\mathcal{J}_H$ and $\mathcal{J}_S$ of $\text{KS}(\mathcal{N})$ on which $S$ and $H$ are continuous. Thus to prove the lemma we must set $\mathcal{J}_S = \mathcal{J}_H \cap \mathcal{J}_S$.

(i) The map $H_n$ is continuous on $\text{KS}(\mathcal{N})$. Let $g' \in \mathcal{N}$ be a sufficiently close approximation of $g$. There is the homeomorphism $h: \Lambda_\mathcal{N} \rightarrow \Lambda_{g'}$ such that $h \cdot g = g' \cdot h$ and $\rho(x, h(x)) < \varepsilon$ for any $x \in \Lambda_\mathcal{N}$, where $\varepsilon \rightarrow 0$ if $g' \rightarrow g$. Thus $\text{dist}(H_n(g), H_n(g')) < \varepsilon$ because $H_{\mathcal{P}g'} = h(H_{\mathcal{P}g})$ and $\text{Per}_n(g') \cap \Lambda_{g'} = h(\text{Per}_n(g) \cap \Lambda_\mathcal{N})$.

(ii) The map $S_n$ is lower semicontinuous. If $g \in \text{KS}(\mathcal{N})$ then $S_n(g) = \{a_1, \ldots, a_k\}$ is hyperbolic set for $g$. Hence from the stability theorem for hyperbolic set one obtains that for any $\varepsilon > 0$ there is a neighbourhood $\mathcal{N}_2$ of $g$ in $\text{Diff}_k^1(\mathcal{M})$ such that $g' \in \mathcal{N}_2$ implies the existence of $\tilde{S} \subset S_n(g')$ with $\text{dist}(\tilde{S}, S_n(g)) < \varepsilon$, so $S_n(g) \subset \text{Li} S_n(g')$. Lemma 2 is proved.

**Lemma 3** Let $g \in \mathcal{N}$, then given any periodical point $p \in \Lambda_\mathcal{N}$ its neighbourhood $U$ and arbitrary neighbourhood $\mathcal{N}_2$ of $g$ in $\text{Diff}_k^1(\mathcal{M})$, there is the diffeomorphism $g' \in \mathcal{N}_2$ which coincides with $g$ on some neighbourhood of $\Lambda_\mathcal{N}$ and has sink $q \in U$.
Proof. Let for \( q_1, q_2 \in \Lambda g \) \( W^s(q_1) \) and \( W^u(q_2) \) have the tangency point \( x \).

(i) It will be shown that there is arbitrarily small perturbation \( g_j \) of \( g \) outside some neighbourhood of \( \Lambda g \) such that for any neighbourhood \( U_1 \) of point \( x \) there is \( x_1 \) - the point of tangency \( W^s(p_1, g_j) \) and \( W^u(p_2, g_j) \), where \( p_1 \) and \( p_2 \) are periodic points of \( g_j \) and \( p_1, p_2 \in \Lambda g \). For sufficiently small \( \alpha, \beta > 0 \) any point \( q \in \Lambda g \) there is stable manifold of size \((\beta, \alpha)\) (see [4]). Global stable and unstable manifolds are defined by the formulas

\[
W^s(q) = \bigcup_{n \geq 0} \phi^{-n}(W^s_{g^n}(q)),
\]

\[
W^u(q) = \bigcup_{n \geq 0} \phi^n(W^u_{g^n}(q)).
\]

We need some notations. Let \( W^S_{n}(q) \) and \( W^U_{n}(q) \) denote

\[
\bigcup_{0 \leq k \leq n} \phi^{-k}(W^S_{g^k}(q)) \quad \text{and} \quad \bigcup_{0 \leq k \leq n} \phi^{k}(W^U_{g^k}(q))
\]

respectively. It is easy to see that \( W^S_{n+1}(q) \supset W^S_n(q), \quad s = u, s. \) There are integers \( n_s, n_u \geq 1 \) such that

\[
x \in W^s_{n_s}(q_1), \quad x \in W^s_{n_s-1}(q_1),
\]

\[
x \in W^u_{n_u}(q_2), \quad x \in W^u_{n_u-1}(q_2).
\]

Now we assert that it is possible to choose sufficiently small disk \( V \) which is the neighbourhood of the point \( x \) in \( M \), \( V \subset U_1 \) and

\[
D^S = V \cap W^S_{n_s}(q_1) \subset W^S_{n_s}(q_1) \cap W^S_{n_s-1}(q_1)
\]

\[
D^U = V \cap W^U_{n_u}(q_2) \subset W^U_{n_u}(q_2) \cap W^U_{n_u-1}(q_2)
\]

where \( D^S \) and \( D^U \) are smoothly embedded disks. Let us choose perio-
dical points \( p_1, p_2 \in \Lambda g \) close enough to points \( q_1 \) and \( q_2 \) for submanifolds \( W_S^s(p_1) \) and \( W_S^u(p_2) \) to be sufficiently close to manifolds \( W_S^s(q_1) \) and \( W_S^u(q_2) \) respectively. Then

\[
\mathcal{D}_i^S = V \cap W_{s}^S(p_i) \subset W_{s}^S(p_i) \setminus W_{s}^S(p_i)
\]

\[
\mathcal{D}_i^\omega = V \cap W_{u}^\omega(p_i) \subset W_{u}^\omega(p_i) \setminus W_{u}^\omega(p_i)
\]

are smoothly embedded disks in \( M \), which are sufficiently close to disks \( \mathcal{D}^S \) and \( \mathcal{D}^\omega \) respectively. Point \( x \) is the point of tangency of \( \mathcal{D}^\omega \) and \( \mathcal{D}^S \) hence there is sufficiently \( C^k \)-close to identity map diffeomorphism \( \omega: M \to M \) such that \( \omega(\mathcal{D}^S) \) and \( \mathcal{D}^\omega \) have a point of tangency \( x_1 \in V \) and \( \omega|_{M \setminus V} = \text{id} \). From the definition of \( V \) one can obtain that \( \omega(\mathcal{D}^S) \subset W^S(p_1, \mathcal{D}^\omega) \) and \( \mathcal{D}^\omega \subset W^\omega(p_2, \mathcal{D}^\omega) \).

Thus \( x_1 \) is the tangency point of \( W^\omega(p_1, \mathcal{D}^\omega) \) and \( W^\omega(p_2, \mathcal{D}^\omega) \).

(ii) Let's show that there is arbitrary small perturbation \( g_\varphi \) of \( g_1 \) in \( \text{Diff}^k(M) \) outside some neighbourhood of \( \Lambda g \) such that manifolds \( W^S(p, \mathcal{D}^\omega) \) and \( W^\omega(p, \mathcal{D}^\omega) \) have a point of tangency \( x_2 \) in an arbitrarily small neighbourhood of point \( x_1 \) where \( p \in \Lambda g \) is periodical point.

