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by

S. E. Newhouse

In this paper we present two new generic properties of C^1 area preserving diffeomorphisms of a compact oriented surface. We obtain a lower bound for the topological entropy of a generic diffeomorphism, and we show that such a diffeomorphism always has closed invariant sets with dense orbits and Hausdorff dimension two.

Before stating our results precisely, let us fix some notation and recall some definitions. Let M be a C^∞ compact, connected, orientable 2-manifold, and let ω be a C^∞ area form on M. That is, ω is a nowhere vanishing differential 2-form of class C^∞. Let $\mathrm{Diff}^1_\omega M$ denote the space of C^1 diffeomorphisms of M which preserve ω, and give $\mathrm{Diff}^1_\omega M$ the uniform C^1 topology.

For f in $\mathrm{Diff}^1_\omega M$, a point $p \in M$ is <u>periodic</u> if $f^n p = p$ for some $n > 0$. Let $\tau(p) = \inf\{n > 0 : f^n p = p\}$. This is the <u>period</u> of p. The periodic point p is <u>hyperbolic</u> if all eigenvalues of $T_p f^{\tau(p)}$ have norm different from one. In our case this means that $T_p f^{\tau(p)}$ has a single eigenvalue of norm bigger than one. Call this eigenvalue $\lambda(p)$. Let $n > 0$ be a positive integer, and let $\mathrm{Hyp}_n f$ denote the set of hyperbolic periodic points of f with period less than or equal to n. Define $s_n(f) = \max\{\frac{1}{\tau(p)} \log|\lambda(p)| : p \in \mathrm{Hyp}_n f\}$, and set $s(f) = \sup_{n \geq 1} s_n(f)$.

Let d be a topological metric on M. For $\varepsilon > 0$, $n > 0$, a set $E \subset M$ is (n,ε)-separated if for any $x \neq y$ in E, there is a $0 \leq j < n$ such that $d(f^j x, f^j y) > \varepsilon$. Let $r(n,\varepsilon,f)$ be the maximal cardinality of an (n,ε)-separated set. The number $h(f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} r(n,\varepsilon,f)$ is the <u>topological entropy</u> of f. It is a rough asymptotic measure of how much f mixes up the points in M. For any C^1 diffeomorphism, $0 \leq h(f) < \infty$.

If $\Lambda \subset M$ is a closed f-invariant set, then $h(f|\Lambda)$ is defined similarly, and it is easy to see that $h(f|\Lambda) \leq h(f)$. Also, for any integer n,

$$h(f^n|\Lambda) = |n| h(f|\Lambda), \text{ and if } \phi : \Lambda \to \Lambda_1$$

is a homeomorphism, then $h(\phi f \phi^{-1}|\Lambda_1) = h(f|\Lambda)$. For more properties of h we refer to [2]. If p is a hyperbolic periodic point of the diffeomorphism f with orbit $o(p)$, we let $H(p,f)$ be the set of transverse homoclinic points of p. Thus $H(p,f)$ is the set of transverse intersections of $W^u(o(p),f)$ and $W^s(o(p),f)$ where $W^u(o(p),f)$ and $W^s(o(p),f)$ are the unstable and stable manifolds of the orbit $o(p)$. Then the closure $\overline{H(p,f)}$ of $H(p,f)$ is a closed f-invariant set on which f has a dense orbit [4].

If E is a closed subset of M and $\alpha > 0$, $\varepsilon > 0$ are positive real numbers, let

$$H^\alpha_\varepsilon(E) = \inf\{ \sum_i (\text{diam } U_i)^\alpha : \{U_i\} \text{ is a countable open covering of } E \text{ each of whose elements has diameter less than } \varepsilon\}.$$

The Hausdorff α-outer measure of E is the number $H^\alpha(E) = \lim_{\varepsilon \to 0} H^\alpha_\varepsilon(E)$. The Hausdorff dimension of E, denoted $HD(E)$, is the number

$$\inf\{\alpha : H^\alpha(E) = 0\} = \sup\{\beta : H^\beta(E) = \infty\} .$$

If $\dim E$ is the topological dimension of E, then $HD(E) \geq \dim E$. Also, $m(E) > 0$ implies $HD(E) = 2$, but not conversely, where $m(E)$ is the Lebesgue measure of E.

