A remark on vector fields on open manifolds

T. Nadzieja

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A REMARK ON VECTOR FIELDS ON OPEN MANIFOLDS

by

T. NADZIEJA

Summary. A partial answer is given to the question in what circumstances for a vector field $X$ on an open manifold $M$ there exists a neighborhood $U$ in $C^0$-Whitney topology, such that for every $Y \in U$ and compact $K \subset M$ the closure of positive semitrajectory $\bigcup_{t \geq 0} Y_t(K)$ is compact.

Let $M$ be an open differentiable manifold / i.e. nen compact without boundary, with a countable basis / and let $X^1(M)$ be the class of all $C^1$ vector fields on $M$ endowed with $C^0$-Whitney topology which is given by the neighborhoods of zero

$$\left\{ X \in X^1(M) : \|X(p)\| \leq \varepsilon(p) \right\}$$

where $\varepsilon$ is a real positive continuous function on $M$ and $\| \|$ is a Riemannian complete metric on $M$.

Throughout the paper $X$ will denote a complete $C^1$ vector field on $M$ and $\{X_t\}_{t \in \mathbb{R}}$ will be the corresponding flow generated by $X$. The positive semitrajectory of a point $p \in M$ will be denoted by $O^+_X(p) = \{X_t(p) : t \geq 0\}$, its $\omega$-limit set by $\omega^X(p)$ and $\alpha$-limit set by $\alpha^X(p)$. 
Most of the notations and definitions used here are as in [1], [3]

DEFINITION 1. We say that $X \in X^1(M)$ is C-stable iff there exists a neighborhood $U$ of $X$ such that for every $Y \in U$ and every compact $K \subset M$ the closure of semitrajectory of $K$ is compact.

EXAMPLES. Let $M = \mathbb{R}^2$ and let flow $\{X_t\}_{t \in \mathbb{R}}$ be given by

$$\frac{d}{dt} x^1 = x^2 \quad \frac{d}{dt} x^2 = -x^1$$

The vector field $(x^2, -x^1)$ is not C-stable. It is easy to see that the vector field on $\mathbb{R}^2$ given by

$$\frac{d}{dt} x^1 = -x^1 \quad \frac{d}{dt} x^2 = -x^2$$

is C-stable.

We ask: Which conditions imply C-stability of $X$?

In this paper we give a partial answer to this question. Proofs and ideas we use in this paper are similar to those used in studying the phenomenon of $\mathcal{L}$-explosion / see [2].

DEFINITION 2. A compact invariant subset $A$ of $M$ is called an attractor iff there exists a neighborhood $V$ of $A$ such that for every $p \in V$ its $\omega$-limit set $\omega^X(p)$ is contained in $A$. The domain of attraction of $A$ is a maximal subset $D$ of $M$ such that for every $p \in D$, $\omega^X(p) \subset A$.

DEFINITION 3. Let $A$ be an attractor. We say that $A$ is uniformly asymptotically stable if for every neighborhood $U$ of $A$ there exists neighborhood $V$ of $A$ such that $V \subset U$ and $X_t(V) \subset V$ for every $t \geq 0$. 

316
Wilson [4] proved that if $A$ is uniformly asymptotically stable and $D$ is a domain of attraction of $A$ then there exists a smooth function $L : D \rightarrow \mathbb{R}^+$ such that:

I. $L|_A = 0$

II. For any $c \in \mathbb{R}^+$ dist$(L^{-1}(c), A) < +\infty$ and

$$\lim_{c \to 0} \text{dist}(L^{-1}(c), A) = 0$$

here dist denotes the Hausdorff distance.

III. $\lim_{p_n \to \infty} L(p_n) = +\infty$ and $\lim_{p_n \to D} L(p_n) = +\infty$

and $p_n \to \infty$ means that dist$(p_n, A)$ tends to $+\infty$.

