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A Differentiable Invariant of Topological Conjugacies and Moduli of Stability

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We discuss here a differentiable invariant that arises when we consider topological conjugacies of a certain class of dynamical systems (diffeomorphisms or vector fields). These dynamical systems exhibit a pair of hyperbolic periodic orbits of saddle type whose stable and unstable manifolds meet non-transversely. This condition is not open or persistent in the set of all dynamical systems. However, this may be the case when we consider one (or more) parameter families of diffeomorphisms or vector fields. Also for holomorphic vector fields near a singularity, when we consider the real flow induced by the intersection of its trajectories with a sphere of small radius centered at the singularity [1], [2]. We think that other interesting situations of a similar nature may arise in dynamical systems, specially when specific subsets of these systems are considered. We point out that most of the results we present here will appear in [7] with more general formulations and several global implications, like the stability of a generic arc of gradient vector fields.

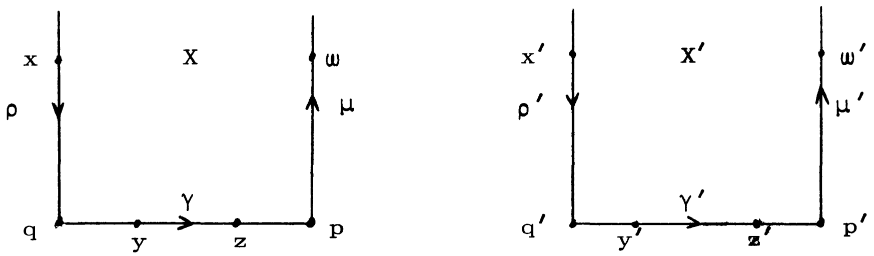
This conjugacy invariant will be used here in quite opposite ways: on one hand we show that certain systems are finitely stable (or have a finite moduli of stability) and on the other there are open sets of diffeomorphisms on the sphere S^2 with an infinite moduli of stability. This last application will make use of diffeomorphisms constructed in [5] and in fact it will provide open sets of diffeomorphisms of S^2 with infinite moduli of Ω-stability. As usual, Ω-stability means stability restricted to the non-wandering set.

We remark that there are quite complicate systems with a finite moduli of stability. This is the case for the Lorenz attractors as recently proved by Guckenheimer and Williams. In [1], [2] moduli of stability are determined for holomorphic vector fields.

Let M be a C^∞ manifold. Two C^r $(r \geq 1)$ diffeomorphisms of M are topologically conjugate if there is a homeomorphisms h of M such that $hf(x) = gh(x)$ for all $x \in M$. A diffeomorphism f is structurally stable if any g C^r near f is conjugate to f. Two C^r vector fields X and Y on M are topologically conjugate if their flows X_t and Y_t are conjugate in the following sense: there is a homeomorphism h of M such that $h\, X_t(x) = Y_t\, h(x)$ for all $x \in M$ and $t \in \mathbb{R}$. The vector fields are topologically equivalent if there is a homeomorphism of M sending trajectories of one onto trajectories of the other. The most natural concept of (structural) stability for vector fields is defined in terms of topological equivalence: X is stable if any C^r close vector field is topologically equivalent to X. In both cases of vector fields and diffeomorphisms the moduli of stability is defined as follows. A dynamical system is k-stable, k a non-negative integer, if in any small C^r neighborhood of this system there is a k-(real) parameter family of topologically distinct systems and every system in the neighborhood is equivalent to one in this family. A stable system is 0-stable and for a system that is not k-stable and for any $k \geq 0$ we say that it has ∞-moduli of stability. Similarly, we can restrict the notions of equivalence, stability and moduli of stability to the non-wandering sets of the systems.