A closed f-invariant set Λ is hyperbolic if there are a continuous splitting $T_\Lambda M = E^s \oplus E^u$, a Riemann norm $|\cdot|$, and a constant $0 < \lambda < 1$ such that $Tf(E^s) = E^s$, $Tf(E^u) = E^u$, $|Tf|E^s| < \lambda$, and $|Tf^{-1}|E^u| < \lambda$. The hyperbolic set Λ is a hyperbolic basic set if $f|\Lambda$ has a dense orbit and there is a compact neighborhood U of Λ such that $\bigcap_{-\infty<n<\infty} f^n U = \Lambda$. For g C^1 near f, there is a hyperbolic basic set $\Lambda(g) = \bigcap_{-\infty<n<\infty} g^n U$ for g such that $f|\Lambda(f)$ and $g|\Lambda(g)$ are topologically equivalent [3].

If M is a hyperbolic set for f, then f is called Anosov.

Theorem. There is a residual set $\mathcal{B} \subset \mathrm{Diff}^1 M$ such that if f is in $\mathcal{B}$, then each set $\overline{H(p,f)}$ has Hausdorff dimension two. In addition, if f is in $\mathcal{B}$ and f is not Anosov, then

$$(*) \qquad h(f) \geq s(f)$$

Recall that a residual set is one which contains a countable intersection of dense open sets. Properties true for residual sets are called generic, and a generic diffeomorphism is defined to be an element of some residual set.

Remarks 1. For an Anosov diffeomorphism f, each $\overline{H(p,f)} = M$, so the first statement of our theorem is trivially true. On the other hand, it is easily seen that there are open sets of Anosov diffeomorphisms for which (*) fails. For instance, if f is linear, then $h(f) = \log|\lambda(p)|$ where $f(p) = p$. However, with a small perturbation, we can increase the expansion at non-fixed periodic points to make (*) fail. With a bit more work one can show that (*) fails for an open dense set of Anosov diffeomorphisms. To see this, consider the function ϕ^u of Bowen and Ruelle [1]. We may suppose that f is C^2, so Lebesgue measure is the unique equilibruim state for ϕ^u. Let μ be the unique invariant measure of maximal entropy for f. Then, $-\int \phi^u d\mu \leq s(f)$. As μ and m are ergodic f-invariant probability measures, they are either equivalent or mutually singular. Using Proposition 4.5 of [1] and simple perturbation techniques, one can show that C^2 generically, μ is singular with respect to m. Then,

$$\begin{aligned} 0 = P_m(\phi^u) &= h_m(f) + \int \phi^u dm \\ &> h_\mu(f) + \int \phi^u d\mu \\ &= h(f) + \int \phi^u d\mu , \end{aligned}$$

so $h(f) < s(f)$. Since $h(f) < s(f)$ is a C^1 open condition for Anosov diffeomorphisms, (*) fails for a C^1 open dense set.

2. It would be nice to know if C^1 generically each set $\overline{H(p,f)}$ has positive measure or if $f|\overline{H(p,f)}$ has positive measure theoretic entropy. Also, what analogs of our results hold for the C^r topology, $r \geq 2$?

We proceed to the proof of the theorem.

In view of remark 1 our theorem only has content for non-Anosov diffeomorphisms. Let A be the set of Anosov diffeomorphisms on M and let $\mathcal{D} = \mathrm{Diff}^1_\omega M - A$. Of course, A is open in $\mathrm{Diff}^1_\omega M$ and is empty unless M is the two-dimensional torus.

For positive integers n and m, let $\mathcal{B}_{n,m}$ be the set of diffeomorphisms f in $\mathcal{D}$ such that there are a p in $\mathrm{Hyp}_n f$ and a hyperbolic basic set $\Lambda \subset \overline{H(p,f)}$ satisfying $h(f|\Lambda) > s_n(f) - \frac{1}{m}$. Analogously, we let $\mathcal{B}'_{n,m}$ be the set of diffeomorphisms f in $\mathcal{D}$ such that $\mathrm{Hyp}_n f \neq \emptyset$, and, for each p in $\mathrm{Hyp}_n f$, there is a hyperbolic basic set $\Lambda \subset \overline{H(p,f)}$ so that $HD(\Lambda) > 2 - \frac{1}{m}$.

