Some results on expanding mappings

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In this note we give some theorems concerning smooth invariant measures for expanding mappings. They complete the results from [3] and [4].

The following notation and terminology will be used:

\( M \) - a compact, connected \( C^\infty \)-manifold,

\( \| \cdot \| \) - a \( C^\infty \)-Riemannian metric on \( M \),

\( \mathcal{B} \) - the family of all Borel subsets of \( M \),

\( M_r \) - the space of all normalized \( C^r \)-measures on \( M \) (\( r = 0, 1, \ldots, \infty \)) with \( C^r \)-topology (\( r = 0, 1, \ldots, \infty \)),

\( E_r \) - the space of all expanding \( C^r \)-mappings of \( M \) (\( r = 1, 2, \ldots, \infty \)), i.e. all \( C^r \)-mappings \( \psi : M \to M \) for which there exist \( a > 0 \) and \( b > 1 \) such that

\[ \| D\psi^n(\alpha)\| \geq ab^n \| \alpha \| \quad \text{for} \quad \alpha \in T(M) \quad \text{and} \quad n \in \mathbb{N}, \]

\( \mu \) - a fixed element of \( M_{r-1} \) (\( r = 1, \ldots, \infty \)),

\( \mathcal{C} \) - the positive \( C^{r-1} \)-function on \( M \) such that if \( \psi|A \) is injective, then

\[ \mu(\psi(A)) = \int_A \psi^\ast \mu \quad \text{where} \quad A \in \mathcal{B} \quad (\psi \in E_r). \]
$U_{\varphi}$ - the mapping of $L(\mu)$ into itself such that

$$U_{\varphi}(f)(x) = \sum_{\bar{x} \in \varphi^{-1}(x)} f(\bar{x})(\mathcal{J}\varphi(\bar{x}))^{-1} \quad \text{for} \quad x \in M,$$

($\varphi \in E_1$),

$\nu_{\varphi}^n$ - the mapping of $L(\mu)$ into itself such that

$$\nu_{\varphi}^n(f) = \frac{1}{n!}U_{\varphi}^{n-1}(f \ln \mathcal{J}\varphi^n) \quad (n \in \mathbb{N}, \ \varphi \in E_1),$$

$C^k(M,\mathbb{R})$ - the space of all real $C^k$-functions on $M$

($k = 0,1,\ldots$),

$\| \cdot \|_k$ - the norm on $C^k(M,\mathbb{R})$,

$C^{k,\alpha}(M,\mathbb{R})$ - the space of all real $C^k$-functions on $M$ whose $k$-th derivatives satisfy a Hölder condition with exponent $\alpha$ ($k = 0,1,\ldots, \alpha \in [0,1]$),

$\| \cdot \|_{k,\alpha}$ - the semi-norm on $C^{k,\alpha}(M,\mathbb{R})$ defined by means of the Hölder constant of $k$-th derivatives,

$\| \cdot \|_{k,\alpha}$ - the norm $\max(\| \cdot \|_k, \| \cdot \|_{k,\alpha})$ on $C^{k,\alpha}(M,\mathbb{R})$

In the sequel $r = 2,\ldots, \alpha \in [0,1]$ and in Theorem 1 $r = 2,\ldots, \infty, \omega$. The first part of Theorem 1 and the second one for $r = 2$ is from [3] and [5].

Theorem 1.

i) For each $\varphi \in E_r$ there exists a unique $\varphi$-invariant measure $\mu_{\varphi} \in \mathcal{M}_{\tau-1}$

ii) If $r \neq \omega$, then the mapping $E_r \ni \varphi \rightarrow \mu_{\varphi} \in \mathcal{M}_{\tau-1}$

is continuous.