We recall that a periodic orbit $o(p)$ of a diffeomorphism f is hyperbolic if $Df^{\ell}(p)$ has all eigenvalues with norm different from one, where ℓ is the period of p. The orbit is called a saddle if some of these eigenvalues have norm bigger than one and some smaller than one. Similar concepts hold for periodic orbits of vector fields by considering the Poincaré mappings associated to local transverse sections. For the singularities of vector fields, we require for hyperbolicity that the eigenvalues of their derivatives at these singularities should have non-zero real part; if some of these real part are negative and some positive, we call the singularity a saddle point. In all cases, we can associate to a hyperbolic periodic or fixed orbit γ a pair of invariant manifolds called the stable and unstable manifolds of γ and denoted by $W^s(\gamma)$ and $W^u(\gamma)$. They are as smooth as the system and are made of orbits that have the periodic or fixed orbit as their ω or α-limit set, respectively.

We can now start presenting the conjugacy invariant we suggested above. Let us first consider C^2 vector fields X and X' on a two-manifold. Let p, q and p', q' be singularities of saddle type for X and X'. Suppose that one component of the stable manifold of p coincides with one compoent of the unstable manifold of q; i.e. there is an orbit γ of X whose ω-limit is p and α-limit is q. The same for X', p', q' with connecting orbit γ'. Finally, let μ, ρ and μ', ρ' be the eigenvalues of the derivatives $DX(p)$, $DX(q)$ and $DX'(p')$, $DX'(q')$ in the transversal directions to γ and γ', respectively.



<u>Theorem</u> - X and X' are conjugate in neighborhoods of γ and γ' if and only if $\rho/\mu = \rho'/\mu'$.

<u>Proof:</u> We observe that C^2 flows on two-manifolds are C^1 linearizable near hyperbolic singularities [3]. Thus we may assume that the flow X_t generated by X is linear in neighborhoods U_p, U_q of p and q. Similarly for X'_t in neighborhoods $U_{p'}$, $U_{q'}$. First, suppose there is a homeomorphism h from a neighborhood of γ onto a neighborhood of γ' such that $h\,X_t = X'_t\,h$. Consider sequences $x_i \to x \in W^s(q) \cap U_q$, $y_i \to y \in$ $\in \gamma \cap U_q$, $z_i \to z \in \gamma \cap U_p$ and $\omega_i \to \omega \in W^u(p) \cap U_p$ such that

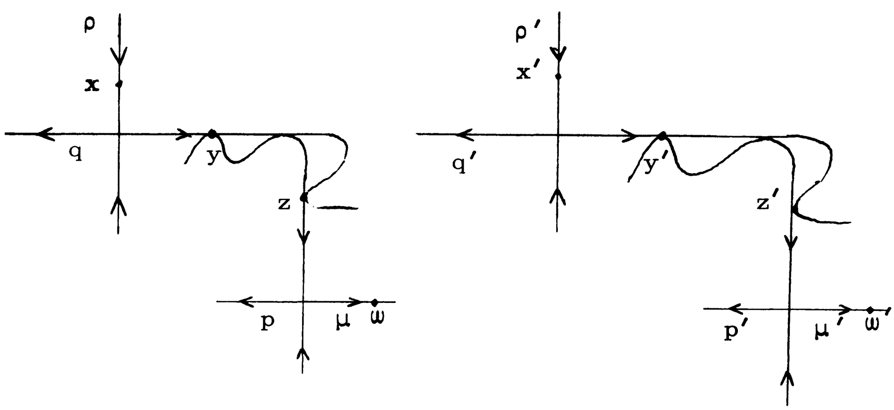
$y_i = X_{t_i}(x_i)$, $z_i = X_s(y_i)$ and $\omega_i = X_{T_i}(z_i)$, where s is constant and $t_i, T_i \to \infty$ as $i \to \infty$. Let $x' = h(x)$, $x'_i = h(x_i)$, $y' = h(y)$, $y'_i = h(y_i)$, $z' = h(z)$, $z'_i = h(z_i)$, $\omega' = h(\omega)$, $\omega'_i = h(\omega_i)$. Let a, a_i be the (vertical) coordinates of x, x_i in the $W^s(q)$ direction and d, d_i (the vertical) coordinate of ω, ω_i (in the $W^u(p)$ direction with a, a_i, d, $d_i > 0$. We get $d_i =$ $= a_i\, e^{\rho t_i}\, k_i\, e^{\mu T_i}$ and $a_i \to a$, $d_i \to d$, $k_i \to k$ as $i \to \infty$. Here, k_i measures the distortion of the normal (vertical) distances in U_q from y_i to γ passing to the normal (vertical) distances in U_p from $z_i = X_s(z_i)$ to γ; k is obtained from $DX_s(y)$ in these normal directions. Thus, we have $\rho t_i + \mu T_i \to$ $\to \log d/ak$ and so $T_i/t_i \to -\rho/\mu$ since $t_i \to \infty$. Similarly $T_i/t_i \to -\rho'/\mu'$, proving that $\rho/\mu = \rho'/\mu'$.