We assert that (1) $\mathcal{B}_{n,m}$ and $\mathcal{B}'_{n,m}$ are dense open sets in $\mathcal{D}$.

The theorem follows from (1) by taking $\mathcal{B} = A \cup \bigcap_{n,m} \mathcal{B}_{n,m} \cap \mathcal{B}'_{n,m}$.

The main step in the proof of (1) is the next result.

<u>Proposition</u>. Suppose p is a hyperbolic periodic point of the diffeomorphism f and $W^u(o(p))$ is tangent to $W^s(o(p))$ at some point. Given $\varepsilon > 0$ and any neighborhood N of f in $\mathcal{D}$, there is a g in N such that p is a hyperbolic periodic point for g, and

(a) g has a hyperbolic basic set Λ in $\overline{H(p,g)}$ on which
$$h(g|\Lambda) > \frac{1}{\tau(p)} \log|\lambda(p)| - \varepsilon$$

(b) each g_1 near g has a hyperbolic basic set $\Lambda(g_1)$ in $\overline{H(p(g_1),g_1)}$ such that $HD(\Lambda(g_1)) > 2 - \varepsilon$.

Before proving the proposition, let us show how we can use it to prove assertion (1).

Let $f \in \mathcal{D}$, and let n and m be positive integers. We may perturb f to f_1 so that the hyperbolic and elliptic periodic points of f_1 are dense in M by theorems (1.3) and Corollary (3.2) in [5]. Using Takens [10], we may also assume $W^u(p,f_1) \cup W^s(p,f_1) \subset \overline{H(p,f_1)}$ for each hyperbolic periodic point p of f_1. Choose $p \in \mathrm{Hyp}_n(f_1)$ so that $\frac{1}{\tau(p)} \log|\lambda(p)| > s_n(f_1) - \frac{1}{2m}$.

Since f_1 has elliptic periodic points, it is in $\mathcal{D}$. If $\overline{H(p,f_1)}$ were hyperbolic, it would have interior (since $W^u(p) \cup W^s(p) \subset \overline{H(p,f_1)}$). But then local product structure [9, Theorem (7.4)] and topological transitivity would imply that $\overline{H(p,f_1)}$ is open and closed in M. So $\overline{H(p,f_1)}$ would equal M, making f_1 Anosov and giving a contradiction. Thus, $\overline{H(p,f_1)}$ is not hyperbolic. Using [5], we can find f_2 C^1 near f_1 so that $p \in \mathrm{Hyp}_n f_2$, and $W^u(o(p))$ has a tangency with $W^s(o(p))$. Applying statement (a) in the proposition enables us to find f_3 C^1 near f_2 so that f_3 has a hyperbolic basic set Λ with entropy larger than $\frac{1}{\tau(p)} \log|\lambda(p)| - \frac{1}{4m}$. Also, $s_n(\cdot)$ is continuous, so if f_3 is near f_1 and f' is near f_3, we have $s_n(f') < s_n(f_1) + \frac{1}{4m}$. But Λ continues to topologically equivalent hyperbolic sets for perturbations f' of f_3. Hence, for f' near f_3,

$$\begin{aligned} h(f') &> \frac{1}{\tau(p)} \log|\lambda(p)| - \frac{1}{4m} \\ &> s_n(f_1) - \frac{3}{4m} \\ &> s_n(f') - \frac{1}{m} . \end{aligned}$$

This proves that $\mathcal{B}_{n,m}$ is dense and open in $\mathcal{D}$. Similarly, we can use statement (b) of the proposition to prove that $\mathcal{B}'_{n,m}$ is dense and open in $\mathcal{D}$.

It remains to prove the proposition. All of our estimates will be with respect to the C^r norm induced from a fixed finite covering by symplectic coordinate charts, $r = 1$ and 2. The C^r norm of a function f will be the maximum of the r^{th} order partial derivatives computed in that covering, and we denote it by $|f|_{C^r}$.