IV. If $p \in D \setminus A$ then $\frac{d}{dt} L(x_t(p)) < 0$

The function $L$ is called Lyapunov function.

**DEFINITION 4.** A filtration for $X$ is a collection

$$\{ M_i : i = 1, 2, \ldots \}$$

of submanifolds of $M$ / with boundaries / such that for every $i$

1. $M_i$ is compact and $M_i \subset \text{Int } M_{i+1}$

2. $x_t(M_i) \subset M_i$ for every $t \geq 0$

3. $X$ is transverse to the boundary $\partial M_i$ of $M_i$

4. $\bigcup_{i \in \mathbb{N}} M_i = M$

It is clear that if a filtration for $X$ exists then for every compact $K \subset M$ $\bigcup_{t \geq 0} x_t(K)$ is compact.

The following property which follows immediately from Definition 4 shows that the existence of filtration is an open property.

**LEMMA 1.** If $\{ M_i : i = 1, 2, \ldots \}$ is a filtration for $X$ then there is a $C^0$-neighborhood $U$ of $X$ such that $\{ M_i : i = 1, 2, \ldots \}$ is also a filtration for every $Y \in U$. 317
In virtue of this Lemma the existence of a filtration implies C-stability of $X$.

**LEMMA 2.** Let $X \in X^1(M)$, $\bigcup_{t \geq 0} X_t(K)$ be compact for every compact $K \subset M$ and let $A$ be an attractor with domain of attraction $M$. Then there is an uniformly assymptotically stable set $B$ such that $A \subset B$.

**PROOF.** We present in detail an argument from [2].

Let $B = \{ p \in M : \chi^X(p) \cap A \neq \emptyset \}$. By definition, $B$ is invariant and closed. Let $W$ be the compact neighborhood of $A$. If $p \in M$ and $\chi^X(p) \cap A \neq \emptyset$ then there exists $t < 0$ such that $X_t(p) \in W$ hence $p \in \bigcup_{t \geq 0} X_t(W)$ so the set $B$ is compact / being a closed subset of compact $\bigcup_{t \geq 0} X_t(W)$ / . It is clear that $B$ is an attractor.

We show that $B$ is uniformly asymptotically stable. Let $U$ be the compact neighborhood of $A$. By definition of the set $B$, all points of $U \setminus B$ have their $\chi$-limit sets empty. For every positive real number $r$ denote $A_r = \bigcap_{0 \leq t \leq r} X_t(U)$. We note that the compact sets $A_r$ are nested. We show that for sufficiently large $r$, $X_t(A_r) \subset \text{Int } A_1$ for $0 \leq t \leq 1$. Consider the sets

$$V_r = \bigcup_{0 \leq t \leq 1} X_t(A_r) \setminus \text{Int } A_1$$

which are a nested family of compact sets with empty intersection hence there exists $\check{r}$ such that $V_{\check{r}} = \emptyset$. For such $r$ and $0 \leq T \leq 1$ $X_T(A_r) \subset A_r$. This implies that $X_T(A_r) \subset A_r$ for every $T \geq 0$. We put $\text{Int } A_r = V$. Then $B \subset V \subset U$ and $X_t(V) \subset V$ for every $t \geq 0$. 

318
**THEOREM 1.** Let \( X \in X^1(M) \) and let for every compact \( K \subset M \) the closure of positive semitrajectory \( \bigcup_{t \geq 0} X_t(K) \) and the set \( F = \bigcup_{p \in M} \omega^X(p) \) be compact. Then the vector field \( X \) is \( C \)-stable.

**PROOF.** \( F \) is an attractor with domain of attraction \( M \). By Lemma 2 there is an uniformly asymptotically stable set \( B \) with domain of attraction \( M \). Therefore \([4]\) there exists a smooth Lyapunov function \( L \) for \( B \). We define \( M_1 = L^{-1}([0, 1]) \). \( M_1 \) is a compact / being a closed, bounded subset of Riemannian manifold with complete metric / submanifold and \( X_t(M_1) \subset M_1 \) for every \( t \geq 0 \).