Conversely, let us suppose that $\rho/\mu = \rho'/\mu'$. We define a conjugacy between X_t and X'_t in neighborhoods of γ and γ' as follows. In U_q, $U_{q'}$ we use the linear structures of X_t and X'_t. We define $h(q) = q'$, $h(a,0) = (a',0)$ for some $(a,0) \in W^s(q)$, $(a',0) \in W^s(q')$ and $h(0,b) = (0,b')$ for some $(0,b) \in \gamma$, $(0,b') \in \gamma'$. Then, for any point $(\tilde{a},\tilde{b}) \in U_q$ we have that $(\tilde{a},0) = X_t(a,0)$ e $(0,\tilde{b}) = X_r(0,b)$ and so we define $h(\tilde{a},\tilde{b}) = (\pi_1\, X'_t(a',0), \pi_2\, X'_r(0,b'))$, where π_1, π_2 are the natural projections. At large, if $v = X_t(u)$ for $u \in U_q$ we set $h(v) = X'_t(h(u))$. We have to show that h extends continuously to a conjugacy from $W^u(p)$ to $W^u(p')$. To see this we fix vertical sections L_q, L'_q in U_q, $U_{q'}$ perpendicular to γ and γ' with $h(L_q) = L_{q'}$. Let $L_p = X_s(L_q)$ and $L_{p'} =$ $= X'_s(L_{q'})$; it follows that $L_{p'} = h(L_p)$. Now consider any sequence $\omega_i \to \omega \in W^u(p)$, ω with vertical coordinate d. We want to show that $h(\omega_i) \to \omega' \in W^u(p')$. As before, we can write $\omega_i = X_{T_i}(z_i)$, $z_i \in L_p$ and $z_i \to z \in \gamma$, $z_i = X_s(y_i)$, $y_i \in L_q$ and $y_i \to y \in \gamma$, $y_i = X_{t_i}(x_i)$, $x_i \to x \in W^s(q)$ with $t_i, T_i \to \infty$ as $i \to \infty$. It is clear that $\omega_i' = h(\omega_i) \to W^u(p')$; the problem is to show that its vertical coordinates converge. With the same notation as above, the vertical coordinate of ω_i is $d_i = a_i e^{\rho t_i} k_i e^{\mu T_i}$ and $d_i \to d$, $a_i \to a$, $k_i \to k$ as $i \to \infty$. The expression for the vertical coordinate d_i' of ω_i' is similar. Using the relation $\rho/\mu = \rho'/\mu'$, we conclude that d_i' converges to $(a'k'/ak)\, e^{\rho'-\rho}\, d$. This shows that h extends continuously to $W^u(p)$ and it is clearly a conjugacy between X_t and X_t'. The proof of the statement is complete.

An obvious consequence of this theorem is that any two such vector fields X and X' with saddle connections γ and γ' are always topologically equivalent near γ and γ', even if $\rho/\mu \neq \rho'/\mu'$. We consider $\tilde{X} = fX$ such that this moduli condition is satisfied for $\tilde{X}$ and X', where f is a positive real function with $f = 1$ outside a neighborhood of a saddle point for X. Since X and $\tilde{X}$ are equivalent and $\tilde{X}$ and X' are conjugate, X and X' are equivalent. This fact was known, but it was proved (see [8], [10]) using proportionality of arc lenghts. We conclude that for vector fields on two manifolds near a saddle-connection, we have a one-parameter family of models up to conjugacy and a unique model up to equivalence.