For simplicity let's consider the case when \( p_1, p_2, p \) are fixed points. There is the disk \( \mathcal{D}^S_i \subset W^S(p) \) arbitrarily close to \( W^S_{p_1} \) (\( W^S_{p_1} \) is a local stable manifold) because \( p \) is homoclinically related to \( p_1 \). By the same argument there is disk \( \mathcal{D}^\omega \subset W^\omega(p) \) arbitrarily close to \( W^\omega_{p_2} \). Let's choose disk \( V \)

which is a neighbourhood of point \( x_1 \) in \( M \) such that

\[
\forall \cap \bigcup_{n>0} \mathcal{D}^S_i = \varnothing \quad \text{and} \quad \forall \cap \bigcup_{n<0} \mathcal{D}^\omega = \varnothing
\]

Let disks \( \mathcal{D}^S \subset W^S(p_1) \cap V \) and \( \mathcal{D}^\omega \subset W^\omega(p_2) \cap V \) be defined as in (i). Then there are disks \( \mathcal{D}^S_2 \subset \mathcal{D}^S_1 \) and \( \mathcal{D}^\omega_2 \subset \mathcal{D}^\omega_1 \)

such that for some integers \( m^S_2, m^\omega_2 \) disks \( \mathcal{D}^S_2 = q_{m^S_2}^{-1}(\mathcal{D}^S_1) \) and \( \mathcal{D}^\omega_2 = q_{m^\omega_2}^{-1}(\mathcal{D}^\omega_1) \).
and $\mathcal{D}_2^{\surd} \cong \mathcal{D}_1^{\surd}(\mathcal{D}_2^{\surd})$ are arbitrary close to $\mathcal{D}_2$ and $\mathcal{D}_3^{\surd}$, but $\mathcal{D}_1^{\surd}(\mathcal{D}_2^{\surd}) \cap \mathcal{D}_3^{\surd} = \emptyset$ for $0 < i < m-1$ and $\mathcal{D}_1^{\surd}(\mathcal{D}_2^{\surd}) \cap \mathcal{D}_3^{\surd} = \emptyset$ for $0 < i \leq m-1$. Thus one can obtain the perturbation $g_2$ as in (i).

(iii) Now we change $g_2$ to $g_3$ for $\mathcal{W}^{\surd}(\mathcal{P}_3, g_3)$ to have a tangency point $x_3$ with $\mathcal{W}^{\surd}(\mathcal{P}_3, g_3)$ near $x_3$. Finally move $g_3$ to $g'$ to introduce a sink $y$ near $x_3$ using theorem A'. Since the sink $y$ may be got arbitrarily close to a certain disk in $\mathcal{W}^{\surd}(\mathcal{P}, g')$ of fixed diameter (depending only on the position of $x_3$), its orbit under $g'$ will get close to $p$. The proof of lemma 3 is completed.

**Proof of theorem A.** Let $g \in \mathcal{N}_1$ where $\mathcal{N}_1 \subset \mathcal{N}_1$ is a residual subset of $\mathcal{N}_1$ on which maps $H, S: \text{Diff}^K(M) \to \mathcal{X}^\mathcal{M}$ are continuous. Fix some topological metric on $M$. Let $P_n \in \mathcal{M}$ be a finite $\mathcal{M}$-net of compact $H(g)$, consisting of periodic points of $g$, where $\epsilon_n > 0$ converges to zero. $N_\mathcal{M}(g)$ denotes the ball of radius $\epsilon$ with its center in $g$. By the lemma 3, given any $n$ there is a diffeomorphism $g_n \in N_\mathcal{M}(g)$ such that for every $P_\mathcal{M}(g_n), P \in P_n$ there is sink $q \in B_\epsilon(P), q \in S(g_n)$. Then $\text{dist}(S(g_n), P_n) \leq \epsilon_n$.

From the continuity of $H, S$ at the point $g$, $P_n \to H(g)$ as $n \to \infty$ and

$$\text{dist}\left(S(g_n), H(g_n)\right) \leq \text{dist}\left(S(g_n), P_n\right) + \text{dist}\left(P_n, H(g)\right) + \text{dist}\left(H(g), H(g_n)\right)$$

one obtains $H(g) = S(g)$, the set $H(g) = \text{cl}\mathcal{H}_g$ being infinite, so this completes the proof of theorem A.

Now we can give the proof of theorem B. We'll consider the case when infinite number of sources appear. We now proceed to state and prove technical lemmas which will be needed for the
LEMMA 4. Let $\mathcal{D}$ be a disk with smooth boundary in the plane $\mathbb{R}^2$ $f: \mathcal{D} \to \text{int}\mathcal{D}$ and $A: \mathbb{R}^2 \to \mathbb{R}^2$ — orientation preserving diffeomorphisms, $A$ has a single nonwandering point $a$, which is a sink $a \in \text{int}(\mathcal{D})$. Then there are disks $\mathcal{D}_1, \mathcal{D}_2$ and diffeomorphism $F: \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\mathcal{D}_2 \ni \mathcal{D} \ni \mathcal{D}_1 \ni f(\mathcal{D}),$$

$$F|_{\mathbb{R}^2 \setminus \mathcal{D}_2} = A, \quad F|_{\mathcal{D}_1} = f$$

and $\nu(F) = \nu(F)$

( here $\cup \subseteq V$ means $\text{clos}U \subseteq \text{int}V$)

Proof. (i) Let's find disks $\mathcal{D}_1$ and $\tilde{\mathcal{D}}$ with smooth boundaries $\mathcal{D}_1 \ni \tilde{\mathcal{D}} \ni \mathcal{D}_1 \ni f(\mathcal{D})$ and extension $\tilde{f}$ of $f$ on the whole plane such that $\tilde{f}: \mathbb{R}^2 \to \text{int} \tilde{\mathcal{D}}$, $\tilde{f}|_{\mathcal{D}_1} = f|_{\mathcal{D}_1}$ and $\nu(\tilde{f}) = \nu(f)$. Consider the diffeomorphisms $h: \mathbb{R}^2 \to \text{int} \tilde{\mathcal{D}}$ which is equal to the identical on $\mathcal{D}_1$ and set $\tilde{f} = f \circ h$. It is obvious that any point $x$ from $\mathbb{R}^2 \setminus \mathcal{D}_1$ is wandering because $\tilde{f}(x) = f \circ h(x) \in f(\mathcal{D}) \subset \mathcal{D}_1$ and $\tilde{f}(\mathcal{D}_1) \subset \text{int} \mathcal{D}_1$.

Hence $\nu(\tilde{f}) \subset \mathcal{D}_1$ and $\nu(f) = \nu(f)$ because $\tilde{f}|_{\mathcal{D}_1} = f|_{\mathcal{D}_1}$.

Using the properties of $A$ one can find disk $\mathcal{D}_2 \ni \tilde{\mathcal{D}}$ such that $\mathcal{D} \in A(\mathcal{D}_2) \subset \mathcal{D}_2$. It is a consequence of the facts that $a$ is a sink for $A$ and thus $A$ is topologically conjugate to linear map with eigenvalues less than 1 and more than $-1$.

Thus there is disk $\mathcal{D}'$ such that $A(\mathcal{D}') \subset \text{int} \mathcal{D}'$. From the relation $W_\mathcal{D}(a,A) = \mathbb{R}^2$ one can obtain some $n \geq 0$ with the property $A^{-n}(\mathcal{D}') \ni \mathcal{D}$, and we set $\mathcal{D}_2 = A^{-n}(\mathcal{D}')$. Let $\tilde{\mathcal{D}}_2$ be a disk $\mathcal{D}_2 \ni \tilde{\mathcal{D}}_2 \ni A \mathcal{D}_2$ and $K, \tilde{K}$ are the rings $\mathcal{D}_2 \setminus \text{int} \tilde{\mathcal{D}}_2$ and $A(\mathcal{D}_2) \setminus \text{int} f(\mathcal{D})$. To prove the lemma it remains to extend diffeomorphism $\tilde{F}: \mathbb{R}^2 \setminus K \to \mathbb{R}^2 \setminus \tilde{K}$ which is
given by the formula

$$ F = \begin{cases} Ax, & x \in \mathbb{R}^2 - \mathcal{D}_2 \\ \frac{1}{\rho} \{ x \}, & x \in \mathcal{D}_2 \end{cases} $$

to the diffeomorphism $F$ of the whole plane so that $F(K) = \widehat{K}$ then $F$ satisfy to the conditions of the lemma, because $F(K) = \mathcal{K}$, $K \cap \mathcal{K} = \emptyset$ and hence all points of $\mathbb{R}^2 - \mathcal{D}_2$ are wandering; thus

$$ \mathcal{N}(F) = \mathcal{N}(\mathcal{K}) = \mathcal{N}(\mathcal{D}_2) $$

(ii) In order to define $F$ we shall consider the standard ring on $\mathbb{R}^2$

$$ T = \{ (\rho, \theta) \mid \rho_0 \leq \rho \leq \rho, \quad 0 \leq \theta < 2\pi \} $$

where $(\rho, \theta)$ are polar coordinates. Let

$$ S_1 = \{ (\rho, \theta) \mid \rho = \rho_0 \} $$

$$ S_2 = \{ (\rho, \theta) \mid \rho = R \} $$

and map some neighbourhoods $U(K)$ and $U(\widehat{K})$ onto the neighbourhood $U(T)$ of $T$ using diffeomorphisms $h_1, h_2$ so that

$$ h_1(K) = h_2(\mathcal{K}) = T $$

$$ h_1(\partial \mathcal{D}_2) = h_2(\partial A(\mathcal{D}_2)) = S_1 $$

$$ h_1(\partial \mathcal{D}_2) = h_2(\partial \rho \{ x \} (\mathcal{D}_2)) = S_2 $$

