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by

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§ 0. Introduction

We introduce the notion of topological conditional entropy for a subset of a compact space and a continuous transformation. It is a generalization of the topological entropy defined by Bowen ([1]) and the topological conditional entropy introduced by Misiurewicz ([3]).

§ 1. Definition

The following notations and terminology will be used:

X - a compact space

$f : X \to X$ - a continuous transformation

$Y \subset X$ - a subset of X

$\mathcal{P}(X)$ - the set of all finite covers of X

$\mathcal{U}(X)$ - the set of all open finite covers of X

N - positive integers

$C \prec A$ - if a subset C of X is contained in some member of $A \in \mathcal{P}(X)$

$A \succ B$ - if $a \prec B$ for every $a \in A$ and $A, B \in \mathcal{P}(X)$

$$A^n = \bigvee_{i=0}^{n-1} f^{-i}A \quad , \quad A^\infty = \bigvee_{i=0}^{\infty} f^{-i}A \quad \text{for } A \in \mathcal{P}(X), \ n \in N$$

Now, for $C \subset X$ and $A \in \mathcal{P}(X)$ let (see [1])

$$n_{f,A}(C) = \begin{cases} \sup\{k \in N : \ C \prec A^k\} & \text{if } C \prec A \\ 0 & \text{if } C \not\prec A \end{cases}$$

$$D_A(C) = e^{-h_{f,A}(C)}$$

$$\mathbf{D}_A(C,\lambda) = \sum_{i=1}^{\infty} D_A(c_i)^{\lambda} \qquad \text{where} \quad C = (c_i)_{i=1}^{\infty} \quad \text{and} \quad \lambda > 0$$

For $b \in B \in \wp(X)$ and $\epsilon > 0$ we define

$$F_{A,b,\lambda,\epsilon}(Y) = \inf_C \left\{ \mathbf{D}_A(C,\lambda) : \bigcup_{i=1}^{\infty} c_i \supset b \cap Y, \quad D_A(c_i) < \epsilon \right\}$$

and

$$F_{A|B,\lambda,\epsilon}(Y) = \max_{b \in B} F_{A,b,\lambda,\epsilon}(Y)$$

It is easy to see that

$$(1.1) \quad F_{A_1|B_1,\lambda_1,\epsilon_1}(Y) \geqslant F_{A_2|B_2,\lambda_2,\epsilon_2}(Y) \quad \text{for} \quad A_1 \geqslant A_2, \; B_1 \leqslant B_2, \; \lambda_1 \leqslant \lambda_2, \; \epsilon_1 \leqslant \epsilon_2$$

Now we can define

$$m_{A|B,\lambda}(Y) = \lim_{\epsilon \to 0} \lim_{n \to \infty} F_{A|B^n,\lambda,\epsilon}(Y) .$$

Notice that

$$m_{A_1|B_1,\lambda_1}(Y) \geqslant m_{A_2|B_2,\lambda_2}(Y) \quad \text{for} \quad A_1 \geqslant A_2, \quad B_1 \leqslant B_2, \; \lambda_1 \leqslant \lambda_2$$

and $m_{A|B,\lambda}(Y) \notin \{0, +\infty\}$ for at most one λ

Define

$$h_{A|B}(f,Y) = \inf \left\{ \lambda ; \; m_{A|B,\lambda}(Y) = 0 \right\} \qquad (\inf \phi = +\infty)$$

From the definition it follows that

$$(1.2) \quad h_{A_1|B_1}(f,Y) \geqslant h_{A_2|B_2}(f,Y) \qquad \text{for} \quad A_1 \geqslant A_2, \quad B_1 \leqslant B_2$$

Now we can take limits (finite or infinite) (see [3])

$$\lim_{A \in \mathfrak{A}(X)} h_{A|B}(f,Y) = \sup_{A \in \mathfrak{A}(X)} h_{A|B}(f,Y) = h(f|B,Y)$$

$$\lim_{B \in \mathfrak{A}(X)} h_{A|B}(f,Y) = \inf_{B \in \mathfrak{A}(X)} h_{A|B}(f,Y) = h^*(f,Y)$$

$h^*(f,Y)$ will be called the topological conditional entropy of f for a subset Y of X.

§ 2. Basic properties

The proofs of the following propositions are simple and therefore we omit them

Proposition 1

a) If $f_1 : X_1 \to X_1$ and $f_2 : X_2 \to X_2$ are topologically conjugate (i.e. there exists a homeomorphism $\Pi : X_1 \to X_2$ such that $\Pi \circ f_1 = f_2 \circ \Pi$) then $h^*(f_1,Y_1) = h^*(f_2,\Pi(Y_1))$ for $Y_1 \subset X_1$

b) If $f : X \to X$ is a homeomorphism, then $h^*(f,f^{-1}(Y)) = = h^*(f,Y) = h^*(f,f(Y))$.