All of our approximations are local and will be done in local coordinates using generating functions. Let us recall the main properties of these functions.

Suppose (x,y) are coordinates in $\mathbb{R}^2$ and $f(x,y) = (\xi(x,y),\eta(x,y))$ is an area preserving C^1 diffeomorphism with $f(o,o) = (o,o)$ and $\frac{\partial\eta}{\partial y}$ nowhere zero. Then we may solve for y as a C^1 function of x and η in the equation $\eta = \eta(x,y)$, and the mapping $(x,\eta) \longrightarrow (x,y(x,\eta))$ allows us to use x and η as coordinates on $\mathbb{R}^2$. Since f preserves area, the 1-form $\alpha = \xi d\eta + y\, dx$ is closed, and we may find a unique C^2 function $S(x,\eta)$ so that $S(o,o) = 0$, $S_x = y$, $S_\eta = \xi$, and $S_{x\eta}$ never vanishes. The function S is called the generating function of f. Conversely, given a C^2 function $S(x,\eta)$ so that $S(o,o) = 0$ and $S_{x\eta}(x,\eta)$ is never zero, we may solve for η as a function of x and y in the equation $S_x(x,\eta) = y$, and obtain an area preserving diffeomorphism by

$$f(x,y) = (S_\eta(x,\eta(x,y)),\ \eta(x,y)) .$$

If g is an area preserving diffeomorphism C^1 near f, then its generating function $\bar{S}$ is C^2 near S, and conversely, The generating function for the identity transformation is $S(x,\eta) = x\eta$.

We now begin the proof of the proposition. Let us assume, at first, for notational simplicity, that p is a fixed point of f, so $\tau(p) = 1$.

Suppose $W^u(p,f)$ is tangent to $W^s(p,f)$ at z_o. With a preliminary C^1 approximation we may make $W^u(p,f)$ and $W^s(p,f)$ coincide on a small curve, say I, around z_o in $W^u(p,f)$. The picture is as follows:

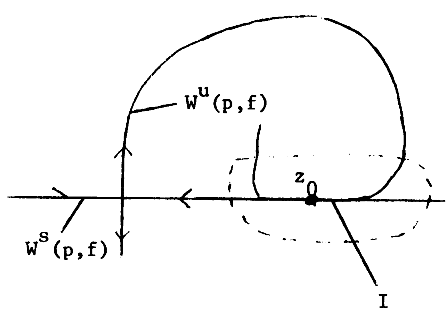

Figure 1

Let U be a small neighborhood around z_0 in M with $f^{-1}U \cap U = \emptyset$ and assume I small enough to be in U. Introduce local symplectic coordinates $z = (x,y)$ about $z_o = (o,o)$ in U so that I is contained in $(y = 0)$. Thus there is a diffeomorphism $\phi : U \to \mathbb{R}^2$ so that $\phi(z_o) = (o,o)$, $\phi(I) \subset \{(x,y) : y = 0\}$, and $\phi^*(dx \wedge dy) = \omega$. Let $a > 0$ be such that $\phi^{-1}([-2a,2a]) \subset I$.

We identify $\mathbb{R}^2$ with U via ϕ in the sequel.

Let $\varepsilon > 0$. We will produce an area preserving C^1 perturbation g of f with $g(z) = f(z)$ for $z \notin f^{-1}U$ such that g has a hyperbolic basic set $\Lambda \subset \overline{H(p,g)}$ such that $h(g|\Lambda) > \log|\lambda(p)| - \varepsilon$.

Intuitively, we obtain Λ in the following way. Introduce a large number of bumps in $W^u(p,g)$ over the interval $[-a,a]$ in I without disturbing the fact that $I \subset W^s(p,g)$. Letting I' denote the piece of $W^u(p,g)$ over I, we arrange for I' to be the graph of the function $x \longrightarrow A\cos(\frac{\pi N x}{2a})$ with $-a \le x \le a$, N a large positive integer, and A a small positive number. The maximum height of I' is A, the minimum is $-A$, and I' has N intersections with I. This gives the next figure