Suppose we shall find a diffeomorphism $g : U(T) \to U(T)$ such that

$$ g = h_2 \circ A \circ h^{-1}_1 $$

on $\{ (\rho, \theta) \mid \rho \geq \rho_0 \}$ and

$$ g = h_2 \circ \frac{1}{\rho} \circ h^{-1}_1 $$

on $\{ (\rho, \theta) \mid \rho \leq \rho_0 \}$ that

$$ F(x) = \begin{cases} Ax, & x \in \mathbb{R}^2 - \mathcal{D}_2 \\ h^{-1}_2 \circ g \circ h_1(x), & x \in K \\ \frac{1}{\rho} \{ x \}, & x \in \mathcal{D}_2 \end{cases} $$
will be a extension of \( P \) that we need. In order to find diffeomorphism \( g \) it is sufficient to find such diffeomorphisms \( w \) and \( w' \) that
\[
\begin{align*}
    w \mid \{ (p, y) \mid p \geq r \} &= A, \\
    w \mid \{ (p, y) \mid p \leq r \} &= \text{id}, \\
    w' \mid \{ (p, y) \mid p \geq r' \} &= \text{id}, \\
    w' \mid \{ (p, y) \mid p \leq r' \} &= \text{id},
\end{align*}
\]
for some \( r \) and \( r' \), \( R_0 < r' < r < R \) and let
\[
    g(p, y) = \begin{cases} 
        w(p, y), & p > r' \\
        w'(p, y), & p < r
\end{cases}
\]
The definition of \( w' \) is the same as that of \( w \) and we shall give one for \( w \) only.

(iii) Let \( h_2 \circ A \circ h_1^{-1} (p, y) = \Psi (p, y) = \\
\begin{cases} 
    \Psi_1 (p, y), & y \geq S_1 \\
    \Psi_2 (p, y), & y < S_1
\end{cases}
\]
where \( \Psi (S_1) = S_1 \) that is \( \Psi_1 (R, y) = R \). \( \Psi \) is orientation preserving hence \( \frac{\partial \Psi_2}{\partial y} (R, y) > 0 \) and there is \( \eta > 0 \) such that
\[
\det \frac{\partial \Psi}{\partial y} (p, y) > \eta, \quad R_0 \leq p \leq R.
\]
Let \( L = \max_{x \in T} \left| \frac{\partial \Psi_2}{\partial y} (x) \right| \),
\[
M = \max_{x \in T} \left| \frac{\partial \Psi_1}{\partial y} (x) \right|, \quad x = (p, y)
\]
and \( 0 < \varepsilon < \eta / 2 \max (LM, L) \). Take \( r \) from the \( \left( \frac{R + R_0}{2}, R \right) \) so close to \( R \) that there is \( \delta > 0 \), for which
\[
\left| \frac{\partial \Psi_1}{\partial y} (p, y) \right| < \varepsilon \quad \text{and} \quad \left| \frac{\partial \Psi_2}{\partial y} (p, y) \right| > \delta \quad \text{when} \quad p \neq R.
\]
The first condition is satisfied, because \( \frac{\partial \Psi_1}{\partial y} (R, y) = 0 \) and the second - because \( \frac{\partial \Psi_2}{\partial y} (R, y) > 0 \) at last
\[
\left| \frac{\partial \Psi_1}{\partial y} \frac{\partial \Psi_2}{\partial y} \right| \geq \eta + \left| \frac{\partial \Psi_1}{\partial y} \frac{\partial \Psi_2}{\partial y} \right| \geq \eta - \left| \frac{\partial \Psi_1}{\partial y} \left| \frac{\partial \Psi_2}{\partial y} \right| \right| > \\
\geq \eta - \varepsilon \cdot L > \frac{\eta}{2}
\]
hence \( \frac{\partial \Psi}{\partial \psi} > \frac{r}{\partial M} \) when \( r < p \leq R \). Choose constants \( \alpha > 0 \) and \( \tau > 1 \) so that \( \alpha \leq \frac{\partial \Psi}{\partial \psi} (R, \psi) \leq \tau \). Let

\[
\alpha (\psi) = \frac{1}{\tau} \frac{\partial \Psi}{\partial \psi} (R, \psi) \quad \text{then} \quad \frac{\alpha}{\tau} \leq \alpha (\psi) \leq 1 .
\]

Let us define a smooth function \( \beta (\psi, \psi) \) such that

\[
\Psi (\psi, \psi) = \Psi (R, \psi) + \frac{\partial \Psi}{\partial \psi} (R, \psi) (\psi - R) + \beta (\psi, \psi) (\psi - R)^2,
\]

\[
= R + \tau \alpha (\psi) (\psi - R) + \beta (\psi, \psi) (\psi - R)^2.
\]

Let's divide segment \((r, R)\) by points \( r = r_{i0} < r_{40} < r_{30} < r_{21} < r_{11} < r_{10} = R \)
on segments \( \tau_i, \Delta_3, \tau_3, \Delta_2, \tau_2, \Delta_1, \tau_1 \) so that

\[
\tau_{40} - \tau_{10} > \frac{\tau - 1}{\tau} (R - r).
\]

For any \( i = 1, 2, 3 \) we shall define the nondecrease \( C^\infty \) function \( \lambda_i (\psi) \) with the following properties

\[
1^o \quad \lambda_i (\psi) = 0 \quad , \quad \psi \leq r_{i0} ; \quad \lambda_i (\psi) = 1 \quad , \quad \psi > r_{i0} ;
\]

\[\frac{\partial \lambda_i}{\partial \psi} (\tau_i) = 0 \quad \forall \quad \psi \in \tau_i
\]

\[0 \leq \frac{\partial \lambda_i}{\partial \psi} (\tau_i) \leq \frac{1 + \tau_i}{\tau_i + \tau_i} = \frac{\lambda_i}{\tau_i - r}
\]

where \( \lambda_i = (1 + \tau_i) \frac{R - r}{r_{i0} - r_{i1}} \), \( \tau > 0 \).

