B. MARCZYŃSKA

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On a generalization of topological conditional entropy

by

B. Marczyńska

§ 0. Introduction

We introduce the notion of topological conditional entropy
for a subset of a compact space and a continuous transformation.
It is a generalization of the topological entropy defined by
Bowen ([1]) and the topological conditional entropy introduced
by Misiurewicz ([3]).

§ 1. Definition

The following notations and terminology will be used:

$X$ - a compact space
$f : X \rightarrow X$ - a continuous transformation
$Y \subseteq X$ - a subset of $X$
$\mathcal{C}(X)$ - the set of all finite covers of $X$
$\mathcal{U}(X)$ - the set of all open finite covers of $X$
$N$ - positive integers
$C \ll A$ - if a subset $C$ of $X$ is contained in some
member of $A \in \mathcal{C}(X)$

$A \succ B$ - if $a \ll B$ for every $a \in A$ and $A, B \in \mathcal{C}(X)$

$A^n = \bigcup_{i=0}^{n-1} f^{-i}A$, $A^\infty = \bigcup_{i=0}^{\infty} f^{-i}A$ for $A \in \mathcal{C}(X)$, $n \in N$

Now, for $C \subseteq X$ and $A \in \mathcal{C}(X)$ let (see [1])

$n_{f,A}(C) = \begin{cases} 
\sup \{k \in N : C \ll A^k\} & \text{if } C \ll A \\
0 & \text{if } C \not\ll A
\end{cases}$
\[ D_A(C) = e^{-\frac{1}{\lambda} A(C) \lambda} \]

\[ D_A(C, \lambda) = \sum_{i=1}^{\infty} D_A(c_i)^\lambda \]

where \( C = (c_i)_{i=1}^{\infty} \) and \( \lambda > 0 \)

For \( b \in B \in \mathcal{P}(X) \) and \( \varepsilon > 0 \) we define

\[ F_{A,b,\lambda,\varepsilon}(Y) = \inf_C \left\{ D_A(C, \lambda) : \bigcup_{i=1}^{\infty} c_i \supset b \cap Y, \quad D_A(c_i) < \varepsilon \right\} \]

and

\[ F_{A,B,\lambda,\varepsilon}(Y) = \max_{b \in B} F_{A,b,\lambda,\varepsilon}(Y) \]

It is easy to see that

(1.1) \( F_{A_1,B_1,\lambda_1,\varepsilon_1}(Y) \geq F_{A_2,B_2,\lambda_2,\varepsilon_2}(Y) \) for \( A_1 \supseteq A_2, B_1 \subseteq B_2, \lambda_1 \leq \lambda_2, \varepsilon_1 \leq \varepsilon_2 \)

Now we can define

\[ m_{A,B,\lambda}(Y) = \lim_{\varepsilon \to 0} \lim_{n \to \infty} F_{A,B^n,\lambda,\varepsilon}(Y). \]

Notice that

\[ m_{A_1,B_1,\lambda_1}(Y) \geq m_{A_2,B_2,\lambda_2}(Y) \] for \( A_1 \supseteq A_2, B_1 \subseteq B_2, \lambda_1 \leq \lambda_2 \)

and \( m_{A,B,\lambda}(Y) \notin \{ 0, +\infty \} \) for at most one \( \lambda \)

Define

\[ h_{A,B}(f,Y) = \inf \left\{ \lambda : m_{A,B,\lambda}(Y) = 0 \right\} \quad (\inf \emptyset = +\infty) \]

From the definition it follows that

(1.2) \( h_{A_1,B_1}(f,Y) \geq h_{A_2,B_2}(f,Y) \) for \( A_1 \supseteq A_2, B_1 \subseteq B_2 \)

Now we can take limits (finite or infinite) (see [3]).
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\[ \lim_{A \in \mathcal{A}(X)} h_{A|B}(f,Y) = \sup_{A \in \mathcal{A}(X)} h_{A|B}(f,Y) = h(f|B,Y) \]

\[ \lim_{B \in \mathcal{B}(X)} h_{A|B}(f,Y) = \inf_{B \in \mathcal{B}(X)} h_{A|B}(f,Y) = h^*(f,Y) \]

\( h^*(f,Y) \) will be called the topological conditional entropy of \( f \) for a subset \( Y \) of \( X \).