The first author's proof of Theorem 1 ($r \neq \omega$) was based on the Banach's theorem on contraction, the Rademacher's
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Theorem on differentiability of Lipschitz functions and the following Proposition. Suppose that $h$ is a real $C^k$-function ($k = 0, 1, \ldots$) from an open subset $A$ of $\mathbb{R}^m$ such that $D^kh$ is Lipschitz. Let $H$ be a continuous mapping from $A$ to the space of $(k+1)$-linear functionals on $\mathbb{R}^m$ such that $D^{k+1}h = H$ almost everywhere. Then $h$ is of class $C^{k+1}$ and $D^{k+1}h = H$.

We now outline this proof for $r = 2$. One first shows that there exists a function $g_{\mathcal{C}} \in C^{0,1}(\mathbb{M}, \mathbb{R})$ such that the measure $\mu_{\mathcal{C}} \in \mathcal{M}_{\mathcal{C}}$ with the density $g_{\mathcal{C}}$ is $\mathcal{C}$-invariant (see b) below). The function $g_{\mathcal{C}}$ satisfies the equation

1) $U_{\mathcal{C}}(g) = g$.

Therefore $Dg_{\mathcal{C}}$ (which exists $\mu$-almost everywhere by the Rademacher theorem) satisfies the equation

2) $A_{\mathcal{C}}(f) = f$

obtained by differentiating both sides of 1) and replacing $g$ and $Dg$ by $g_{\mathcal{C}}$ and $f$ respectively. Using the Banach's theorem on contraction one proves that 2) has a unique solution in the space of continuous sections of $T^\mathbb{M}$ as well as in the space of essentially bounded ones. This and Proposition imply that $g_{\mathcal{C}} \in C^1(\mathbb{M}, \mathbb{R})$. It is easy to see that in view of b) the above reasoning also gives ii). This method can also be used for $r > 2$ but is complicated in technical respect.

Let us remark that the above method can be used in proof of the stable manifold theorem for hyperbolic sets.

In the second author's proof one first shows the following
Theorem 2. For each $\varphi \in E_r$ and each $\alpha$ there exist a neighbourhood $\mathcal{Y}$ of $\varphi$ in $E_r$, numbers $\delta > 0$ and $q \in ]0,1[$ such that for $\psi \in \mathcal{Y}$ and $n \in \mathbb{N}$

$$\|\tilde{U}_n^\delta\|_{r-2,\alpha} \leq \delta q^n.$$ 

In the above theorem for $\varphi \in E_r$ and $n \in \mathbb{N}$ $\tilde{U}_n^\varphi$ denotes the operator of $L(\mu)$ into itself such that $\tilde{U}_n^\varphi(f) = U_{\varphi n}(f) - \int f \, d\mu$, where $\varphi \in C^{r-2,\alpha}(M,\mathbb{R})$ is the density of the $\varphi$-invariant normalized measure $\mu_\varphi$ with respect to $\mu$. $\tilde{U}_n^\varphi$ also acts in $L^p(\mu)$ ($p \geq 1$), $C^{r-2,\alpha}(M,\mathbb{R})$ and in $C^{r-1}(M,\mathbb{R})$, if in addition $\varphi \in C^{r-1}(\tilde{\mu},\mathbb{R})$.

The norms $\| \cdot \|_{r-2,1}$ and $\| \cdot \|_{r-1}$ are equivalent in $C^{r-1}(M,\mathbb{R})$ and $C^{r-1}(M,\mathbb{R})$ is closed in $C^{r-2,1}(M,\mathbb{R})$. Therefore Theorem 2 implies Theorem 1 i) $(r \neq \omega)$ and the following Corollary 1.

Corollary 1. For each $\varphi \in E_r$ there exist a neighbourhood $\mathcal{Y}$ of $\varphi$ in $E_r$ and numbers $\delta > 0$, $q \in ]0,1[$ such that for $\psi \in \mathcal{Y}$ and $n \in \mathbb{N}$

$$\|\tilde{U}_n^\delta\|_{r-1} \leq \delta q^n.$$ 

It is easy to see that the mapping $E_r \ni \varphi \mapsto U_\varphi(1) \in C^{r-1}(\tilde{\mu},\mathbb{R})$ is continuous. Therefore Corollary 1 gives Theorem 1 ii).