Let's define the increasing \( C^\infty \) function \( \mu (\psi) \) such that \( \mu (\psi) = \psi \)

for \( \psi < r \) and \( \mu (\psi) = r \psi + R (1 - \tau) \quad \text{for} \quad \psi > r_{40} \)

The existence of such a function follows from relations

\[
\mu (r_{40}) = r \quad r_{40} + R (1 - \tau) > r \quad r + (\tau - 1) (R - r) +
\]

\[
+ R (1 - \tau) = r = \mu (r_{41})
\]

Let's define following maps

\[
\Psi^1 (\psi, \psi) = \left( R + \tau \alpha (\psi) (\psi - R) + \lambda_i (\psi) \beta (\psi, \psi) (\psi - R)^2, \Psi_i (\psi, \psi) \right)
\]

\[
\Psi^2 (\psi, \psi) = \left( R + \tau \left[ 1 - \lambda_2 (\psi) (\alpha (\psi) - 1) \right] (\psi - R), \Psi_2 (\psi, \psi) \right)
\]
\[ \psi^3 (\rho, \gamma) = \left( R + \tilde{\alpha} (\rho - R), \psi + \lambda_3 (\rho) \left( \psi_2 (\rho, \gamma) - \phi \right) \right) \]

\[ \psi^4 (\rho, \gamma) = \left( \rho (\rho), \gamma \right) \]

It is easy to see that \( \psi' = \psi \) outside the circle of radius \( R \)
\( \psi^4 = \text{id} \) in the circle of radius \( r \) and \( \psi^i (\rho, \gamma) = \psi^{i+1} (\rho, \gamma) \)
when \( \rho \in \Delta_i \). Thus the lemma will be proved if we find that
for \( i=1,2,3,4 \) \( \psi^i \) is a diffeomorphism on \( \{ (\rho, \gamma) \mid \tau \leq \rho \leq R \} \)
when \( R-r \) is sufficiently small.

\( (iy) \) \( \psi^4 \) is a diffeomorphism because the function \( \rho (\rho) \) is monotonous. If \( R-r \) is sufficiently small then the degree of \( \psi^i \)
\( (i=1,2,3) \) is equal to that of \( \psi' \) which is 1. Thus to prove
the lemma it remains to prove that \( \det \frac{D(\psi^i, \gamma)}{D(\rho, \gamma)} \neq 0 \) when \( \tau \leq \rho \leq R \).
and \( R-r \) is sufficiently small. Indeed,
\[ \frac{\partial \psi_1}{\partial \rho} = \tau \alpha (\gamma) + \lambda_1 (\rho) \beta (\rho, \gamma) (\rho - R)^2 + \lambda_1 (\rho) \frac{\partial \beta}{\partial \rho} (\rho - R)^2 + \]
\[ + 2 \lambda_1 (\rho) \beta (\rho, \gamma) (\rho - R) \equiv \delta + O (\rho - R) , \]
\[ | \lambda_1 (\rho) \beta (\rho, \gamma) (\rho - R)^2 | < K_1 | \beta (\rho, \gamma) | | \rho - R | = O (\rho - R) \]
then
\[ \frac{\partial \psi_2}{\partial \rho} = \tau \alpha' (\gamma) (\rho - R) + \lambda_1 (\rho) \frac{\partial \beta}{\partial \rho} (\rho - R)^2 = O (\rho - R) \]

Thus
\[ \det \frac{D(\psi_1, \psi_2)}{D(\rho, \gamma)} = \frac{\partial \psi_1}{\partial \rho} \frac{\partial \psi_2}{\partial \rho} - \frac{\partial \psi_1}{\partial \gamma} \cdot \frac{\partial \psi_2}{\partial \rho} \geq \]
\[ \equiv \delta \frac{\partial \psi_2}{\partial \rho} + O (\rho - R) \geq \delta \delta + O (\rho - R) \quad (1) \]

Further
\[ \frac{\partial \psi_2}{\partial \rho} = \tau \left[ 1 + \lambda_2 (\rho) (\alpha (\gamma) - 1) \right] + \]
\[ + \tau \lambda_2 (\rho) (\alpha (\gamma) - 1) (\rho - R) \geq \tau \left[ 1 + \lambda_2 (\rho) (\alpha (\gamma) - 1) \right] = \]
\[ \tau (1 - \lambda_2 (f)) + \tau \lambda_2 (f) \alpha (y) \geq \tau - \tau \lambda_2 (f) + 2 \lambda_2 (f) = 0 \]

\[ \tau + \tau (1 - \lambda_2 (f)) - 2 (1 - \lambda_2 (f)) = 0 \]

\[ \tau + (\tau - 2) (1 - \lambda_2 (f)) \geq 2 \quad \frac{\partial \psi_2}{\partial y} = -\tau \lambda_2 (f) \omega '(f, R) = 0 (f - R) \]

\[ \text{hence} \quad \det \frac{\partial (\psi_2^2, \psi_2^3)}{\partial (f, y)} = \frac{\partial \psi_2^2}{\partial y} + \frac{\partial \psi_2^3}{\partial y} = 0 \]

\[ \text{(2)} \]

\[ \text{Finally, from} \quad \frac{\partial \psi_3^3}{\partial y} = 0 \quad \text{one obtains} \]

\[ \det \frac{\partial (\psi_3^2, \psi_3^3)}{\partial (f, y)} = \frac{\partial \psi_3^2}{\partial f} + \frac{\partial \psi_3^3}{\partial y} = (1 - \lambda_3 (f)) + \lambda_3 (f) \delta > 0 \]

\[ \text{(3)} \]

From (1), (2), (3) it follows that \( \psi_1^1, \psi_2^2, \psi_3^3 \) are diffeomorphisms when \( B - r \) is small enough. The lemma is proved.

**Lemma 5.** Given any \( C^0 \)-neighbourhood \( U \) of any diffeomorphism \( f \) \( \text{Diff}^r_{x} (M) \) for a set \( \Sigma \) of \( n \) different positive values distinguished from 1, there is a diffeomorphism \( f_{1} \in U \) having periodical point \( p \) in whose neighbourhood \( f_{1} \) is smoothly conjugate to the linear map with a real spectrum and the set of absolute values of elements from this spectrum is \( \Sigma \).

**Proof.** (i) \( M \) is compact hence there is a nonwandering point of \( f \). Then there is arbitrarily small approximation \( f_{1} \) of \( f \) such that \( f_{1} \) has periodical point \( p \). We may assume for simplicity that \( f_{1} \) has a fixed point \( p \).

(ii) We shall show that there is arbitrarily \( C^1 \)-close to \( f \) diffeomorphism \( f_{1} \) with the derivative \( (Df_{1})_{p} : R^n \rightarrow R^n \) having a simple spectrum. In some coordinate neighbourhood of \( p \) \( f_{1} \),
has the following form \( f_r(x) = Ax + \phi(x) \) where \( \phi \in C^r_0 \), \( \phi(0) = 0 \) and \((D\phi)_0 = 0\), \( A \) is a real Jordan form of operator \((Df)_p\). Let \( A = \text{diag}(A_1, \ldots, A_s, B_1, \ldots, B_t) \), where \( A_m \) is Jordan cell corresponding to the real eigenvalue multiple \( n_m = 1, \ldots, s \), \( B_j \) is a cell corresponding to complex eigenvalue \( \lambda_j + i \eta_j \) multiple \( m_j \), \( j = 1, \ldots, t \). Let \( \epsilon > 0 \) be sufficiently small and real values \( \epsilon_{m, k} (m = 1, s; k = 1, n) \) and \( \delta_j \epsilon \) \( (j = 1, t; \epsilon = 1, m_j) \) from the interval \((0, \epsilon)\) be such that all values \( |\lambda_m + \epsilon_{m, k}|, |\delta_j + i \eta_j| \) are different. Let \( E_m = \text{diag}(\epsilon_{m, 1}, \ldots, \epsilon_{m, n_m}) \), \( F_j = \text{diag}(\delta_j, \delta_j, \ldots, \delta_j) \), \( \epsilon_{j, k} = \delta_{j, k} \) and \( \epsilon = \text{diag}(E_1, \ldots, E_s, F_1, \ldots, F_t) \). Let \( r(t), t \in \mathbb{R} \) be monotonous \( C^r \)-function with properties

(a) \[ r(t) = \begin{cases} 1 & \text{when } t < \frac{\epsilon}{2} \\ 0 & \text{when } t > \epsilon \end{cases} \]