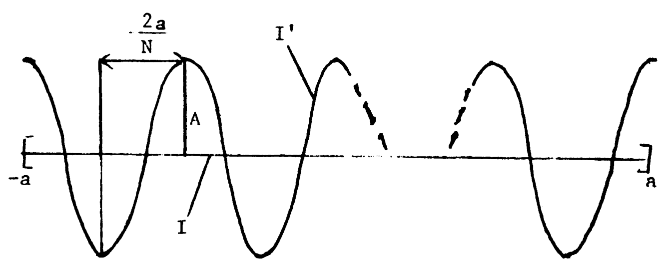

Figure 2

To do this with g $\delta - C^1$ close to f we will need $2A(\frac{2a}{N})^{-1} < K_1\delta$ for some constant K_1 independent of N. Suppose we take $A = \frac{K_1\delta a}{2N}$. Since $I \subset W^s(p,g)$ and $I' \subset W^u(p,g)$, we will be able to find a rectangle D_A with distance around $\frac{A}{4}$ units from I whose image under g^n for some large n is around $\frac{A}{4}$ units from I' as in the next figure

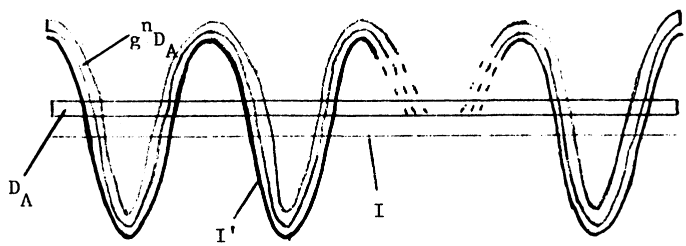

Figure 3

The "around" in the preceding statement means we are ignoring constants independent of N. Then, if Λ_1 is the largest invariant set for $g^n|D_A$, Λ_1 will be hyperbolic for g^n and $h(g^n|\Lambda_1) = \log N$. This gives us $\Lambda = \bigcup_{0 \leq j < n} g^j\Lambda_1$ hyperbolic for g and $h(g|\Lambda) = \frac{1}{n}\log N$. From the construction, g has a periodic point in Λ which is homoclinically related to p, so $\Lambda \subset \overline{H(p,g)}$. Except, for constants independent of N, we will have $|\lambda(p)|^{-n} = \frac{A}{4} = \frac{K_1\delta a}{8N}$. Thus, $-n \log|\lambda(p)| = \log\frac{K_1\delta a}{8} - \log N$ or $\log|\lambda(p)| = -\frac{1}{n}\log\frac{K_1\delta a}{8} + \frac{\log N}{n}$. Choosing N very large forces n to be large, so we can get

$$h(g|\Lambda) = \frac{1}{n}\log N > \log|\lambda(p)| - \varepsilon.$$

Let us now specify more precisely how we obtain g.

Let $\alpha(x,y)$ be a C^∞ function from U to $\mathbb{R}$ so that $\alpha(x,y) = 1$ on a neighborhood U_1 of I and $\alpha(x,y) = 0$ off a slightly larger neighborhood contained in U. Given the neighborhood N of f, let $\delta > 0$ be small enough so that any g which is δ - C^1-close to f must be in N. Let A be a small constant, and consider the area preserving transformation $\xi(x,y) = (x, A\cos\frac{\pi x N}{2a} + y)$. It carries the line segment $-a \leq x \leq a$, $y = 0$ onto a curve I' as described earlier.

The generating function for ξ is $S(x,\eta) = x\eta - \int_0^x A\cos(\frac{\pi s N}{2a})ds$ where $\zeta(x,y) = x$ and $\eta(x,y) = A\cos(\frac{\pi x N}{2a}) + y$. Note that $S_{x\eta} = 1$ throughout the region, so (x,η) is a good coordinate system throughout.