We now outline the proof of Theorem 2. One first shows

a) For each $\varphi \in E_2$ and each $\alpha$ there exist a neighbourhood $\mathcal{Y}$ of $\varphi$ in $E_2$, numbers $q \in ]0,1[$, $L > 0$ such that for $\psi \in \mathcal{Y}$ and $n \in \mathbb{N}$
3) \( \| U \psi_n \|_\infty \leq L \),
4) \( \| U \psi_n(f) \|_{0,\alpha} \leq L(q^n \| f \|_{0,\alpha} + \| f \|_0) \)
for \( f \in C^{0,\alpha}(M,R) \).

From a) it follows

b) For each \( \Upsilon \in E_2 \) there exists a unique \( \Upsilon \)-invariant measure \( \mu_{\Upsilon} \in \mathcal{M}_0 \). The mapping \( E_2 \ni \Upsilon \rightarrow \mu_{\Upsilon} \in \mathcal{M}_0 \) is continuous.

For the proof of b) let us remark that in view of a) the set \( \{ H(\Upsilon) \} \), \( \Upsilon \in \mathcal{N}_0 \), is relatively compact in \( C(M,R) \), where

\( H(\Upsilon) = n^{-1} \sum_{k=0}^{n-1} U \Upsilon^k \).

Therefore there exists a sequence \( (H_n(\Upsilon)) \) convergent in \( C(M,R) \) to a function \( \Upsilon \Upsilon \).

The measure \( \mu_{\Upsilon} \) such that \( \frac{d\mu_{\Upsilon}}{d\mu} = \Upsilon \Upsilon \) has the required properties. The uniqueness of \( \mu_{\Upsilon} \) follows from \([2]\). In the proof of the second part of b) one uses relative compactness in \( C(M,R) \) of the set \( \{ \Upsilon \Upsilon \} \), \( \Upsilon \in \mathcal{N}_0 \), and the uniqueness of \( \mu_{\Upsilon} \), where \( \mathcal{N}_0 \) is from a).

We now show
c) For each \( \Upsilon \in E_2 \)

\[ \sup_{\| f \|_{0,\alpha} \leq 1} \| U_{\Upsilon}^n(f) \|_0 \rightarrow 0. \]

For the proof let us remark that the dynamical system

\( (M,\mu_{\Upsilon},\Upsilon) \) is exact \([2]\). Hence \( U_{\Upsilon}^n(f) \rightarrow 0 \) for \( f \in L(\mu_{\Upsilon}) \). This and relative compactness in \( C(M,R) \) of the set \( \{ U_{\Upsilon}^n(f) \} \), \( n \in \mathbb{N} \), for \( f \in C^{0,\alpha}(M,R) \) imply that \( U_{\Upsilon}^n(f) \rightarrow 0 \).

To complete the proof it is sufficient to use a).
From a) - c) it follows Theorem 2 for \( r = 2 \). To show this let \( q_1, q_2, 0 < q_1 < q_2 < 1 \) be such that there exists \( n \in \mathbb{N} \) that

\[
5) \quad 3q^n_2 + Lq_1 \leq q_2 .
\]

Then, in view of c) one may assume that

\[
6) \quad \sup_{\|f\|_0,\alpha \leq 1} \| \tilde{u}^i_{\alpha} (f) \|_0 < q_1
\]

for \( i = \tilde{n}, 2\tilde{n} \). Now let us remark that the mapping \( E_2 \ni \psi \rightarrow U_{\psi} \in L_\infty (c^{0,\alpha}(M,R), C(M,R)) \) is continuous. Therefore in view of b) and 6) we may assume that for \( \psi \in \mathcal{V} \) and \( i = \tilde{n}, 2\tilde{n} \)

\[
7) \quad \sup_{\|f\|_0,\alpha \leq 1} \| \tilde{u}^i_{\psi} (f) \|_0 < q_1
\]

and in view of a) and b) - that for \( \psi \in \mathcal{V} \)

\[
8) \quad \sup_{\|f\|_0,\alpha \leq 1} \| \tilde{u}^i_{\psi} (f) \|^{0,\alpha} \leq 3L .
\]