Now define a sequence of submanifolds \( \{M_i : i = 1, 2, \ldots\} \) by putting \( M_i = X_{-1}(M_i) \). It is then clear that \( \{M_i : i = 1, 2, \ldots\} \) is a filtration for \( X \), and by Lemma 1, \( X \) is \( C \)-stable.

Suppose that for a vector field \( X \) the set \( F = \bigcup_{p \in M} \omega^X(p) \) is a union of compact, invariant, isolated subset \( \omega_i \) i.e.

\[(\ast) \quad F = \omega_1 \cup \omega_2 \cup \omega_3 \cup \ldots\]

Let \( W^s \omega_1 = \left\{ p \in M : \omega^X(p) \subset \omega_1 \right\} \) and \( W^u \omega_1 = \left\{ p \in M : \omega^X(p) \supset \omega_1 \right\} \) and define in \( \{\omega_i : i = 1, 2, \ldots\} \) the relation

\[\omega_i \leq \omega_j \quad \text{iff} \quad W^s \omega_i \cap W^u \omega_j \neq \emptyset\]

**LEMMA 3.** Let \( \omega_1, \omega_2, \omega_3 \) be any sets appearing in \((\ast)\) and suppose that there exists a point \( p_0 \in \omega_2 \) such that \( \omega^X(p_0) = \omega_2 \).

If \( \omega_1 \supset \omega_2 \supset \omega_3 \) then in every neighborhoods \( U \) and \( V \) of \( X \) and \( \omega_1 \) respectively there are \( Y \in U \) and \( p \in V \) such that \( \omega^Y(p) \subset \omega_3 \).

This Lemma is consequence lemma 9 from \([2]\).
THEOREM 2. Let $X$ be a vector field, $F = \bigcup_{p \in M} \omega_X(p) = \omega_1 \cup \omega_2 \cup \ldots$ be a union of infinitely many compact, invariant, isolated sets and let for every $\omega_1$ there exists $p_i \in \omega_1$ such that $\gamma_X(p_i) = \omega_1$. Moreover let $\bigcup_{t \geq 0} X_t(K)$ be compact for every compact $K$. Then $X$ is stable iff no infinite sequence

$$\omega_{i_1} \geq \omega_{i_2} \geq \omega_{i_3} \geq \ldots$$

exists.

PROOF. If suffices to show the existence of a filtration for $X$. We define a set

$$A_1 = \{ \omega_k : \text{there is a sequence } \omega_{i_1}, \ldots, \omega_{i_j} \text{ such that } \omega_{i_1} = \omega_1, \omega_{i_j} = \omega_k \text{ and } \omega_{i_j} \geq \omega_{i_2} \geq \ldots \geq \omega_{i_j} \}$$

Due to our assumption there is no infinite sequence

$$\omega_{i_1} \geq \omega_{i_2} \geq \omega_{i_3} \geq \ldots$$

and since $\bigcup_{t \geq 0} X_t(K)$ is compact for every compact $K$, $A_1$ is finite and $C_1 = \bigcup_{A_1} \omega_1$ is compact, therefore there is a neighborhood $V$ of $C_1$ such that for every $p \in V$ $\omega_X(p)$ is contained in $C_1$. Hence $C_1$ is an attractor with a domain of attraction $D_1 = \bigcup_{A_1} \omega_1$. Let $B_1 = \{ p \in D_1 : \alpha_X(p) \cap C_1 \neq \emptyset \}$.

The set $B_1$ is compact invariant / see Lemma 2 / and $B_1 \subset D_1$ by definition of the relation $\leq$. In the same way as in Lemma 2 we may show that $B_1$ is uniformly asymptotically stable. Let $L_1$ be a Lyapunov function for $B_1$ and $M_1 = L_1^{-1}(\omega_1, \overline{1})$.

