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SINGULARITY THEORY AND EQUIVARIANT DYNAMICAL SYSTEMS

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In this note we wish to describe some recent results in the theory of equivariant dynamical systems. The theorems we describe are, for the most part, firmly rooted in the singularity theory of differentiable maps as developed by Malgrange, Mather, Thom and many others. Apart from applications to problems in dynamical systems with symmetry, we expect applications to problems involving "breaking of symmetry" as well as to the topology of G-manifolds.

We start by reviewing a little of the theory of G-manifolds and establishing some notation. Let M denote a compact C^∞ manifold without boundary and G be a compact Lie group acting differentiably on M. If $x \in M$, we let $G(x)$ denote the G-orbit through x and G_x denote the isotropy group of G at x. The equivariant diffeomorphism type of $G(x)$ is uniquely determined by the conjugacy class of G_x in G and clearly $G(x)$ is equivariantly diffeomorphic to the homogeneous space G/G_x. Only finitely many conjugacy classes of isotropy subgroups occur for any given G-action on M and, in the obvious way, we obtain a finite decomposition of M into points of the same "orbit type" :

$$M = M_1 \cup \dots\dots \cup M_N .$$

Much of the difficulty encountered in the study of G-manifolds arises from the fact that the sets M_i, though submanifolds, need not be closed. Put another way, the orbit space M/G can be highly singular. Fix $x \in M$ and take a G-invariant Riemannian metric on M and corresponding tubular neighbourhood of $G(x)$. We obtain a linearization of the action on G_x on a transverse disc to $G(x)$. This observation (the equivariant slice theorem) enables one to apply results on linear G-actions and brings in the representation theory of G. However, in the geometrically most interesting cases, G_x is not usually connected and the powerful and elegant representation theory of compact connected Lie groups does not generalize at all easily to the non-connected case. For a detailed introduction to the theory of compact transformation groups we refer to the book by Bredon [1].

In earlier work [2, 3] we proved various genericity results for equivariant diffeomorphisms and vector fields. For simplicity we restrict attention here to equivariant diffeomorphisms and we shall let $\mathrm{Diff}_G^r(M)$ denote the space of C^r equivariant diffeomorphisms of M.

If x is a fixed (periodic) point of $f \in \mathrm{Diff}_G^r(M)$, then so is $g(x)$, $g \in G$. In other words, we have to allow for G-orbits to be fixed sets rather than isolated points (at least, if dimension$(G) \geq 1$). Recall that a fixed set $G(x)$ for f is said to be generic if f is "normally hyperbolic" on $G(x)$ (see [3] for a complete definition and references). With a similar definition of genericity for periodic points one can then prove that any equivariant diffeomorphism $f \in \mathrm{Diff}_G^r(M)$ can be C^r approximated by an equivariant C^∞ diffeomorphism with all fixed and periodic points generic. One also has the usual finiteness and isotopy results. For example, given $T \geq 0$, the number of points of period $\leq T$ is finite (mod G) for a generic diffeomorphism.

If $G(x)$ is a generic fixed set for $f \in \mathrm{Diff}_G^r(M)$ one may construct the stable and unstable manifolds of $G(x)$ in the usual way (that they are immersed manifolds follows from [6]). To obtain a "reasonable" geometric description of an equivariant dynamical system, one needs a good definition of transversality of stable and unstable manifolds. In particular, transversality must be an open condition. For most of the remainder of this paper, we wish to describe our concept of "G-transversal". Full details will appear elsewhere [4] as will examples of equivariant dynamical systems satisfying these conditions [5]. We remark the following theorems about the existence of equivariant dynamical systems (proofs in [5]).

Theorem.

Let $f \in \mathrm{Diff}_G^r(M)$. *We may* C^r *approximate* f *by a* C^∞ *equivariant diffeomorphism* f' *such that*

1. *The fixed and periodic points of* f' *are generic.*
2. *Stable and unstable manifolds meet* G-*transversally.*

Theorem.