Now let \( \psi \in \mathcal{V} \) and \( \| f \|_0,\alpha \leq 1 \). Then a), 5), 7) and 8) imply

\[
9) \quad \| \tilde{u}^{2\tilde{n}}_{\psi} (f) \|_0,\alpha \leq q_2 .
\]

Notice that there exist a neighbourhood \( \mathcal{V}_1 \) of \( \psi \) in \( E_2 \) and \( c > 0 \) such that \( \text{for } \psi \in \mathcal{V}_1 \quad \| \tilde{u}^i_{\psi} \|_0,\alpha \leq c \).

From this and 9) it follows Theorem 2 for \( r = 2 \). Further the proof goes by induction on \( r \) for all \( \alpha \). In the inductive step the proof is based on
d) For each \( \psi \in E_r \) and each \( \alpha \) there exist a neighbourhood \( V \) of \( \psi \) in \( E_r \), numbers \( L > 0, \ q \in [0,1[ \) and \( \bar{n} \in N \) such that for \( \psi \in V \) and \( f \in C^{r-2,\alpha}(M,\mathbb{R}) \)

\[
\| U^{\bar{n}}(f) \|_{r-2,\alpha} \leq q \| f \|_{r-2,\alpha} + L \| f \|_{r-2} .
\]

Now, suppose that Theorem 2 is true for the number \( r-1 \), where \( r > 2 \). Then using the induction hypothesis and d) one shows that there exist a neighbourhood \( V_1 \) of \( \psi \) in \( E_r \) and number \( L_1 > 0 \) such that for \( \psi \in V_1 \), \( n \in N \) and \( f \in C^{r-2,\alpha}(M,\mathbb{R}) \)

\[
\| U^{\bar{n}}(f) \|_{r-2,\alpha} \leq q^n \| f \|_{r-2,\alpha} + L_1 \| f \|_{r-2} .
\]

Further reasoning is similar to that for \( r = 2 \).

Theorem 2 also implies

Corollary 2. For each \( \psi \in E_2 \) there exists a neighbourhood \( V \) of \( \psi \) in \( E_2 \) such that for \( f \in C(M,\mathbb{R}) \)

\( f \in L^p(\mu) \), where \( p \in [1, +\infty[ \) the sequence \( (U^{\bar{n}}(f)) \)

is convergent in \( C(M,\mathbb{R}) \) \( (L^p(\mu)) \) to 0 uniformly in \( \psi \in V \).

Corollary 3. For each \( \psi \in E_2 \) and each \( \alpha \) there exist a neighbourhood \( V \) of \( \psi \) in \( E_2 \) and numbers \( \delta > 0, \ q \in [0,1[ \) such that for \( \psi \in V \) and \( n \in N \)

\[
\sup_{\| f \| \leq 1} \| f \|_{L(\mu)} \leq 1 \quad \left\| \int_M \psi^ng d\mu - \int_M f d\mu \right\|_\infty \leq \delta q^n .
\]

Corollary 4. For each \( \psi \in E_2 \) there exist a neighbourhood \( V \) of \( \psi \) in \( E_2 \) and numbers \( \delta > 0, \ q \in [0,1[ \) such
that for $\Psi \in \mathcal{Y}$, $n \in \mathbb{N}$ and $A \in \mathcal{B}$

$$|\mu(\psi^{-n}(A)) - \mu_{\Psi}(A)| \leq q^n \mu(A).$$

We now outline the proof of Theorem 1 for $r = \omega$. For this purpose let us remark that $\Pi : M^m \to M$ is a $C^\omega$-
universal covering of $M$, where $m = \dim M$ [9]. Then there exists a $C^\omega$-diffeomorphism $\hat{\psi}$ of $\hat{M}^m$ such that $\Pi \circ \hat{\psi} = \Psi \circ \Pi$.