§ 2. Basic properties

The proofs of the following propositions are simple and therefore we omit them.

Proposition 1

a) If \( f_1 : X_1 \rightarrow X_1 \) and \( f_2 : X_2 \rightarrow X_2 \) are topologically conjugate (i.e. there exists a homeomorphism \( \Pi : X_1 \rightarrow X_2 \) such that \( \Pi \circ f_1 = f_2 \circ \Pi \)) then \( h^*(f_1,Y_1) = h^*(f_2,\Pi(Y_1)) \) for \( Y_1 \subset X_1 \).

b) If \( f : X \rightarrow X \) is a homeomorphism, then \( h^*(f,f^{-1}(Y)) = h^*(f,Y) \).

Proposition 2

If \( n \in \mathbb{N} \) and \( A,B \in \mathcal{A}(X) \), then

a) \( h_{A^n|B}(f^n,Y) = nh_{A|B}(f,Y) \)

b) \( h(f^n|B,Y) = nh(f|B,Y) \)

c) \( h^*(f^n,Y) = nh^*(f,Y) \)

Proposition 3

If \( n \in \mathbb{N} \) and \( A,B \in \mathcal{A}(X) \), then

a) \( h_{A|B}(f, \bigcup_{i=1}^{\infty} Y_i) = \sup_{i} h_{A|B}(f,Y_i) \)

b) \( h^*(f, \bigcup_{i=1}^{\infty} Y_i) = \max_{1 \leq i \leq n} h^*(f,Y_i) \)
From Proposition 3 it follows that if $Y$ is countable, then $h^*(f,Y) = 0$.

§ 3. Connection with topological conditional entropy

First let us recall the definition of the topological conditional entropy $h^*(f)$, ([3]).

$$h^*(f) = \lim_{\mathcal{B} \in \mathcal{U}(X)} h(f|\mathcal{B}) = \inf_{\mathcal{B} \in \mathcal{U}(X)} h(f|\mathcal{B})$$

where

$$h(f|\mathcal{B}) = \lim_{n \to \infty} \frac{1}{n} \log N(A^n|B^n).$$

Moreover for $A,B \in \mathcal{O}(X)$

$$N(A|B) = \max_{b \in B} N(b,A),$$

where for $b \subset X$, $b \neq \emptyset$

$$N(b,A) = \min \{ \text{Card } C; C \subset A, \cup C = b \}$$

and $N(\emptyset,A) = 1$.

Theorem 1

$$h^*(f,X) = h^*(f)$$

Proof

It is sufficient to prove that

a) if $A,B \in \mathcal{O}(X)$, then $h_{A|B}(f,X) \leq h(f,A|B)$

b) $h^*(f,X) \geq h^*(f)$

For the proof of a) let $\lambda > 0$, $\epsilon > 0$ and $e^{-n} < \epsilon$,

where $n \in \mathbb{N}$. Moreover let $b^n \in B^n$ be such, that

$F_{A|B^n,\lambda,\epsilon}(X) = F_{A,b^n,\lambda,\epsilon}(X)$ and let $C \subset A^n$ be a cover of

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\( b^n \) with \( N(b^n, A^n) \) members. Then

\[
D_A(C, \lambda) \leq \text{Card } C \cdot e^{-n\lambda} \leq N(A^n|B^n)e^{-n\lambda}
\]

and

\[
P_{A|B^n, \lambda, \varepsilon}(X) \leq P_{A|B^n, \lambda, \varepsilon-n}(X) \leq D_A(C, \lambda) \leq \\
\leq \frac{1}{n} \log N(A^n|B^n) - \lambda^n
\]