On any compact G-*manifold* M, *we can find an "equivariant Morse-Smale diffeomorphism". That is, there exists* $f \in \mathrm{Diff}_G^\infty(M)$ *such that*

1. $\Omega(f)$, mod G, *is finite and consists of generic fixed and periodic points.*
2. *Stable and unstable manifolds meet* G-*transversally.*

Similar theorems hold for equivariant vector fields.

Deferring our definition of transversality until later we may summarize our main result by

Theorem.

Suppose that V *and* M *are compact* G-*manifolds and that* W *is a compact* G-*invariant submanifold of* M . *Let* $C^{\infty}_{G}(V,M;s)$ *denote the space of* C^{∞} *equivariant maps from* V *into* M *with the* C^{s} *topology. We may find* $r \geq 1$ *and an open dense subset* $X \subset C^{\infty}_{G}(V,M;r)$ *such that if* $f \in X$, *there exists an open neighbourhood* N *of* f *in* X *such that* $g^{-1}(W)$ *is continuously equivariantly isotopic to* $f^{-1}(W)$ *in* V *for all* $g \in N$. *Moreover, if* $f \in X$, *the intersection* $f^{-1}(W)$ *is given locally by equisingular families of real algebraic varieties. In general we cannot require that intersections are differentiably stable.*

After circulating the preprint for the first half of [4], I learnt from E. Bierstone that he had proved results similar to the theorem above, though with a slightly different definition of G-transversality.

We shall start by giving one or two rather simple examples of G-transversality including an example showing that one cannot require intersections to be differentiably stable. We conclude by giving some indication of the role of equisingularity theory in our definition of G-transversality.

Example 1. Perhaps the most remarkable feature of the transversality theory of G-manifolds is that a decade ago even the simplest examples that we shall now describe could not have been given a rigourous presentation. Even now, there is no C^{r} theory, $r < \infty$[1] . The geometry implicit in the G-transversality theorem seems to admit of many applications even for relatively simple group actions and we shall start by giving an example of Z_2 transversality (Z_2 de-

1. The C^{r} theory goes through if the C^{r} version of G.Schwarz' theorem is true [11]. See also [9].

notes the cyclic group of order 2).

Give $R \times R$ the coordinates (t,x) and R the coordinate (y). We let Z_2 act on $R \times R$ as $(t,x) \longmapsto (t,-x)$ and on R as multiplication by -1. Give $R \times R \times R$ the coordinates (t,x,y) and let X denote the Z_2-invariant submanifold $\{(t,x,0): t,x \in R\}$ of $R \times R \times R$. Let $\emptyset: R \times R \longmapsto R$ be a $C^\infty Z_2$-invariant map. We shall consider the Z_2-transversality of the graph of $\emptyset$ to X along the subset $R \times \{0\}$ of $R \times R$. First note that for all $\emptyset, R \times \{0\} \subset \text{graph}(\emptyset) \cap X$. Associated to $\emptyset$ we define the map

$$\gamma(\emptyset): R \longmapsto R$$

by

$$\gamma(\emptyset)(t) = D_2\emptyset_{(t,0)}$$

($D_2\emptyset_{(t,0)}$ denotes the partial derivative with respect to x at $(t,0)$). We say that graph($\emptyset$) is Z_2-transversal to X along $R \times \{0\}$ if $\gamma(\emptyset)$ is transversal to $0 \in R$. Notice that if $\gamma(\emptyset)(t) \neq 0$, then graph($\emptyset$) is transversal to X at $(t,0,\emptyset(t,0))$-usual definition. However, we will in general have points of non-transversality. We now examine some consequences of our transversality definition. Suppose that we have Z_2-transversality at 0 and that $\gamma(\emptyset)(0) = 0$. The transversality condition on $\gamma(\emptyset)$ implies that we can find $a \neq 0$ and a $C^\infty Z_2$-invariant function $b: R \times R \longrightarrow R$ such that

1) $\emptyset(t,x = axt + b(t,x)$, for all $(t,x \in R \times R$.

2) $D_{12}b_{(0,0)} = 0$.

($D_{12}b_{(0,0)}$ denotes the mixed partial derivative $D_1(D_2b)_{(0,0)}$).