(b) \[ |r'(t)| < \frac{3}{\epsilon} \]

Let \( \hat{f}_2(x) = f_1(x) + r(1) \xi \), then \( \text{spec}(Df_2) = \{ \lambda_m + \epsilon_{m, k}, \delta_j + i \eta_j \} \) and \( |f_1(x) - \hat{f}_2(x)| \leq |r(1) \xi| E \xi| = 0 \) when \( |x| > \epsilon \), \( |f_1(x) - \hat{f}_2(x)| \leq |E \xi| \leq ||E|| |x| \leq ||E|| \epsilon \) when \( |x| < \epsilon \), one can obtain that \( |f_1(x) - \hat{f}_2(x)| \) is sufficiently small if \( \epsilon \) is small enough. Consider

\[ \left| \frac{\partial \hat{f}_2}{\partial \lambda_j} - \frac{\partial f_2}{\partial \lambda_j} \right| = \begin{cases} \left| \mu'(|x|) \xi_j + \mu(\xi) \xi_j \right| \leq 3 \epsilon & \text{when } |x| \leq \epsilon \\ 0 & \text{when } |x| > \epsilon \end{cases} \]

hence when \( \epsilon \) is small enough \( \det(D\hat{f}_2(x)) \neq 0 \), thus \( f_2 \) is diffeomorphism sufficiently \( C^r \)-close to \( f_1 \). Choosing values \( \epsilon_{m, k} \) and \( \delta_j \) by the some special way one can obtain apply Sternberg's linearization theorem and thus we may assume that \( f_2(x) = Ax \) where \( A \) has a simple spectrum.
(iii) Let's show that there is arbitrarily \( C \) - small perturbation \( f \) of \( f \) such that \( (D_{f,s})_p \) has a real spectrum.

Consider the invariant subspace \( L \) corresponding to the complex eigenvalue \( \sigma + i\tau \) of matrix \( A \), then

\[
B(x) = A \bigg|_L = \left( 2x_1 + \tau x_2, -\tau x_1 + \tau x_2 \right)
\]

In polar coordinates the map \( B \) has the form \( B(f, \Psi) = (\alpha f, \Psi - \lambda(t) \Psi) \), where \( \alpha = \sqrt{\lambda^2 + \tau^2}, \quad t_0 \Psi = \frac{\tau}{\alpha}. \) Let \( \varepsilon > 0 \) and \( \lambda(t) \) be a nondecreasing smooth function, which is equal to 0 when \( t < \frac{\varepsilon}{\lambda} \) and 1 when \( t > \varepsilon \). Let \( B_1(f, \Psi) = (\alpha f, \Psi - \lambda(f) \Psi) \), it is easy to see that \( B_1 \) is diffeomorphism because \( \det \frac{D_B(\beta_1, \beta_2)}{D_B(\beta_1, \beta_2)} = \alpha \), when \( f < \varepsilon \) we have \( B_1(f, \Psi) = (\alpha f, \Psi) \) or in previous coordinates \( B_1 x = \alpha x \). Extend \( B_1 \) on the whole subspace so that outside some neighbourhood of the disk \( \{ x \in L \mid |x| < \varepsilon \} \) it is equal to \( f \). We shall do the same procedure with other invariant subspaces corresponding to the complex values. Then using the procedure of (ii) one can obtain diffeomorphism \( f_3 \) which is smoothly equivalent to the linear map with a simple real spectrum in some neighbourhood of \( p \).

(iv) Let \( L \) be an one-dimensional invariant subspace corresponding to eigenvalue \( \lambda \) of the operator \( A \) with the simple real spectrum. For any real \( t \) with \( \text{sgn} t = \text{sgn} \lambda \) and any \( \varepsilon > 0 \) there is \( \delta \in (0, \varepsilon) \) and strictly monotonous smooth function \( \sqrt{t}, t \in \mathbb{R} \) such that \( \sqrt{t} = t \) when \( t < \delta \) and \( \sqrt{t} = t \) when \( t > \varepsilon \). Then using arbitrary small \( C^0 \)-perturbation \( \forall \) one can obtain that \( f_3 \bigg|_L = \sqrt{t} \) in some neighbourhood of \( p \), but other eigenvalues of \( (D_{f_3})_p \) are the same. Lemma is proved.

**Lemma 6.** Let \( f_i : E_i \to E_i \) \((i=1, 2)\) be \( C^1 \)-diffeomorphisms of Banach spaces \( E_i \) such that
(a) \( f_1 \) is Lipschitz and for some \( c_1 > 0 \)
\[ f_1 (E_1(c_1)) \subset \text{int } E_1(c_1) \]

(b) \( 0 \in E_2 \) is a fixed point for \( f_2 \).

Let \( L(f_i^{-1}) \) denotes the Lipschitz constant of \( f_i^{-1} \) where \( E_2 \) is a fixed point for \( f_2 \).

Then for any \( \varepsilon > 0 \) there is \( \delta > 0 \) with the following property. If \( L(f_i^{-1}) < \delta \) then there is a neighbourhood \( U \) of \( E_i \) of diffeomorphism \( f_i : E_i \to E_i \) in \( \text{Diff}'(E_i, E_i) \)
such that for any \( F \in U \) there is a map \( \varphi : E_i(c) \to E_i \)
with Lipschitz constant less than \( \varepsilon \) and graph \( \varphi \) is invariant under \( F \).

**Proof.** (i) For definiteness we shall suppose that \( E_i \) has
the norm \( ||(x, y)|| = \max \{ |x|, |y| \} \).

Let \( f(x, y) = (f_1(x), f_2(y)) \), \( x \in E_i \), \( y \in E_i \),
\( F(x, y) = (f_1(x) + \delta_1(x, y), f_2(y) + \delta_2(x, y)) \)
where \( \delta_i : E_i x E_i \to E_i \) are \( C^1 \) maps and
\[ \Delta = \max \{ ||\delta_1||_{C^1}, ||\delta_2||_{C^1} \} < 1 \]

Consider the space of continuous maps from \( E_i(c) \) in \( E_i(c) \)
with Lipschitz constant less than 1. We shall denote this space by \( \mathcal{H} \)
and let \( \mathcal{H} \) have uniform topology. \( \mathcal{H} \) is a complete metric
space. We define the graph transform \( G_F : \mathcal{H} \to \mathcal{H} \) by
\[ G_F(\varphi) = f_2^{-1} \circ \left[ \varphi \circ F \circ (1, \varphi) - \delta_2 \circ (1, \varphi) \right] \]

Let's prove that \( G_F \) is well defined map \( \mathcal{H} \to \mathcal{H} \) when \( \Delta \) is \( C^1 \)-small. First we must show that \( G_F(\varphi) \) is defined for any
point of \( E_i(c) \). This is obvious because
\[ f_i(x, y_\alpha) = f_1(x) + \delta_1(x, y_\alpha) \]
and \( f_i \) maps the ball \( E_i(c) \) into \( \text{int } E_i(c) \) thus for any \( x \in E_i(c) \)
Let's evaluate the Lipschitz constant of $\mathcal{F}(\varphi)$

$$L(\mathcal{F}(\varphi)) = L(f_2^{-1} \circ F_2 \circ (I, \varphi) - \delta_2 \circ (I, \varphi)) \leq L(f_2^{-1}) L(\varphi \circ F_1 \circ (I, \varphi) - \delta_1 \circ (I, \varphi)) \leq$$

$$\leq \delta [L(\varphi) L(F_1) L(I, \varphi) + L(\delta_1) L(I, \varphi)] \leq \delta [L(F_1) + L(\delta_2)] \leq$$

$$\leq \delta [L(f_1) + L(\delta_1) + L(\delta_2)] \leq \delta [L(f_1) + 2 \Delta]$$

hence $L(\mathcal{F}(\varphi)) < 1$ when $\delta$ is sufficiently $C'$-small. Thus $\mathcal{F}$ is well defined map of $\mathcal{H}$ into itself.