If \( \lambda > h(f, A|B) \), then \( \lim_{n \to \infty} P_{A|B^n, \lambda, \varepsilon}(X) = 0 \). Since \( \varepsilon \) was arbitrary, it follows that

\( m_{A|B, \lambda}(X) = 0 \), hence \( h(f, A|B) \geq h_{A|B}(f, X) \)

For the proof of b) let \( m_{A|B, \lambda}(X) = 0 \) where \( A, B \in \Omega(X) \)

and \( \lambda > 0 \) then for fixed \( 0 < \eta < 1 \) and \( 0 < \varepsilon < 1 \) there exists \( n_0 \in \mathbb{N} \) such that

\[ F_{A|B^{n_0}, \lambda, \varepsilon}(X) < \eta \]

Hence for every \( b^{n_0} \in B^{n_0} \) there exists a cover \( C(b^{n_0}) = \bigcup_{i=1}^{\infty} (c_i) \) of \( b^{n_0} \) with \( D_A(c_i) < \varepsilon \) (so \( n_{f, A}(c_i) > 0 \)) and \( D_A(C(b^{n_0}), \lambda) < \eta \). We may assume that \( c_i \) is open.

Let \( D \in \Omega(X) \) and \( \tilde{D} \supseteq B^{n_0} \), where \( \tilde{D} = \{ \tilde{d} : d \in D \} \).

Since for \( d \in D \) \( \tilde{d} \) is compact, therefore there exists a finite, open cover \( C^\infty(d) \) of \( \tilde{d} \) such that \( C^\infty(d) \subseteq C(b^{n_0}) \) for some \( b^{n_0} \in B^{n_0} \).

If \( C = \{ c \in C^\infty(d) : d \in D \} \) and \( K = \text{Card } C \), then \( 0 < k < \infty \).

Let \( n \in \mathbb{N} \) and for \( c_1, \ldots, c_s \in C \) such that
\[ n_{f,A}(c_1) + \ldots + n_{f,A}(c_{s-1}) < n \quad \text{and} \]
\[ n \leq n_{f,A}(c_1) + \ldots + n_{f,A}(c_s) \]

\[ Z(c_1, \ldots, c_s) = \left\{ x \in c_1 : f^{n_{f,A}(c_1)} + \ldots + n_{f,A}(c_r)(x) \in c_{r+1} \right. \]
\[ \quad \text{for } r = 1, \ldots, s-1 \}

then
\[ Z(c_1, \ldots, c_s) \subset \mathbb{A}^n \]

If \( d^n \in \mathcal{D}^n \) and \( d^n = \bigcap_{j=0}^{\infty} f^{-j}d_j \), where \( d_j \in \mathcal{D} \) for \( j=0, \ldots, n-1 \)

then
\[ Z = \left\{ Z(c_1, \ldots, c_s) : s \geq 1, \ c_1 \in \mathcal{C}(d_0), \ c_{r+1} \in \mathcal{C}(d_{n_{f,A}(c_1)} + \ldots + n_{f,A}(c_r)) \right. \]
\[ \quad \text{for } r = 1, \ldots, s-1 \}

covers \( d^n \).

Hence
\[ N(d^n, \mathbb{A}^n)e^{-\lambda n} \leq \text{Card } Z e^{-\lambda n} \leq \]
\[ \leq \sum_{c \in \mathcal{C}} \left( 1 + \sum_{s=2}^\infty \frac{\sum_{(c_1, \ldots, c_s) \in Z} e^{-\lambda n_{f,A}(c_1)} \ldots \sum_{c_{s-1}} e^{-\lambda n_{f,A}(c_{s-1})}}\right) \]
\[ \leq \sum_{c \in \mathcal{C}} \left( 1 + \sum_{s=2}^\infty \sum_{c_1} e^{-\lambda n_{f,A}(c_1)} \ldots \sum_{c_{s-1}} e^{-\lambda n_{f,A}(c_{s-1})} \right) \]
\[ \leq K \sum_{s=0}^\infty \eta^s = M < \infty \]