The intersection of the graph of $\emptyset$ with X is given by the set of zeroes of $\emptyset(t,x) = 0$. That is, (t,x) belongs to the intersection if and only if

$$axt + b(t,x) = 0 .$$

Now the Z_2-invariance of b, together with the fact that $b(t,0) = 0$ for all $t \in R$, implies that b is divisible by x : This is an easy consequence of the Malgrange division theorem. That is, there exists a (unique) C^∞ function $f: R \times R \longrightarrow R$ such that

$$b(t,x) = xf(t,x), \quad \text{for all } (t,x) \in R \times R .$$

Substituting for b, we find that the intersection of graph($\emptyset$) with X is given by the set of solutions of

$$x(at + f(t,x)) = 0 .$$

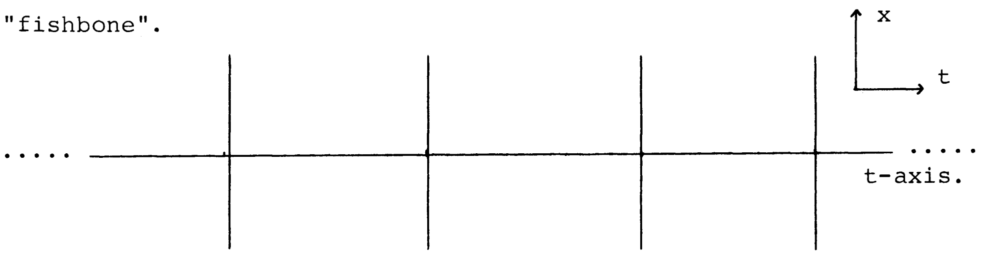
We already knew that $x = 0$ (the fixed point set of Z_2) lay in the intersection and so we are left with the problem of analysing the solutions of the equation

$$at + f(t,x) = 0 .$$

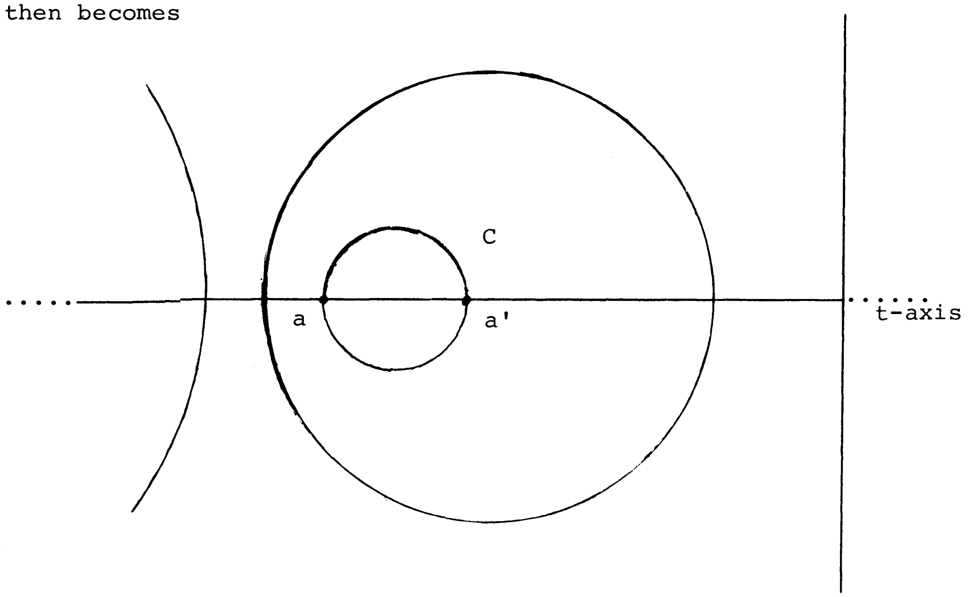
Since $D_{12}b_{(0,0)} = 0, D_1 f_{(0,0)} = 0$. By the implicit function theorem, we may therefore find an open neighbourhood $U \times V$ of $0 \in R \times R$ and a C^∞ function $g: V \longrightarrow R$ such that $\{(g(x),x); x \in V\}$ gives all the solutions of the equation $at + f(t,x) = 0$ in $U \times V$. Hence the intersection of graph($\emptyset$) with X is given in a neighbourhood of zero by the line $x = 0$ and the curve $t = g(x)$.