Proposition 2

If $n \in N$ and $A,B \in \mathcal{P}(X)$, then

a) $h_{A^n|B}(f^n,Y) = nh_{A|B}(f,Y)$

b) $h(f^n|B,Y) = nh(f|B,Y)$

c) $h^*(f^n,Y) = nh^*(f,Y)$

Proposition 3

If $n \in N$ and $A,B \in \mathcal{P}(X)$, then

a) $h_{A|B}(f, \bigcup_{i=1}^{\infty} Y_i) = \sup_i h_{A|B}(f,Y_i)$

b) $h^*(f, \bigcup_{i=1}^{n} Y_i) = \max_{1 \leq i \leq n} h^*(f,Y_i)$

From Proposition 3 it follows that if Y is countable, then $h^*(f,Y) = 0$.

§ 3. Connection with topological conditional entropy

First let us recall the definition of the topological conditional entropy $h^*(f)$, ([3]).

$$h^*(f) = \lim_{B\in\mathfrak{A}(X)} h(f|B) = \inf_{B\in\mathfrak{A}(X)} h(f|B)$$

where $h(f|B) = \lim\limits_{A\in\mathfrak{A}(X)} h(f,A|B) = \sup\limits_{A\in\mathfrak{A}(X)} h(f,A|B)$

and $h(f,A|B) = \lim\limits_{n\to\infty} \frac{1}{n}\log N(A^n|B^n)$.

Moreover for $A,B \in \wp(X)$ $\quad N(A|B) = \max\limits_{b\in B} N(b,A)$,

where for $b \subset X$, $b \neq \phi$ $\quad N(b,A) = \min\{\text{Card } C;\ C\subset A,\ \bigcup c_i \supset b$

and $N(\phi,A) = 1$.

Theorem 1

$$h^*(f,X) = h^*(f)$$

Proof

It is sufficient to prove that

a) if $A,B \in \wp(X)$, then $h_{A|B}(f,X) \leqslant h(f,A|B)$

b) $h^*(f,X) \geqslant h^*(f)$

For the proof of a) let $\lambda > 0$, $\epsilon > 0$ and $e^{-n} < \epsilon$,
where $n \in N$. Moreover let $b^n \in B^n$ be such, that
$F_{A|B^n,\lambda,\epsilon}(X) = F_{A,b^n,\lambda,\epsilon}(X)$ and let $C\subset A^n$ be a cover of

b^n with $N(b^n, A^n)$ members. Then

$$D_A(C, \lambda) \leq \text{Card } C \cdot e^{-n\lambda} \leq N(A^n | B^n) e^{-n\lambda}$$

and

$$F_{A|B^n, \lambda, \epsilon}(X) \leq F_{A|B^n, \lambda, e^{-n}}(X) \leq D_A(C, \lambda) \leq$$

$$\leq (e^{\frac{1}{n}\log N(A^n|B^n) - \lambda})^n$$

If $\lambda > h(f, A|B)$, then $\lim_{n\to\infty} F_{A|B^n, \lambda, \epsilon}(X) = 0$. Since ϵ was arbitrary, it follows that

$$m_{A|B, \lambda}(X) = 0, \quad \text{hence} \quad h(f, A|B) \geqslant h_{A|B}(f, X)$$

For the proof of b) let $m_{A|B,\lambda}(X) = 0$ where $A, B \in \mathfrak{A}(X)$ and $\lambda > 0$ then for fixed $0 < \eta < 1$ and $0 < \epsilon < 1$ there exists $n_o \in N$ such that

$$F_{A|B^{n_o}, \lambda, \epsilon}(X) < \eta \quad .$$

Hence for every $b^{n_o} \in B^{n_o}$ there exists a cover $C(b^{n_o}) = (c_i)_{i=1}^{\infty}$ of b^{n_o} with $D_A(c_i) < \epsilon$ (so $n_{f,A}(c_i) > 0$) and $D_A(C(b^{n_o}), \lambda) < \eta$. We may assume that c_i is open. Let $D \in \mathfrak{A}(X)$ and $\bar{D} \geqslant B^{n_o}$, where $\bar{D} = \{\bar{d} : d \in D\}$. Since for $d \in D$ $\bar{d}$ is compact, therefore there exists a finite, open cover $C^*(d)$ of $\bar{d}$ such that $C^*(d) \subset C(b^{n_o})$ for some $b^{n_o} \in B^{n_o}$.