and
\[ (e^{-\lambda + \frac{1}{n} \log N(\mathbb{A}^n|\mathcal{D}^n)})^n \leq M \]
Therefore if \( m_{A|B} (X) = 0 \), then \( h(f, A|D) \leq \lambda \)
(for some \( D \) which depends on \( B \)).
This implies that
\[
h(f, A|D) \leq h_{A|B}(f, X) \quad \text{and therefore} \quad h_S(f) \leq h^S(f, X).
\]

Remark.
If \( Y \) is closed and \( f \)-invariant then \( h^S(f, X) = h^S(f|Y) \).
The proof is like that of Theorem 1 and is close to the proof of Proposition 1 [1].

§ 4. Connection with topological entropy introduced by Bowen ([1]).
R. Bowen defined the topological entropy \( h(f, Y) \) in the following way:
\[
h(f, Y) = \sup_{A \in \mathcal{O}(X)} h_{A}(f, Y)
\]
where for \( A \in \mathcal{O}(X) \) \( h_{A}(f, Y) = \inf \{ \lambda ; m_{A, \lambda} (Y) = 0 \} \)
and for \( \lambda > 0 \)
\[
m_{A, \lambda} (Y) = \lim_{\epsilon \to 0} \inf \{ D_A(C, \lambda) : \bigcup_{i=1}^{\infty} c_i \supset Y, D_A(c_i) < \epsilon \}
\]
It is easy to see that \( h(f, Y) \geq h^S(f, Y) \).
Theorem 2 gives another relation between these notions.

Theorem 2
Let \( A \in \mathcal{O}(X) \) and \( B \in \mathcal{G}(X) \). Then
\[
h_{A|B}(f, Y) \geq \sup_{B \in \mathcal{G}(X)} h_{A}(f, Y \cap B).
\]
If in addition \( Y \) is closed and \( B \) is a closed cover then the reverse inequality also holds.
Proof.
If $B \in \mathcal{F}(X)$ and $n \in \mathbb{N}$ then for any $b^\infty \in B^\infty$ exists $b^n \in B^n$ such that $b^\infty \subset b^n$.
Therefore for $\varepsilon > 0$
\[
\inf \left\{ D_A(C, \lambda) : \bigcup_{i=1}^\infty c_i \supset Y \cap b^\infty, \ D_A(c_i) < \varepsilon \right\} \leq \inf \left\{ F_A, b^n, \lambda, \varepsilon \right\} (Y) \leq F_A | B^n, \lambda, \varepsilon (Y)
\]
and $h_A(f, Y \cap b^\infty) \leq h_A | B(f, Y)$

This gives the required inequality.
For the proof of the second part let $\varepsilon > 0$ and $\lambda > 0$.
Then for any $b^\infty \in B^\infty$ there exists a cover $\mathcal{C}(b^\infty) = \{c_i(b^\infty)\}_{i=1}^\infty$ of $b^\infty \cap Y$ such that $D_A(c_i(b^\infty)) < \varepsilon$
for $i = 1, 2, \ldots$ and $D_A(\mathcal{C}(b^\infty), \lambda) \leq F_A, b^\infty, \lambda, \varepsilon (Y) + \varepsilon$

We may assume that $c_i(b^\infty)$ are open.
Let $U(b^\infty) = \bigcup_{i=1}^\infty c_i(b^\infty)$ and $U = \{ U(b^\infty) \}_{b^\infty \in B^\infty}$, then $U$ covers $Y$.
For $W = \{ d^n \subset Y \cap B^n : n \in \mathbb{N} \}$ and $V = \{ d^n \in W : d^n \nsubseteq U \}$
we define $g : V \rightarrow W$ in the following way:
if $d^{n+1} \in V$, where $d^{n+1} = d^n \cap f^{-n}b_n$ for $d^n \in W$ and $b_n \in B$,
then $g(d^{n+1}) = d^n$.

