EIRA J. SCOURFIELD

On the property \((f(n), g(n)) = 1\) for certain multiplicative functions


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ON THE PROPERTY \((f(n),g(n)) = 1\) FOR CERTAIN MULTIPLICATIVE FUNCTIONS

by

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The problem of investigating the sum

\[ \Sigma_n(x) = \sum_{\substack{n \leq x \\text{ is an integer} \\text{ such that} \ (n,h(n)) = 1}} 1 \]  

for certain integer-valued arithmetic functions \(h\) has been considered by several authors in cases when the arithmetic properties of \(n\) and \(h(n)\) are not too closely related, and the expected result

\[ \Sigma_n(x) \sim \frac{6x}{\pi^2} \]

established; for references, see this author's paper [2]. Multiplicative functions, however, present a rather different problem, and in 1948 [1], Erdös obtained the result

\[ \sum_{\varphi(x)} \sim e^{-\gamma} x / \log \log x \quad \text{as} \quad x \to \infty \]

for Euler's function \(\varphi\). In [2], we considered the sum (1) for a class of integer-valued multiplicative functions, called polynomial-like, that includes \(\varphi\) and the divisor functions \(\sigma_v (v \geq 0)\); \(f\) is polynomial-like if there exists a polynomial \(W \in \mathbb{Z}[x]\) such that
\[ f(p) = W(p) \quad \text{for all primes } p. \quad (2) \]

For these functions, we proved in [2]:

**THEOREM 1.** If the polynomial \( W \) of (2) satisfies \( \deg W > 0 \), \( W(0) \neq 0 \), then there exist constants \( C > 0 \), \( \lambda \) \( (0 < \lambda \leq 1, \lambda \text{ rational}) \), depending on \( f \), such that

\[
\sum_f(x) \sim C x (\log \log \log x)^{-\lambda} \quad \text{as } \quad x \to \infty.
\]

If \( W \) is a non-zero constant, then there exists a constant \( C \) \( (0 < C \leq 1) \) such that

\[
\sum_f(x) \sim C x \quad \text{as } \quad x \to \infty.
\]

If \( W(0) = 0 \),

\[
\sum_f(x) = O(x^{1/2}).
\]

Example. \( f = \sigma_v \) \( (v > 0) \). For \( v \) odd, \( \lambda = 1 \), \( C = e^{-v} \), whilst for \( v \) even, \( \lambda = 2^{-\beta} \), where \( 2^\beta \| v \).

We obtain a generalization of the sum in theorem 1 by noting that \( n \) itself is a polynomial-like multiplicative function. Let \( f, g \) be multiplicative polynomial-like functions, and let \( W_1, W_2 \in \mathbb{Z}[x] \) be the polynomials such that

\[
f(p) = W_1(p), \quad g(p) = W_2(p) \quad \text{for all primes } p.
\]

Suppose that the following conditions hold:

(i) \( \deg W_i > 0 \quad (i = 1, 2); \)

(ii) \( W_1(x) = x^{a_1} W_1^*(x) \) where \( a > 0 \), \( W_1^*(0) \neq 0 \), \( \deg W_1^* > 0 \), and \( W_2(0) \neq 0 \);

(iii) \( W_1, W_2 \) are coprime polynomials.

It follows from (iii) that the set
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\[ S_0 = \{ p : p | (f(q), g(q)) \text{ for all primes } q \neq p \} \]

of primes is finite (possibly empty). If \( p \in S_0 \), \( p | (f(n), g(n)) \) whenever there exists a prime \( q \neq p \) with \( q \| n \), and hence for "most" \( n \). This suggests that for our generalization of the sum \( \Sigma_f(x) \), we consider

\[ \Sigma_{f, g}(x) = \sum_{p | (f(n), g(n)) \forall p \in S_0} 1. \]

Using results from sieve theory, we can prove

**THEOREM 2.** If conditions (i), (ii), (iii) above hold, there exist constants \( C > 0 \), \( \lambda (0 < \lambda \leq 1 \text{, } \lambda \text{ rational}) \) such that

\[ \frac{x}{\log x} \log \log x \ll \Sigma_{f, g}(x) \ll \frac{x}{\log x} \exp \left( \frac{C \log \log x}{(\log \log \log x)^\lambda} \right). \]

Conditions (i), (ii), (iii) ensure that the sum \( \Sigma_{f, g}(x) \) is not too small and does not reduce to the sum considered in theorem 1 or in other published papers.

**Examples.**

(i) \( f = \varphi \), \( g = \sigma_v \) \((v > 0)\), when \( S_0 = \{2\} \), \( \lambda = 2^{-\beta} \) where \( 2^\beta \| v \).

(ii) \( f = \sigma_v \), \( g = \sigma_x \) \((v, x > 0 \text{, } \beta > \gamma \text{, where } 2^\beta \| v, 2^\gamma \| x)\), when \( S_0 = \{2\} \), \( \lambda = 2^{-\beta} \).

The method used to prove the upper bound in theorem 2 also establishes

**THEOREM 3.** The number of positive integers \( n \leq x \) with the property that \( n \) does not have a prime divisor in every residue class \((\text{mod } p)\) coprime to \( p \) for any odd prime \( p \) is

\[ \ll \frac{x}{\log x} \exp \left( \frac{B \log \log x}{(\log \log \log x)} \right), \]

where \( B > 0 \) is constant.
BIBLIOGRAPHY


Eira J. SCOURFIELD
Westfield College
Department of Mathematics
LONDON NW3 7ST, England U.K.