We may assume that

10) \[ \| D\hat{\psi}(\sigma) \|^{*} \geq b \| \sigma \|^{*} \]

for $\sigma \in M^m$, where $b > 1$ and $\| \cdot \|^{*}$ is the lifting of $\| \cdot \|$ to $M^m$. Let $\Gamma$ be the group of the deck transfor-
mations of the covering $\Pi$. Then $\mathcal{G} \to \hat{\psi} \circ \gamma \circ \hat{\psi}^{-1}$ is a
homomorphism of $\Gamma$ onto a subgroup $\Gamma_1$ of $\Gamma$. Let $\Gamma_2$
be a set such that its intersection with each right coset of $\Gamma_1$ in $\Gamma$ is a one-point set. Then $\text{Card} \Gamma_2$ is equal to the
multiplicity of $\Psi$. If $\hat{C}(M^m, \mathbb{R})$ is the set of all $\Gamma$-in-
vARIANT $f \in C(M^m, \mathbb{R})$, then $C(M, \mathbb{R}) \ni f \to \hat{f} = f \circ \Pi \in \hat{C}(M^m, \mathbb{R})$
is a bijection. Therefore

11) \[ \hat{U}_{\hat{\psi}}(\hat{f}) = \hat{U}_{\hat{\psi}}(\hat{f}) \]
defines the operator of $\hat{C}(M^m, \mathbb{R})$ into itself. It turns out
that for $n \in \mathbb{N}$

12) \[ (\hat{U}_{\hat{\psi}})^n(1) = \sum_{(\hat{y}_{1}, ..., \hat{y}_{n}) \in \Gamma_2^n} \prod_{i=1}^{n} (\hat{\psi}^{-1})_{\hat{y}_{i-1}} \hat{\psi}_{\hat{y}_{i-1}}^{\hat{y}_{i-1}} \hat{\psi}_{\hat{y}_{i}}^{\hat{y}_{i}}, \]

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where $\mathcal{J}$ $\Phi^{-1}$ is defined in the similar way as $\mathcal{J} \Psi$. In the definition one replaces $\mu$ by the $C^\infty$-measure $\hat{\mu}$ on $\mathbb{R}^m$ such that $\pi$ transforms locally $\hat{\mu}$ on $\mu$. Then

Corollary 2 and 11) imply

13) $((\hat{\Psi})^n(1))$ is uniformly convergent to $\hat{\Psi}$.

Using 10) one shows that there exist open sets $G_1, G_2 \subset (\mathbb{R}^m)^2 = \mathbb{C}^m$, $G_1 \subset G_2$ such that $\gamma \in \Gamma_2$, $\Phi^{-1}$ and $\mathcal{J} \Phi^{-1}$ restricted to $G_2 \cap \mathbb{R}^m$ have the complex analytic extensions $f_\gamma, g$ and $h$ to $G_2$ respectively. Moreover, $\pi(G_1 \cap \mathbb{R}^m) = M$ and for each $i \in \mathbb{N}$, if $\gamma_1, ..., \gamma_i \in \Gamma_2$, then for $z \in G_1$

$f_\gamma^i (g(f_\gamma^{i-1}(...g(f_\gamma^1(z))))))$ is well-defined and belongs to $G_2$. Moreover, one shows that $(F_n)$ is uniformly bounded in $G_1$, where for $z \in G_1$

$F_n(z) = \sum_{(\gamma_1,...,\gamma_n) \in \Gamma_2^n} \prod_{i=1}^{n} h(f_\gamma^i (g(f_\gamma^{i-1}(...g(f_\gamma^1(z))))))$.

This 12), 13) and the Montel theorem for complex analytic functions give the required result.

It turns out that Theorem 1 i) is false for $r = 1$. In fact the following theorem is true.

Theorem 3. The set of all $\Phi \in E_1$ for which there exists a $\Phi$-invariant $\gamma \in \mathcal{N}_{\mathcal{E}_0}$, is of the first category in $E_1$.