We shall show that this map is a contraction when $\delta$ is sufficiently $C'$-small. Let $\varphi, \psi \in \mathcal{H}$ then

$$\| \mathcal{F}(\varphi) - \mathcal{F}(\psi) \| \leq \delta \| \varphi \circ F_1 \circ (I, \varphi) - \delta_1 \circ (I, \varphi) - \varphi \circ F_1 \circ (I, \psi) + \delta_1 \circ (I, \psi) \| \leq$$

$$\leq \delta [L(\varphi) \| \varphi - \psi \| + L(F_1) \| \varphi \circ F_1 \circ (I, \varphi) - \varphi \circ F_1 \circ (I, \psi) \| + L(\delta_1) \| \varphi \circ F_1 \circ (I, \psi) - \varphi \circ F_1 \circ (I, \psi) \|] \leq$$

$$\leq \delta [\Delta \| \varphi - \psi \| + L(\mathcal{F}_1) \| \varphi - \psi \| + L(\delta_1) \| \varphi - \psi \|] \leq$$

$$\leq \delta [1 + 2 \Delta + L(\mathcal{F}_1)] \| \varphi - \psi \|$$

and one can obtain that $L(\mathcal{F}) < 1$ as $\delta$ is $C'$-small. Thus there is the unique fixed point $\mathcal{F}$ of $\mathcal{F}$ by the contracting map theorem. $\mathcal{F}$ satisfies the equation

$$f_2^{-1} \circ [\mathcal{F}_1 \circ F_1 \circ (I, \varphi) - \delta_2 \circ (I, \varphi)] = \varphi$$

hence

$$\mathcal{F}_1 \circ F_1 \circ (I, \varphi) = F_2 \circ (I, \varphi)$$

which implies the invariance of $gr \mathcal{F}$ under $F$:

$$F(x, \varphi, y) = (y, \varphi, y)$$

for any $x \in E_i(\mathfrak{h})$ where $y = F_i(x, \varphi, y)$

(ii) Next we investigate the differentiability of $\mathcal{F}$. Assume $\mathcal{G}$ is the space of continuous bounded maps $h : E_i(\mathfrak{h}) \rightarrow L(E_i, E_i)$ such that $\|h(x)\| \leq 1$ for any $x \in E_i(\mathfrak{h})$ where $L(E_i, E_i)$ is the
space of continuous linear maps acting from $\mathbb{K}$ to $\mathbb{R}^2$. The map

$$\Gamma^*_F: \mathbb{K} \times \mathcal{Y} \to \mathbb{K} \times \mathcal{Y}$$

is well defined by the formula

$$\Gamma^*_F(\varphi, h) (\omega) = (\Gamma_F(\varphi(\omega), D\varphi(\omega)) \{ h(\varphi) D \varphi - D_{\varphi z} \delta_z \} (1, h), h)$$

where $\delta_z = (\varphi, h) \times (\varphi, h)$, if $L(\lambda)$ is small enough. Let $\pi_2: \mathbb{E}_z \times \mathbb{E}_z \to \mathbb{E}_z$ be projection.

We shall show that $\pi_2 \Gamma^*_F(\varphi, h): \mathcal{Y} \to \mathcal{Y}$ is a contraction. For any $h, h_2 \in \mathcal{Y}$ we have

$$\| \pi_2 \Gamma^*_F(\varphi, h) (\omega) - \pi_2 \Gamma^*_F(\varphi, h_2) (\omega) \| \leq$$

$$\leq \| D_{\varphi z} \delta_z (1, h_2(z)) \| \leq L(\lambda)(\| h_2(z) D \varphi - D_{\varphi z} \delta_z (1, h_2(z)) -$$

$$- D_{\varphi z} \delta_z (1, h_2(z)) \| \leq L(\lambda)(\| h_2(z) D \varphi - D_{\varphi z} \delta_z (1, h_2(z)) -$$

$$- h_2(z) D \varphi - (1, h_2(z)) \| + \| D_{\varphi z} \delta_z (1, h_2(z) - h_2(z)) \| \leq$$

$$\leq \delta (\| h_2(z) \| D \varphi + \| h_2(z) \| h_2(z) - h_2(z)) \| + \| h_2(z) - h_2(z) \| \leq$$

$$\| \pi_2 \Gamma^*_F(\varphi, h) (\omega) - \pi_2 \Gamma^*_F(\varphi, h_2) (\omega) \| \leq$$

Thus $\Gamma^*_F$ is a contraction on the second coordinate provided $\delta$ is small. Therefore, by the fiber contraction theorem $[\mathcal{Y}]$ there is an unique attractively fixed point $\Gamma^*_F, h_F$ of $\Gamma^*_F$. For smooth map $\varphi, (\varphi, D \varphi) \in \mathcal{Y} \times \mathcal{Y}$ and

$$\Gamma^*_F(\varphi, D \varphi) = \Gamma_F \varphi, D \Gamma_F(\varphi)$$
it follows from the definition of $F^*$. Indeed
\[ H \circ F^*(\psi, D\psi)(x) = D_1 \left( f^* \right) \left\{ D_{f^*} \psi - D_{f^*} \delta_1 \right\} (1, D_x \psi) = \]
\[ D_x \left[ f^* \left( f^* \left( f^* \right) \right) - \delta_2 \left( f^* \right) \right] = D_x \left( f^* \right) (1, D_x \psi) \]
converges to $(\psi_F, h_F)$ as $n \to \infty$. Thus $\psi_F$ is the limit of the sequence $F^n$, and $h_F$ is the limit of the sequence of their derivatives $D [ F^n ]$, hence $D \psi_F = h_F$ (see [6]).

(iii) To prove the lemma it remains to show that $\psi_F$ is $C$-close to 0 when $\Delta = \max \left\{ |\psi_F(x)| \right\}$ is small. For any $x \in E_1(\tau)$ we have
\[ |\psi_F(\infty)| = |F^* (\psi_F(\infty))| = |f^* \left( f^* (x, \psi_F(\infty)) + \delta_1 (x, \psi_F(\infty)) \right) - \]
\[ |f^* (x)| = |f^* \left( f^* (x, \psi_F(\infty)) + \delta_1 (x, \psi_F(\infty)) \right) - \]
\[ \Delta \left( f^* \right) \left( f^* (x) + \Delta \right), \]
so that
\[ |\psi_F(\infty)| \leq L \left( f^* \right) \left( f^* (x) \right) \left( \max_{y \in E_1(\tau)} |\psi_F(y)| + \Delta \right) \]
and
\[ \sup_{x \in E_1(\tau)} |\psi_F(\infty)| \leq \frac{\Delta}{1 - L \left( f^* \right)} \]
Further, for any $x \in E_1(\tau)$
\[ \| D_x \psi_F \| = \| H \circ F^* \| (\psi, D\psi)(x) \| \leq \]
\[ \leq \| D_x \psi_F \| \left\{ \| D_{f^*} \psi \| + \| D_{f^*} \delta_1 \| + \| D_{f^*} \delta_2 \| \right\} \]
\[ + \left\{ \| D_{f^*} \psi \| \right\} + \| D_{f^*} \psi \| \| \psi_F \| \leq \]
\[ \leq L \left( f^* \right) \left( \max_{y \in E_1(\tau)} \| D_{f^*} \psi \| + \Delta \right) \sup_{x \in E_1(\tau)} \| D_x \psi_F \| + \Delta \]
\[ \sup_{x \in E_1(\tau)} \| D_x \psi_F \| \leq \frac{\Delta}{1 - [\Delta + L \left( f^* \right)] L \left( f^* \right)} \]

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as
\[ L(f_2) < \frac{1}{\Lambda + L(f_1)} \]

And this completes the proof of the lemma.