Starting with $M_1$ we construct a filtration $M_1, M_2, M_3, \ldots$ by induction. Suppose that the submanifolds $M_1, M_2, M_3, \ldots, M_k$ are already done. To define $M_{k+1}$, put $N_k = X_{-1}(M_k)$. 
We choose a set \( \omega_i \) not contained in \( N_k \) and define
\[
A_{k+1} = \left\{ \omega_k : \text{there is a sequence } \omega_{i_1}, \ldots, \omega_{i_j} \text{ such that } \omega_{i_1} = \omega_{i_0} \text{ and } \omega_k = \omega_{i_j} \text{ and } \omega_{i_1} \gg \omega_{i_2} \gg \ldots \gg \omega_{i_j} \right\}
\]
Again
\[
C_{k+1} = \bigcup_{\omega_i \in A_{k+1}} \omega_i \cup B_k \text{ is an attractor.}
\]
Let \( B_{k+1} \) be the uniformly asymptotically stable set such that \( C_{k+1} \subset B_{k+1} \) and let \( L_{k+1} \) be a Lyapunov function for \( B_{k+1} \).

Put \( M_{k+1} = L_{k+1}^{-1} \left( [0, c_k] \right) \) where \( c_k \) is chosen such that \( N_k \subset M_{k+1} \).

The sequence \( M_1, M_2, M_3, \ldots \) of compact submanifolds is a filtration for \( X \), for it is clear that \( X_t(M_1) \subset M_1 \) for \( t > 0 \) and \( X \) is transverse to \( \partial M_1 \). To verify that \( i = 1 \) \( M_i = M \),
take \( p \in M \), by assumption \( \omega^X(p) \neq \emptyset \), hence there exists \( k \)
such that \( p \) is contained in domain of attraction of \( B_k \). By our construction \( X_i(M_k) \subset M_{k+1} \) so there exists \( l \) such that \( p \in M_1 \), hence \( \bigcup_{i=1}^{l} M_i = M \).

Suppose now that there exists an infinite sequence
\[
\omega_{i_1} \gg \omega_{i_2} \gg \omega_{i_3} \gg \ldots \ . \text{ We will show then that in every neighborhood of the vector } X \text{ there is a vector field } Y \text{ and a point } p \in M \text{ such that } \omega^Y(p) = \emptyset \ . \text{ Since the closure of the semitrajectory of every compact set is compact, we may choose from the sequence } \omega_{i_1}, \omega_{i_2}, \ldots \text{ a sequence } \omega_{i_k} \gg \omega_{i_2} \gg \omega_{i_3} \gg \ldots \text{ such that if only } k > l+1 \text{ then } \overline{\omega_{i_k}} \cap \overline{\omega_{i_l}} = \emptyset \}
Then choose for every $\tilde{\omega}_k$ a neighborhood $V_{ik}$ of $\tilde{\omega}_k$ such that $V_{ik} \cap V_{i1} = \emptyset$ if $k \neq 1$. By Lemma 3 there are a vector field $Y_1$ and $p_{i1} \in V_{i1}$ such that $\omega Y_1(p_{i1}) \subseteq \tilde{\omega}_{i3}$.

$Y_1 = X$ off $V_{i1}$ and $\sup_{p \in M} ||Y_1(p) - X(p)||$ is arbitrarily small. Similarly we change the vector field $Y_1$ on $V_{i3}$ so that for this new vector field $Y_2$, $\omega Y_2(p_{i1}) \subseteq \tilde{\omega}_{i4}$ and then repeat this procedure for $V_{i4}, V_{i5}, \ldots$. In this way we get a vector field $Y$ which is arbitrarily near to $X$ and such that $\omega Y(p_{i1}) = \emptyset$.

This proves the necessity part of our theorem.

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Tadeusz Nadzieja
Instytut Matematyczny, Uniwersytet Wrocławski
Pl. Grunwaldzki 2/4 50 234 Wrocław