For the proof let $\Phi \in E_1$ and let $\gamma \in \mathcal{N}_{\mathcal{E}_0}$ be $\Phi$-invariant. Then $\Phi \circ n_\Phi (\frac{d\gamma}{d\mu}) = \frac{d\gamma}{d\mu}$ for $n \in \mathbb{N}$. Therefore there exists $k \in \mathbb{N}$ such that

3) We identify $\mathbb{R}^m$ with $\mathbb{R}^m \times \{0\}$. 

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Then, for each \( k \in \mathbb{N} \) the set \( A_k \) of all \( \varphi \in E_1 \) for which (14) is satisfied, is closed. Moreover, by a perturbation of \( \varphi \in A_k \) in a neighbourhood of \( \bigcup_{i=1}^{n} \varphi^{-1}(x_0) \), where \( x_0 \) is a fixed point of \( M \) and \( n \) is sufficiently large, one shows that \( A_k \) is also a boundary set. This proves the theorem.

It is easy to see that in the above theorem one may replace \( \mathcal{H}_0 \) by the set of all normalized measures \( \nu \) absolutely continuous with respect to \( \mu \) and such that \( \varepsilon^{-1} \leq \frac{d
u}{d\mu} \leq \varepsilon \) \( \mu \)-almost everywhere for a certain number \( \varepsilon > 0 \).

The following theorem says that in Corollary 2 in the case of the space \( L^p(\mu) \) one cannot replace convergence in \( L^p(\mu) \) by \( \mu \)-almost everywhere convergence.

**Theorem 4.** Let \( \varphi \in E_2 \) and \( p \in [1, +\infty[. \) Then the set of all \( f \in L^p(\mu) \) for which \( \mu \)-almost everywhere

\[
\sup_{n \geq 1} \left| U_{\varphi_n}(f) \right| < +\infty
\]

is of the first category in \( L^p(\mu) \).

For the proof one may assume that \( \mu = \mu_\varphi \). Then, let \( G \subset M \) be a non-empty connected open set with sufficiently small diameter and let \( k \) be a natural number. Then there exists a component \( G_k \) of \( \varphi^{-k}(G) \) such that

\[
\mu_\varphi(G_k) \leq N^{-k} \mu_\varphi(G),
\]

where \( N \) is the multiplicity of \( \varphi \). From this it follows that

\[
\sup_{\| f \|_{L^p(\mu)} \leq 1} \mu_\varphi(\{ x \in M : \sup_{n \geq 1} \left| U_{\varphi_n}(f)(x) \right| \geq N^p(\mu_\varphi(G))^{-1} \}) = 1.
\]
This and the Banach's principle [1] give the required result.

We now pass on to some results concerning convergence to invariant smooth measure for expanding mapping in which the rate is linear. For this purpose let for \( \mathcal{C} \in \mathfrak{E}_r \) and \( n \in \mathbb{N} \)
\[
\nabla^n_{\mathcal{C}} \text{ be the operator of } L(\mu) \text{ into itself such that }
\]
\[
\nabla^n_{\mathcal{C}}(f) = \mathcal{C}^n_{\mathcal{C}}(f) - \mathcal{C}^{n-1}_{\mathcal{C}}(f) d\mu,
\]
where \( \mathcal{C}^{n}_{\mathcal{C}}(f) = \ln \mathcal{C}_n^p f \mathcal{C}_n^q \mathcal{C}_n^r \).

\( \nabla^n_{\mathcal{C}} \) also acts in \( L^p(\mu) \) \((p > 1)\), \( C^{r-2,\alpha}(\mathfrak{M}, \mathbb{R}) \) and \( C^{r-1}(\mathfrak{M}, \mathbb{R}) \). Then the following theorem is true.