A sequence $(d^n)_{n=1}^m$ with $g(d^{n+1}) = d^n$ for $n = 1, \ldots, m-1$
is called a branch of length $m$ with initial vertex $d^1$. 

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We notice that for every $d^n \in W$, $g^{-1}(d^n)$ is finite and $d^1$ is from a finite family $Y \cap B$.

Suppose that for every $m \in \mathbb{N}$ there is a branch of length $m$. By means of König's Lemma (\cite{2}, p. 104) there exists an infinite branch, that is a sequence $(d^n)_{n \in \mathbb{N}}$ with $d^{n+1} \subseteq d^n$ and $d^n \not\subseteq U$ for $n > 1$. There exists $b^\infty \in B^\infty$ such that $\bigcap_{n \in \mathbb{N}} d^n = Y \cap b^\infty \subseteq U(b^\infty)$.

Since each $d^n$ is closed, so $d^n \subseteq U(b^\infty)$ for sufficiently large $n$. This is impossible, because $d^n \subseteq V$ for $n > 1$.

We have proved that there exists $n_0 \in \mathbb{N}$ such that all branches have length less than $n_0$. Hence for $b^{n_0} \in B^{n_0}$ such that $F_{A|B^{n_0}}, \lambda, \epsilon$ we have $Y \cap b^{n_0} \not\subseteq U$.

so $Y \cap b^{n_0} \subseteq U(b^\infty)$ for some $b^\infty \in B^\infty$.

From this and (1.1) it follows that for $n \geq n_0$

$$F_{A|B^\infty}, \lambda, \epsilon (Y) \leq F_{A|B^{n_0}}, \lambda, \epsilon (Y) \leq D_A(C(b^\infty), \lambda) \leq F_{A, b^{n_0}}, \lambda, \epsilon (Y) + \epsilon$$

This implies that $m_{A|B, \lambda} (Y) \leq m_{A, \lambda} (Y \cap b^\infty)$ and this gives the required inequality.

Corollary 1.

$$h^\infty(f, Y) \geq \inf_{B \in \mathcal{O}(X)} \sup_{b^{n_0} \in B^\infty} h(f, Y \cap b^{n_0}) \text{ for any } Y \subseteq X$$

Corollary 2.

If $h^\infty(f, \overline{Y} - Y) = 0$, then $h^\infty(f, Y) = \inf_{B \in \mathcal{O}(X)} \sup_{b^{n_0} \in B^\infty} h(f, Y \cap b^{n_0})$.
Proof

In view of Proposition 3 b) § 2, Proposition 2 c) [1] and Corollary 1 it is sufficient to prove that if \( Y = Y \) then

\[
    h^X(f,Y) = \inf_{B \in \mathcal{C}(X)} \sup_{b^D} h(f,Y \cap b^D).
\]

Let \( Y = Y \). For any \( B \in \mathcal{C}(X) \) there exists \( C_B \in \mathcal{C}(X) \) such that \( C_B \supseteq B \). Then by Theorem 2b) and (1.2) we obtain

\[
    h^X(f,Y) \leq \inf_{B \in \mathcal{C}(X)} \sup_{A \in \mathcal{C}(X)} h_A(f,Y \cap c^D) = \inf_{B \in \mathcal{C}(X)} \sup_{A \in \mathcal{C}(X)} h(f,Y \cap b^D) = \inf_{B \in \mathcal{C}(X)} \sup_{b^D} h(f,Y \cap b^D)
\]

In view of Corollary 1 this completes the proof.

References


B. Marczynska
Institute of Mathematics
Warsaw University