The theorem will be generalized to higher dimensions in [7] together with several global applications. Among other things, it will be proved the stability of a generic one-parameter family of gradient vector fields.

Let us consider now a similar situation for diffeomorphisms on two-manifolds. We take diffeomorphisms f, f' of class C^2 with hyperbolic fixed (or periodic) points p and q, p' and q' of saddle type. Suppose that $W^u(q)$ and $W^s(p)$, $W^u(q')$ and $W^s(p')$ have one orbit γ, γ' of quasi-transversal (parabolic) intersection, respectively. Denote by ρ, ρ' the eigenvalues of $Df(q)$, $Df'(q')$ with norm less than one and μ, μ' the eigenvalues of $Df(p)$, $Df'(p')$ with norm bigger than one.



Theorem - If f and f' are conjugate in neighborhoods of γ and γ' then $\log \rho/\log \mu = \log \rho'/\log \mu'$.

Proof: As before, take neighborhoods U_q, U_p and $U_{q'}$, $U_{p'}$ where f and f' are C^1 linearizable [3]. Take points $y \in \gamma \cap U_q$ and y' in $\gamma' \cap U_{q'}$. On a C^1 curve L perpendicular to $W^u(q)$ in U_q at y take a sequence $y_i \to y$. Consider sequences $x_i \to x$, $z_i \to z$ and $\omega_i \to \omega$ such that

$y_i = f^{n_i}(x_i)$, $z_i = f^s(y_i)$, $\omega_i = f^{m_i}(z_i)$ with $n_i, m_i \to \infty$, s a fixed integer and $x \in W^s(q) \cap U_q$, $z \in U_p$ and $\omega \in W^u(p) \cap U_p$. If a, a_i are the vertical (along $W^s(q)$) coordinates of x, x_i in U_q and d, d_i are the horizontal (along $W^u(p)$) coordinates of ω, ω_i in U_p, we have $d_i = a_i \rho^{n_i} k_i \mu^{m_i}$. Since $d_i \to d$, $a_i \to a$ and $k_i \to k$, all positive numbers, we get $m_i/n_i \to$ $\to - \log \rho/\log \mu$ as $i \to \infty$. The constant k is given by $Df^s(y)$ along L expressed in terms of the linearizing system for f in U_p. Taking the images of these sequences by the conjugacy and using similar notation, we have $\mathrm{dist}(z_i', W^u(q')) = a_i' \rho'^{n_i} k_i$ with $a_i' \to a' > 0$, $k_i' \to k' > 0$ and $\mathrm{dist}(z_i', W^s(p')) \geq$ $\geq \mathrm{dis}(z_i', W^u(q'))$. Passing to the sequence $\omega_i' \to \omega'$ and taking coordinates, we get $a_i' \rho'^{n_i} k_i' \mu'^{m_i} \leq d_i' \to d'$. So $\rho'^{n_i} \mu'^{m_i}$ is bounded above. Since $m_i/n_i \to - \log \rho/\log \mu$, we conclude that $\log \rho'/\log \mu' \leq \log \rho/\log \mu$. Reversing the argument, we get $\log \rho/\log \mu \leq \log \rho'/\log \mu'$. Thus $\log \rho/\log \mu = \log \rho'/\log \mu'$, proving the statement.