Let $\beta(x,\eta) = \alpha(x,y(x,\eta)) = \alpha(x,\eta - A\cos\frac{\pi x N}{2a})$, and let $S_1(x,\eta) = \beta(x,\eta)(S(x,\eta) - x\eta) + x\eta$. The reader may check that as AN approaches 0, the function $S(x,\eta) - x\eta$ approaches 0 in the C^2 topology. Thus, for AN small, $S_{1x\eta}(x,\eta) \neq 0$ for all x,η. We may find a C^1 function $\eta_1(x,y)$ so that $S_{1x}(x,\eta_1(x,y)) = y$, and $\eta_1(x,y)$ approaches $\eta(x,y)$ in the C^1 topology as $AN \to 0$. Let $\psi(x,y) = (S_{1\eta}(x,\eta_1(x,y)),\eta_1(x,y))$ be the area preserving transformation induced by S_1, and let $g = \psi\circ f$. For some small constant $K_1 > 0$, if we put $A = \frac{K_1\delta a}{2N}$, then $|g - f|_{C^1} < \delta$ and $g = f$ off $f^{-1}U_1$ as required.

We now construct the rectangle D_A. Let $W^s_{loc}(p,g)$ be a closed interval in $W^s(p,g)$ containing p and I in its interior, and let V be a tubular neighborhood of $W^s_{loc}(p,g)$. We assume that U is contained in V. For a set E and a point z in E, let C(z,E) be the connected component of E which contains z. Let γ_1 be the curve in U given by $x = -a$, $0 \leq y \leq 2A$, and let γ_2 be the curve given by $x = a$, $0 \leq y \leq 2A$. Set $\{z_1\} = \gamma_1 \cap I'$ and $\{z_2\} = \gamma_2 \cap I'$. Since $I' \subset W^u(p,g)$, parts of backward iterates of γ_1 and γ_2 will accumulate on $W^s_{loc}(p,g)$ by the λ-lemma [8]. Also, there are constants $K_2, K_3 > 0$ so that if $g^j(z) \in V$ for $0 \leq j \leq m$, then

$$K_2|\lambda(p)|^{-m} \leq \text{dist}(z, W^s_{loc}(p,g)) \leq K_3|\lambda(p)|^{-m},$$

and if $g^{-j}(z) \in V$ for $0 \leq j \leq m$, then

$$K_2|\lambda(p)|^{-m} \leq \text{dist}(z,C(p,W^u(p,g) \cap V)) \leq K_3|\lambda(p)|^{-m}.$$

For this step it is convenient to assume via a preliminary approximation that f is C^2. Then g is C^2 as well and hence C^1 linearizable on $W^s(p,g)$ and $W^u(p,g)$ near p.

For n large the curves γ_1, γ_2, $C(g^{-n}z_1, g^{-n}\gamma_1 \cap V)$, and $C(g^{-n}z_2, g^{-n}\gamma_2 \cap V)$ will enclose a rectangle R_n in U near I. Let γ_1' and γ_2' be the pieces of γ_1 and γ_2 in that rectangle as indicated in figure 4.

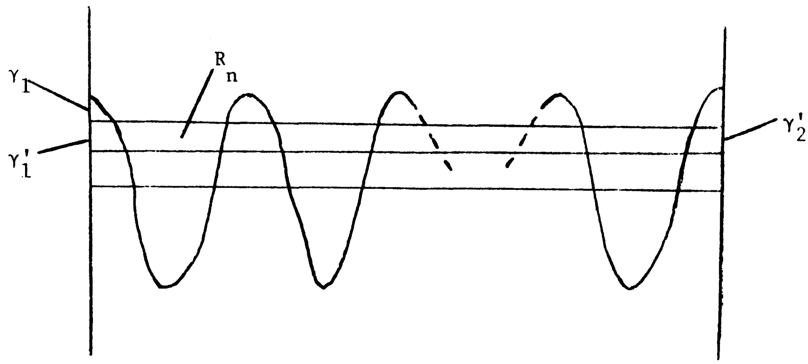

Figure 4

Let n be the smallest positive integer such that $C(g^{-n}z_1, g^{-n}\gamma_1 \cap V)$ and $C(g^{-n}z_2, g^{-n}\gamma_2 \cap V)$ are C^1 closer to $W^s_{loc}(p,g)$ than $\frac{A}{4}$ and $g^n\gamma_1'$ and $g^n\gamma_2'$ are C^1 closer to I' than $\frac{A}{4}$. There are constants K_4, $K_5 > 0$ so that $K_4|\lambda(p)|^{-n} \leq A \leq K_5|\lambda(p)|^{-n}$. Set $D_A = R_n$, $\Lambda_1 = \bigcap_{-\infty<j<\infty} g^{jn}D_A$, and $\Lambda = \bigcup_{0\leq j\leq n} g^j\Lambda_1$. For N large, the reader may verify, with estimates similar to those in [7] and [6], that Λ is hyperbolic basic set for g. Clearly, $\Lambda \subset \overline{H(p,g)}$ and, as we have indicated, $h(g|\Lambda) = \frac{1}{n}\log N > \log|\lambda(p)| - \varepsilon$. This proves statement (a) of the proposition when $\tau(p) = 1$.