If $C = \{c \in C^*(d) : d \in D\}$ and $K = \text{Card } C$, then $0 < k < \infty$. Let $n \in N$ and for $c_1, \ldots, c_s \in C$ such that

$$n_{f,A}(c_1) + \ldots + n_{f,A}(c_{s-1}) < n \quad \text{and}$$

$$n \leqslant n_{f,A}(c_1) + \ldots + n_{f,A}(c_s)$$

$$Z(c_1,\ldots,c_s) = \left\{ x \in c_1 :\ f^{n_{f,A}(c_1)+\ldots+n_{f,A}(c_r)}(x) \in c_{r+1} \right.$$
$$\left. \text{for } r = 1,\ldots,s-1 \right\}$$

then

$$Z(c_1,\ldots,c_s) \prec A^n$$

If $d^n \in D^n$ and $d^n = \bigcap_{j=0}^{n-1} f^{-j} d_j$, where $d_j \in D$ for $j=0,\ldots,n-1$ then

$$Z = \left\{ Z(c_1,\ldots,c_s) :\ s \geqslant 1, \quad c_1 \in C^*(d_0), \right.$$
$$c_{r+1} \in C^*(d_{n_{f,A}(c_1)+\ldots+n_{f,A}(c_r)})$$
$$\left. \text{for } r = 1,\ldots,s-1 \right\} \quad \text{covers } d^n.$$

Hence

$$N(d^n,A^n)e^{-\lambda n} \leqslant \text{Card } Z\ e^{-\lambda n} \leqslant$$

$$\leqslant \sum_{c \in C} \Big(1 + \sum_{s \geqslant 2} \sum_{Z(c_1,\ldots,c_{s-1},c) \in Z} e^{-\lambda(n_{f,A}(c_1)+\ldots+n_{f,A}(c_{s-1}))}$$

$$\leqslant \sum_{c \in C} \Big(1 + \sum_{s \geqslant 2} \sum_{c_1} e^{-\lambda n_{f,A}(c_1)} (\ldots(\sum_{c_{s-1}} e^{-\lambda n_{f,A}(c_{s-1})})\ldots)\Big)$$

$$\leqslant K \sum_{s=0}^{\infty} \eta^s = M < \infty$$

and

$$\left(e^{-\lambda + \frac{1}{n}\log N(A^n|D^n)}\right)^n \leqslant M$$

Therefore if $m_{A|B,\lambda}(X) = 0$, then $h(f,A|D) \leqslant \lambda$ (for some D which depends on B).

This implies that

$h(f,A|D) \leqslant h_{A|B}(f,X)$ and therefore $h^*(f) \leqslant h^*(f,X)$.

Remark.

If Y is closed and f - invariant then $h^*(f,Y) = h^*(f|Y)$.

The proof is like that of Theorem 1 and is close to the proof of Proposition 1 [1] .

§ 4. Connection with topological entropy introduced by Bowen ([1]).

R. Bowen defined the topological entropy $h(f,Y)$ in the following way:

$$h(f,Y) = \sup_{A \in \mathfrak{A}(X)} h_A(f,Y)$$

where for $A \in \mathfrak{A}(X) h_A(f,Y) = \inf \{\lambda ; m_{A,\lambda}(Y) = 0\}$

and for $\lambda > 0$

$$m_{A,\lambda}(Y) = \lim_{\epsilon \to 0} \inf\left\{D_A(C,\lambda) : \bigcup_{i=1}^{\infty} c_i \supset Y, \quad D_A(c_i) < \epsilon\right\}$$

It is easy to see that $h(f,Y) \geqslant h^*(f,Y)$

Theorem 2 gives another relation between these notions

Theorem 2

Let $A \in \mathfrak{A}(X)$ and $B \in \mathcal{G}(X)$. Then

$$h_{A|B}(f,Y) \geqslant \sup_{b^\infty \in B^\infty} h_A(f,Y \cap b^\infty).$$

If in addition Y is closed and B is a closed cover then the reverse inequality also holds.

Proof.

If $B \in \mathcal{P}(X)$ and $n \in N$ then for any $b^\infty \in B^\infty$ exists $b^n \in B^n$ such that $b^\infty \subset b^n$.

Therefore for $\epsilon > 0$

$$\inf\left\{D_A(C,\lambda) : \bigcup_{i=1}^{\infty} c_i \supset Y \cap b^\infty , \ D_A(c_i) < \epsilon\right\} \leqslant$$

$$\leqslant F_{A,b^n,\lambda,\epsilon}(Y) \leqslant F_{A|B^n,\lambda,\epsilon}(Y)$$

and $\quad h_A(f, Y \cap b^\infty) \leqslant h_{A|B}(f,Y)$

This gives the required inequality.

For the proof of the second part let $\epsilon > 0$ and $\lambda > 0$.