Proof of theorem B.

(1) Let \( M \) be a smooth manifold, \( \text{dim} M > 3, f: M \to M \) be a \( C^r \)-diffeomorphism, \( r \geq 2 \). Using the lemma 5 one obtains that for any \( C^\infty \)-neighbourhood of \( f \) there is diffeomorphism \( f' \) in this neighbourhood such that

1) \( f' \) has periodic point \( p \) of period \( \nu \),
2) in local coordinates of some neighbourhood of \( p \) \( f' \) has the form
\[ (f')^\nu(x, y) = (A_1x, Ay), \quad x \in \mathbb{R}^2, \quad y \in \mathbb{R}^{n-2} \]

where \( A_1: \mathbb{R}^2 \to \mathbb{R}^2, A: \mathbb{R}^{n-2} \to \mathbb{R}^{n-2} \) are linear maps and
\[ \| A_1 \| < 1 \quad \text{while} \quad \| A \| > 1 \]

is sufficiently large.

A point of \( \mathbb{R}^n \) we shall denote in two ways; first \((x, y)\) where \( x \in \mathbb{R}^2, y \in \mathbb{R}^{n-2} \), and the second \((x_1, x_2, y)\) where \( x_1, x_2 \in \mathbb{R}^2 \). In other words, we shall use two decompositions of \( \mathbb{R}^n \) is direct sum
\[ \mathbb{R}^n = \mathbb{R}^2 \oplus \mathbb{R}^{n-2} = \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}^{n-2}. \]

Further for simplicity we shall suppose that the point \( p \) is a fixed point \((\nu = 1)\).

There is an arbitrarily small \( C^\infty \)-perturbation \( f'' \) of \( f' \) which has a form \( f''(x, y) = (g(x), Ay) \) in a neighbourhood of the point \( p \) which is smaller than one described in 2). Diffeomorphism \( g: D^2 \to \text{int} D^2 \) where \( D^2 \) is two-dimensional disk, has the following properties:
1) \( g \) has the basic set \( \Lambda \subset \text{int} D^2 \), \( \Lambda \) is a sink;
2) \( g \) is absolutely structurally stable;
3) \(0 \in \Lambda\) is the fixed point of \(g\) and in some two-dimensional disk \(0 \in B \subset \text{int } D\) \(g\) has the form \(g(x_1, x_2) = (\mu x_1, \lambda x_2)\), where \(0 < \mu < 1\), \(\lambda > 1\), \(\mu > 1\).

Examples of diffeomorphisms with such properties are given in [10] and in \(\Lambda\) at the beginning of the present paper.

Further we shall denote \(f^\prime\) by \(f\). Thus \(\Lambda\) is the basic set of
diffeomorphism \(f\) (we don’t make distinction between the sets
\(\Lambda \subset \mathbb{R}^2\) and \(\Lambda \times \{0\} \subset \mathbb{R}^n, 0 \in \mathbb{R}^{n-2}\)) and the segment \(\mathcal{J}\) of the
line \(\{(x_1, x_2, y) | x_2 = 0, y = 0\}\) lying in the disk \(B\) is in \(\Lambda\).

If \(D^{n-2}\) is sufficiently small neighboorhood of \(0 \in \mathbb{R}^{n-2}\) then the
set \(\mathcal{J} \times D^{n-2}\) is local unstable manifold of point 0. Every seg­
ment \(\mathcal{J}^\prime\) of the line \(\{(x_1, x_2, y) | x_2 = t, y = 0\}\) lying in \(B\)
is local stable manifolds of the point \(Z_\varepsilon = (0, 0, 0) \in \mathcal{J}\). Thus
the disk \(B\) is fibered by the family of the segments \(\mathcal{J}^\prime\) of
stable manifolds of points from \(\Lambda\).

(ii) Now we shall apply the construction due to C. Simon.

We shall describe it in \(\mathbb{R}^n\) keeping in mind that it takes place
in a small coordinate neighboorhood of the point \(p \in M\).

Let \(U\) be a sufficiently small neighboorhood of \(\Lambda\) in \(D^2\) such that
\(g(U) \subset \text{int } U\) ; \(W_1\) - two-dimensional subdisk of \(B\) lying
in the set \(g^{-1}(U) \setminus U\). Consider \((n-1)\)-dimentional subdisk \(W_2\)
of the disk \(D^{n-1}\) = \(\{(x_1, y) | (x_1) \in B, y \in D^{n-2}\}\),
and let \(A^{-k}(\pi_y W_2) \cap \pi_y W_2 = \phi\) for any \(k \geq 0\), where
\(\pi_y : \mathbb{R}^2 \oplus \mathbb{R}^{n-2} \to \mathbb{R}^{n-2}\) is a natural projection.

Let \(V_1\) be a smoothly imbedded in \(\mathbb{R}^n\) \((n-1)\)-dimentional disk
which is transversal to \(D^2\) and let the intersection of that
disk with \(D^2\) be a smoothly imbedded in \(D^2\) circle \(S^1\) \(S^1 \cap \text{int } W_1\).

\(V_2\) denotes \((n-1)\)-dimentional disk for which \(V_2 \cap V_1 = \phi\) and
\(V_2 \subset \text{int } W_1\). Let \(K\) be smoothly imbedded in \(\mathbb{R}^n\) \(n\)-dimentional
disk,
Finally find diffeomorphism $h: R^n \to R^n$ with the following properties:

1) $h$ is identical out of $K$,
2) $h(V_2) = V_1$ and $h(W_2 \setminus V_2) \cap D^2 = \emptyset$.

It is easy to see that $\Lambda$ is a basic set for $h \circ f$, because $h \circ f = f$ in some neighbourhood of $\Lambda$.

Consider $(h \circ f)^\ast(x,y) = (h \circ f)^{-k} \circ f^{-1}(u,v) = (h \circ f)^{-k}(0, \lambda^{-1}u, A^{-1}v) = f^{-k}(0, \lambda^{-1}u, A^{-1}v)$.

The last relation can be easily obtained by induction using that $h \circ f = f$ out of $K$, $K \cap D^{-1} = W_2$, $A^{-1}v \in \mathcal{X}_y W_2$.

$(0, \lambda^{-1}u, A^{-1}v) \in W^u(h \circ f)$, hence $(h \circ f)^{-k}(x,y)$ $\to 0$ when $k \to \infty$.

It implies that $(x,y) \in W^u((h \circ f))$ and hence $S \subset W^u((h \circ f))$.

By the same arguments $f_t \cap W_1 \subset W^S(Z_t, h \circ f)$.

$Z_t = (0, f_t, 0) \in f_t$. This follows from the fact that $x \in W_1$ implies $f(x) \in U$, $U \cap W_t = \emptyset$ and hence $(h \circ f)(x) = f(x)$.

At the same time $U$ is $f$-invariant neighbourhood, hence

$\mathcal{F}^k(x) = f^{-k}(x)$ for $x \in W_1$. And we obtain that from

$x \in W^S(Z_t, f) \cap W_1$ follows $x \in W^S(Z_t, h \circ f)$.

$S$ is smoothly imbedded circle in $U$, hence it has tangency with some segment $f_{t_0}$, but $S \subset W^u(0, h \circ f)$ and $f_{t_0} \subset W^S(Z_{t_0}, h \circ f)$.

hence manifolds $W^u((h \circ f))$ and $W^S(\Lambda, h \circ f)$ have a point of tangency.

(iii) It is necessary to prove that the property of tangency is $C^k$-stable for $k \geq 1$. We shall show that for any diffeomorphism $\omega$ from the sufficiently small $C'$-neighbourhood the property of
tangency of stable and unstable manifolds for basic set takes place. Thus the theorem will be proved.