When $\tau(p) > 1$, the proof is analogous except that z_0 will be in $W^s(p,f) \cap W^u(f^k p,f)$, $[0 \leq k < \tau(p)]$. The n above may then be chosen of the form $n = \tau(p)n_1 + k$, and we have the estimate $K_4|\lambda(p)|^{-n_1} \leq A \leq K_5|\lambda(p)|^{-n_1}$. We obtain Λ and g near f so that $h(g|\Lambda) = \frac{1}{n}\log N = \frac{1}{\tau(p)n_1+k}\log N$, and $\frac{1}{\tau(p)n_1+k}\log N \to \frac{1}{\tau(p)}\log|\lambda(p)|$ as $N \to \infty$.

We now move on to statement (b) of the proposition. We assume $\tau(p) = 1$ leaving the remaining generalization to the reader.

Consider the rectangle D_A and the mapping g^n. It is clear from figure 3 that $g^n D_A \cap D_A$ has N components. These are slanted "rectangles" joining the top and bottom of D_A as in the next figure.

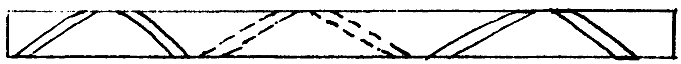

Figure 5

Also, $g^{-n}(D_A) \cap D_A$ consists of N rectangular strips stretching across D_A. In the standard way, this implies that for $k > 0$, $\bigcap_{-k \leq j \leq 0} g^{jn} D_A$ consists of N^k thin rectangular strips joining the sides of D_A, and $\bigcap_{0 \leq j \leq k} g^{jn} D_A$ consists of N^k thin slanted rectangular strips joining the top and bottom of D_A. Each component of $\bigcap_{-k \leq j \leq k} g^{jn} D_A$ is a small disk whose diameter is larger than $(K_6 |\lambda(p)|^{-n})^k$ with $K_6 > 0$ independent of N. There are N^{2k} such components and their diameters approach zero as $k \to \infty$.

From this it follows that the Hausdorff dimension α of $\bigcap_{-\infty < j < \infty} g^{jn} D_A$ satisfies

$$\alpha \geq \alpha_1 = \inf\{\beta : \inf_{k \geq 0} N^{2k}(K_6 |\lambda(p)|^{-n})^{k\beta} = 0\}.$$

Now α_1 is given by $N^2(K_6 |\lambda(p)|^{-n})^{\alpha_1} = 1$ or $\alpha_1 = \dfrac{2 \log N}{n \log|\lambda(p)| - \log K_6}$.

But for some constant $K_7 > 0$ independent of N, $n \log|\lambda(p)| < K_7 + \log N$, so $\alpha_1 > \dfrac{2 \log N}{K_7 + \log N - \log K_6}$. Thus $\alpha_1 \to 2$ as $N \to \infty$, so $\alpha \to 2$. Given $\varepsilon > 0$, we choose N_1 large enough so that $\dfrac{2 \log N_1}{K_7 + \log N_1 - \log K_6} > 2 - \varepsilon$. Then,

$HD(\Lambda) > 2 - \varepsilon$ with $\Lambda = \bigcup_{0 \le j \le n} g^j \left(\bigcap_{-\infty < k < \infty} g^{nk} D_A \right)$. For g_1 near g, each component of $\bigcap_{-k \le j \le k} g_1^{jn} D_A$ has diameter larger than $(K_6 |\lambda(p)|^{-n} - \varepsilon_1)^k$ with ε_1 small, so we can insure that $HD(\Lambda(g_1)) > 2 - \varepsilon$. This completes the proof of the proposition.

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