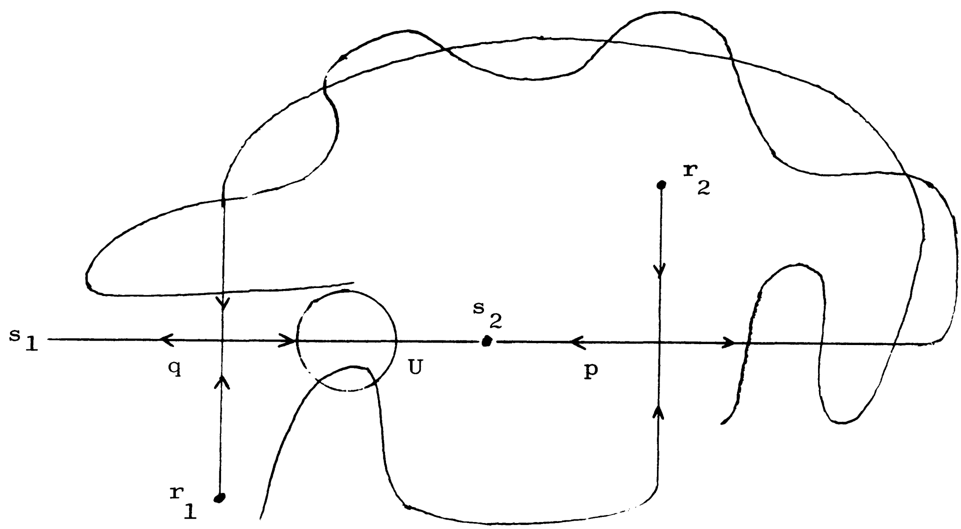
This result can be generalized to higher dimensions [7]. At this writing, W. Melo [4] proved the converse for diffeomorphisms above on two-manifolds; i.e., if $\log \rho/\log \mu = \log \rho'/\log \mu'$ then we can construct a conjugacy between the diffeomorphisms in neighborhoods of the quasi-transversal orbits.

Using the conjugacy invariant of the previous theorem and results of Newhouse in [5], we can now exhibit open sets of diffeomorphisms on the sphere S^2 with infinite moduli of stability and, in fact, of Ω-stability. The examples can be constructed on every manifold of dimension bigger than one.

Previously, Robinson and Williams [9] also exhibit open sets of diffeomorphisms with infinite moduli of stability. However, their reasoning is of different nature and requires surfaces of genus two or higher and for Ω-stability higher dimensional manifolds.

<u>Theorem</u> - There are open sets of C^∞ diffeomorphisms of S^2 with infinite moduli of stability.

<u>Proof</u>: Let f be a C^∞ Morse-Smale diffeomorphism of S^2 as follows. The limit set of f consists of six fixed points, all hyperbolic and their stable and unstable manifolds are in general position. See the picture below, where r_1 and r_2 are sources, s_1 and s_2 are sinks and p and q are saddles with $W^s(q)$ intersecting transversely $W^u(p)$.



For $t \in [0,1]$, we consider $\varphi_t = \rho_t \circ f$, where ρ_t is an isotopy with support in U as above and ρ_o is the identity map. Thus,

$\varphi_t = \varphi_o = f$ outside U for all $t \in [0,1]$. This isotopy is such that, for some $b \in (0,1)$, φ_t is Morse-Smale and conjugate to f for $0 \leq t < b$ and for φ_b we create a generic (parabolic) orbit of tangency between $W^u(q)$ and $W^s(p)$ (see [6]). We observe, as shown in [6], that for many values of $t > b$ and near b, φ_t is structurally stable with infinitely many periodic orbits. However, it follows from [5] that, for some values of t near b, there are open neighborhoods V of φ_t of C^∞ diffeomorphisms with the following remarkable property. For any integer $K > 0$, is dense in V the subset of diffeomorphisms having more than K distinct pairs of hyperbolic periodic points p_i, q_i with a quasi-transversal orbit of intersection γ_i in $W^s(p_i) \cap W^u(q_i)$. This fact is proved in [5] for $K = 1$, but for $K > 1$ one can use the same argument inductively. The previous theorem implies that each such pair of periodic points yields a positive real number as a conjugacy invariant. We conclude that every $g \in V$ has moduli of stability bigger than K, for all integers $K > 0$. Therefore, all $g \in V$ have infinite moduli of stability.

We notice that these open sets of C^∞ diffeomorphisms of S^2 have infinite moduli of Ω-stability. This follows from the fact that the orbits of tangency in $W^s(p_i) \cap W^u(q_i)$ above is non-wandering because, by construction, $W^u(p_i)$ and $W^s(q_i)$ have a non-empty transversal intersection. Finally, we remark, based in [5], that a large number of these examples can be constructed on every manifold of dimension bigger than one.

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