**Theorem 5.** For each \( \mathcal{C} \in \mathfrak{E}_r \) and each \( \alpha \) there exist a neighbourhood \( \mathcal{V} \) of \( \mathcal{C} \) in \( \mathfrak{E}_r \) and number \( \varepsilon > 0 \) such that for \( \mathcal{C} \in \mathcal{V} \) and \( n \in \mathbb{N} \)
\[
\|\nabla^n_{\mathcal{C}}\|_{r-2,\alpha} \leq n^{-1}\varepsilon.
\]

For the proof let us remark that
\[
15) \quad h_{\mathcal{C}} = \int_M \ln \mathcal{C} d\mu_{\mathcal{C}}, \quad \text{for } \mathcal{C} \in \mathfrak{E}_r,
\]
\[
16) \quad \text{the mapping } \mathfrak{E}_r \ni \mathcal{C} \rightarrow h_{\mathcal{C}} \in \mathbb{R}, \text{ is continuous.}
\]

Moreover, for \( \mathcal{C} \in \mathfrak{E}_r, f \in C^{r-2,\alpha}(\mathfrak{M}, \mathbb{R}) \) and \( n \in \mathbb{N} \)
\[
\nabla^n_{\mathcal{C}}(f) = n^{-1} \nabla^n_{\mathcal{C}}(f) + h_{\mathcal{C}} \nabla^n_{\mathcal{C}}(f),
\]
where \( \nabla^n_{\mathcal{C}}(f) = \mathcal{C}_n^p (f(\ln \mathcal{C}^n f - nh \mathcal{C})). \) Therefore in view of 16) and Theorem 2, it is sufficient to prove
there exist a neighbourhood \( Y \) of \( \Psi \) in \( E_r \) and number \( c > 0 \) such that for \( \Psi \in Y \) and \( n \in \mathbb{N} \)
\[
\| \nabla^n \|_{r-2, \infty} \leq c.
\]

For this purpose let us remark that in addition \( m \in \mathbb{N} \), then
\[
\nabla^{n+m} = \nabla^n (\nabla_m^m (f)) + \sum_{i=1}^n U_{\psi_i} (\delta_{\psi_i} (\ln \Psi - h \psi)) + \nabla^n (\nabla_m^m (f)) + \delta_{\psi} \sum_{i=0}^{m-1} \left( \int \ln \Psi^i \Psi^i \, dm - h \psi \right) \int f \, dm.
\]

This, 15), 16), Theorem 1 ii), Theorem 2 and Corollary 3 imply the existence of a neighbourhood \( Y \) of \( \Psi \) in \( E_r \), numbers \( \delta > 0 \) and \( q \in ]0, 1[ \) such that for \( \Psi \in Y \) and \( n, m \in \mathbb{N} \)
\[
\| \nabla^{n+m} \|_{r-2, \infty} \leq \delta (q^m \| \nabla^n \|_{r-2, \infty} + q^n \| \nabla^m \|_{r-2, \infty}^{m+1}).
\]

It is easy to see that 18) gives 17). This completes the proof.

It can be shown that the rate of convergence in Theorem 5 is the best possible. Moreover it turns out that for a fixed \( \Psi \in E_r \), the rate of convergence in this theorem is the best possible unless \( \mu \), \( \mu_\Psi \) and the measure with maximal entropy for \( \Psi \) are equal.

Corollaries 1-4 and Theorem 4 have their counterparts in the case of the sequence \( \left( \nabla^n \Psi \right) \) and \( \left( \nabla^n \Psi \right) \) respectively.

We end this note by stating the following theorem whose proof is rather long.
Theorem 6. For each \( \xi \in E_2 \) and each \( \alpha \) there exist numbers \( \delta > 0 \) and \( q \in ]0,1[ \) such that for \( n \in \mathbb{N} \)

\[
\sup_{\|f\|_{0,\alpha} \leq 1} \left| \sum_{x \in \text{Fix}(\xi^n)} f(x) \left| \det(D\xi^n(x)) \right|^{-1} - \int_M f \, d\mu_{\xi} \right| \leq \delta q^n.
\]

The complete proofs will be contained in [6], [7] and [8].

References


Karol Krzyżewski
Instytut Matematyki
Uniwersytet Warszawski
00-901 Warszawa