Then for any $b^\infty \in B^\infty$ there exists a cover $C(b^\infty) = \{c_i(b^\infty)\}_{i=1}^{\infty}$ of $b^\infty \cap Y$ such that $D_A(c_i(b^\infty)) < \epsilon$ for $i = 1,2,\ldots$ and $D_A(C(b^\infty),\lambda) \leqslant F_{A,b^\infty,\lambda,\epsilon}(Y) + \epsilon$

We may assume that $c_i(b^\infty)$ are open.

Let $U(b^\infty) = \bigcup_{i=1}^{\infty} c_i(b^\infty)$ and $U = \{U(b^\infty)\}_{b^\infty \in B^\infty}$, then U covers Y.

For $W = \{d^n \in Y \cap B^n : n \in N\}$ and $V = \{d^n \in W : d^n \not\prec U\}$ we define $g : V \to W$ in the following way:

if $d^{n+1} \in V$, where $d^{n+1} = d^n \cap f^{-n}b_n$ for $d^n \in W$ and $b_n \in B$, then

$$g(d^{n+1}) = d^n.$$

A sequence $(d^n)_{n=1}^{m}$ with $g(d^{n+1}) = d^n$ for $n = 1,\ldots,m-1$ is called a branch of lenght m with initial vertex d^1.

We notice that for every $d^n \in W$ $g^{-1}(d^n)$ is finite and d^1 is from a finite family $Y \cap B$.

Suppose that for every $m \in N$ there is a branch of lenght m. By means of König's Lemma ([2], p.104) there exists an infinite branch, that is a sequence $(d^n)_{n \in N}$ with $d^{n+1} \subset d^n$ and $d^n \not\prec U$ for $n > 1$. There exists $b^\infty \in B^\infty$ such that $\bigcap_{n \in N} d^n = Y \cap b^\infty \subset U(b^\infty)$.

Since each d^n is closed, so $d^n \subset U(b^\infty)$ for sufficiently large n. This is impossible, because $d^n \in V$ for $n > 1$. We have proved that there exists $n_o \in N$ such that all branches have lenght less then n_o. Hence for $b^{n_o} \in B^{n_o}$ such that $F_{A|B^{n_o},\lambda,\epsilon}(Y) = F_{A,b^{n_o},\lambda,\epsilon}(Y)$ we have $Y \cap b^{n_o} \prec U$, so $Y \cap b^{n_o} \subset U(b^\infty)$ for some $b^\infty \in B^\infty$.

From this and (1.1) it follows that for $n \geqslant n_o$

$$F_{A|B^n,\lambda,\epsilon}(Y) \leqslant F_{A|B^{n_o},\lambda,\epsilon}(Y) \leqslant D_A(C(b^\infty),\lambda) \leqslant$$

$$\leqslant F_{A,b^\infty,\lambda,\epsilon}(Y) + \epsilon$$

This implies that $m_{A|B,\lambda}(Y) \leqslant m_{A,\lambda}(Y \cap b^\infty)$ and this gives the required inequality.

Corollary 1.

$$h^*(f,Y) \geqslant \inf_{B \in \mathfrak{A}(X)} \sup_{b^\infty \in B^\infty} h(f, Y \cap b^\infty) \text{ for any } Y \subset X$$

Corollary 2.

If $h^*(f,\bar{Y} - Y) = 0$, then $h^*(f,Y) = \inf_{B \in \mathfrak{A}(X)} \sup_{b^\infty \in B^\infty} h(f, Y \cap b^\infty)$

Proof

In view of Proposition 3 b) § 2, Proposition 2 c) [1] and Corollary 1 it is sufficient to prove that if $\bar{Y} = Y$ then

$$h^*(f,Y) = \inf_{B \in \mathfrak{A}(X)} \sup_{b^\infty \in B^\infty} h(f, Y \cap b^\infty) .$$

Let $\bar{Y} = Y$. For any $B \in \mathfrak{A}(X)$ there exists $C_B \in \mathfrak{A}(X)$ such that $\bar{C}_B \geqslant B$. Then by Theorem 2b) and (1.2) we obtain

$$h^*(f,Y) \leqslant \inf_{B \in \mathfrak{A}(X)} \sup_{A \in \mathfrak{A}(X)} h_{A|\bar{C}_B}(f,Y) \leqslant \inf_{B \in \mathfrak{A}(X)} \sup_{A \in \mathfrak{A}(X)} \sup_{c \in \bar{C}_B^\infty} h_A(f, Y \cap c) \leqslant$$

$$\leqslant \inf_{B \in \mathfrak{A}(X)} \sup_{A \in \mathfrak{A}(X)} \sup_{b^\infty \in B^\infty} h_A(f, Y \cap b^\infty) =$$

$$= \inf_{B \in \mathfrak{A}(X)} \sup_{b^\infty \in B^\infty} h(f, Y \cap b^\infty)$$

In view of Corollary 1 this completes the proof.

References

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