Let diffeomorphism \( \omega \) be defined by the formula

\[
\omega(x,y) = (gx + \delta_1(x,y), Ay + \delta_2(x,y)),
\]

where \( x \in \mathbb{D}^2, y \in \mathbb{D}^{n-2} \) and \( \delta_1: \mathbb{D}^n \to \mathbb{D}_2, \delta_2: \mathbb{D}^n \to \mathbb{D}_2^{n-2} \) are smooth and sufficiently \( C \)-small. It was mentioned above that \( g \) maps \( \mathbb{D} \) into \( \text{int} \mathbb{D} \), \( A \) was linear map with the norm large enough, hence by the lemma 6 for \( f = g \times A \) (setting \( E_1 = \mathbb{R}^1, E_2 = \mathbb{R}^{n-2} \) and so on) one can find a smooth map \( \varphi: \mathbb{D}^2 \to \mathbb{R}^{n-2} \) which is sufficiently \( g^k \)-close to the identity and such that its graph \( \mathbb{D}_2^n = \{ \varphi(x) \mid x \in \mathbb{D}^2 \} \) is invariant under \( \omega \).

Let \( \xi: \mathbb{D}^2 \times \mathbb{R}^{n-2} \to \mathbb{D}^2 \times \mathbb{R}^{n-2} \) be new coordinates given by the formula

\[
(\xi(x,y)) = (x, y - \varphi(x)).
\]

One can obtain

\[
\omega(u,v) = \xi \circ \omega \circ \xi^{-1}(u,v) = (gu + \delta_1(u,v + \varphi(u)), Av + \delta_2(u,v + \varphi(u)) - \varphi(gu + \delta_1(u,v + \varphi(u))) - \varphi(gu + \delta_1(u,v + \varphi(u)) - \varphi(gu + \delta_1(u,v + \varphi(u))).
\]

It is obvious that \( \mathbb{D}^i \) is invariant under \( \tilde{\omega} \), because \( \mathbb{D}^i = \xi(\mathbb{D}^2) \) then

\[
\omega(u,v) = (g(u) + \delta_1(u,v), 0) = (g - \delta_1(1, \varphi)u, 0).
\]

There is a homeomorphism \( \eta: \mathbb{D}^i \to \mathbb{D}^1 \), which is sufficiently close to the identity and such that \( g + \delta_1(1, \varphi) = \eta \circ g \circ \eta^{-1} \) because \( g \) is absolutely structurally stable and \( \delta_1(1, \varphi) \) is \( C \)-small. 

The point \( \eta(\lambda) \) is a fixed point for \( \tilde{\eta}, \tilde{\Lambda}^\ast \eta(\Lambda) \) is one-dimensional sink for \( \tilde{\eta}, \tilde{U} \cdot \eta(u) \) is invariant under \( \tilde{\eta} \) neighbourhood of \( \tilde{\Lambda} \).
The set $\Lambda_{\omega} = \xi(\tilde{\Lambda}) = \{(x, y) \mid x \in \tilde{\Lambda} \} \subset D^2$
is the basic set for $\omega$ and $U_{\omega} = \xi(\tilde{\Omega}, 0)$ is its invariant
neighbourhood in $D^1$ under $\omega$.

The set $\tilde{W}_1 = \tilde{\Omega} \cap K$ is smoothly embedded in $D^1$ disk
which is sufficiently close to $W_1$ when $\delta_1$ and $\delta_2$ are $C^1$-small
enough. Thus we can consider that

$W_1 \subset \text{int} \left( \eta \circ g^{-1}(\Omega) \setminus \eta(\Omega) \right) = \text{int} \left( \tilde{g}^{-1}(\tilde{\Omega}) \setminus \tilde{\Omega} \right)$,

because $\eta$ is small. $W_1$ and $U_\omega$ are close to $\tilde{W}_1$ and $\tilde{\Omega}$ respec-
tively and we obtain that

$\tilde{W}_2 \subset \text{int} \left( \omega^{-1}(U_{\omega}) \setminus U_{\omega} \right)$.

It is easy to see that each segment of the stable manifold $W^s(\Lambda_{\omega}, \omega)$ of the family fibering the disk $\tilde{W}_1$ is also a segment of the stable manifold $W^s(\Lambda_{h^\omega}, h^\omega)$,

$\Lambda_{h^\omega} = \Lambda_{\omega}$.

Consider the map $\omega - T$, where

$T(x, y) = (\mu x_1, \lambda x_2, Ay)$

for $(x, y) \in B \times R^{n-2}$ we have

$(\omega - T)(x, y) = (\delta_1(x, y), \delta_2(x, y))$,

hence Lipschitz constant $L(\omega - T)$ is small because $\delta_1, \delta_2$ are $C^1$-small. There is function $x_\omega = \psi(x, y), (0, x_2, y) \in D^{n-1}$ such that

$\text{gr} \psi = \{(\psi(x, y), x_\omega, y) \mid (0, x_2, y) \in D^{n-1}\} = \bigcap_{k \geq 0} \omega^K(D^n)$.

It follows from the stable manifold theorem for a point ( see [4],[6] ).
\( \rho \omega = (r(0), \varphi \cdot q(0)) \in \mathbb{D}^n \) is a fixed point for \( \omega \), \( gr \varphi \) is the local unstable manifold \( W'_{uc}(\rho \omega, \omega) \).

We can treat that function \( \varphi \) is sufficiently \( C' \)-small, because it smoothly depends on \( \omega \) in \( C' \)-topology and \( \omega \) is \( C' \)-close to \( T \).

In such a case choosing \( \omega \) \( C' \)-close to \( f \) we can obtain that the set \( \tilde{W}_2 = gr \varphi \cap K \) is smoothly embedded in \( \mathbb{R}^{n-1} \) disk which is \( C' \)-close to \( W_2 \). As in (ii) we can obtain that the disk \( h(\tilde{W}_2) \) is smoothly embedded in \( W^u(\rho \omega, \ell^{-} \omega) \).

Thus, if \( \omega \) is \( C' \)-close to \( f \) then

1) disks \( W_1 \) and \( \tilde{W}_1 \) are close to each other in \( C' \)-topology and are smoothly embedded,

2) smoothly embedded in \( \mathbb{R}^n \) disks \( W_1 \) and \( \tilde{W}_2 \) are \( C' \)-close to each other, hence \( (n-1) \)-disks \( h(W_2) \) and \( h(\tilde{W}_2) \) are close too,

3) disk \( W_1 \) is transversal to \( h(W_2) \) and their intersection is the circle \( S \) smoothly embedded in \( \mathbb{R}^n \).

Therefore if the diffeomorphism \( \omega \) is close to \( f \) in \( \text{Diff}(\mathbb{M}) \) then

1) disks \( \tilde{W}_1 \) and \( h(\tilde{W}_1) \) are transversal,

2) their intersection is smoothly embedded in \( \mathbb{M} \) circle \( \tilde{S}' \) which is \( C' \)-close to \( S \).

From the preceding remarks it is clear that \( \tilde{S}' \subset W^u(\rho \omega, \ell^{-} \omega) \) the disk \( \tilde{W}_1 \) being fibered by the smooth family of segments of stable manifolds of points of \( \Lambda_{\widetilde{h} \omega} \), alike the (ii) one can find that \( W^s(\Lambda_{h \omega}, h^{-} \omega) \) has a point of tangency with \( W^u(\Lambda_{h \omega}, h^{-} \omega) \).

Finally, notice that values \( \mu(\rho_{h \omega}) \) and \( \lambda(\rho_{h \omega}) \) are approximately equal to \( \lambda(\mu, \|A\|) \) respectively and \( \|A\| \) is large enough, hence \( \mu(\rho_{h \omega}) \lambda(\rho_{h \omega}) > 1 \) and the theorem B has been proved.
REFERENCES