Using the implicit function theorem, with parameters, it follows easily from the above argument that we have local stability of Z_2-transversal intersections. Indeed, if $\emptyset'$ is C^2 close to $\emptyset$, then the intersections are C^1 close (for more details we refer to [4]).

Along $R \times \{0\}$, the intersection of graph($\emptyset$) and X is a "fishbone".



Now we can always perturb $\emptyset$ to achieve transversality (in the usual sense) outside of the fixed point set of Z_2 . The picture then becomes



Observe that we may cancel adjacent points of non-transversal intersection on the t-axis by equivariantly isotoping. For example, with the notation of the diagram, the points a and a' can be moved together and cancelled by "unfolding" the fold in the graph of $\emptyset$ which gives the circle C in the intersection. Suppose that Z_2 acts on a 3-manifold and has fixed point set F diffeomorphic to S^1 . Let X and Y be Z_2-invariant submanifolds containing F . Working in a tubular neighbourhood of F (diffeomorphic to a solid torus) it is easily seen that we can equivariantly isotop X and Y so that they are Z_2-transversal on F with $2|m-n|$ points of non-transversal intersection, where m and n denote the number of "twists" of X and Y around F . See also [7,8].

Example 2. We shall let C_p denote the complex plane, C, together with the S^1 action $e^{pi\theta}$. Suppose that $pq \neq 0$ and let

$\emptyset : C_p \longrightarrow C_q$ denote an S^1 invariant C^∞ map. Set $W = \text{graph}(\emptyset) = \{(z,\emptyset(z)) \in C_p \times C_q\}$. W is an S^1 invariant submanifold of $C_p \times C_q$. Since $pq \neq 0$, W passes through the origin of $C_p \times C_q$. Let X denote the S^1 invariant submanifold $C_p \times \{0\}$ of $C_p \times C_q$. We shall investigate the S^1 transversality of W and X. Ignoring the case $p = q$ (which is easily dealt with) we shall suppose $p \neq q$. Since the representations of S^1 on C_p and C_q are irreducible and different, we find that W has tangent space $C_p \times \{0\}$ at zero. In other words, we cannot perturb $\emptyset$ so as to obtain transversality at zero. Since $\emptyset(gz) = g\emptyset(z)$ for all $g \in S^1$ and $z \in C_p$, it follows that if $\emptyset \not\equiv 0$, then $Z_p \subset Z_q$ (Z_p and Z_q denote the isotropy groups of the actions of S^1 on C_p and C_q respectively). Now $Z_p \subset Z_q$ if and only if p divides q. Hence if p does not divide q, $\emptyset$ is identically zero. But this implies $W = X$. Obviously the intersection of W and X is highly stable ! Finally, we turn to the most interesting case when p divides q. Suppose that $q = pk, k > 1$. Using the Malgrange division theorem one may easily show that any S^1 invariant C^∞ map $\psi : C_p \longrightarrow C_q$ can be written in the form

$$\psi(z) = p(z)z^k + q(z)iz^k ,$$

where p and q are $C^\infty S^1$ invariant real valued functions on C_p. In this case our S^1 transversality condition requires that $(p(0),q(0)) \neq 0$. If this condition is satisfied, the intersection at zero is the isolated point $(0,0)$.

<u>Example 3</u>. In this example we show that we cannot expect <u>differential</u> stability for intersections of G-transversal maps.

Let S^1 act on $C \oplus C$ as $(e^{i\theta}, e^{i\theta})$ and on C as $e^{4i\theta}$. Working with complex coefficients, the general equivariant polynomial of degree 4 mapping from $C \oplus C$ to C is given by

$$\sum_{j=0}^{4} c_j z_1^j z_2^{4-j} ,$$

where $c_j \in C$. The discriminant locus of quintic polynomials defines a proper algebraic subset of R^{10} ($\cong C^5$). Off the discriminant locus, every such polynomial has four distinct roots corresponding to four distinct complex lines in $C \oplus C$. In general, if we are given two distinct sets of four complex lines in $C \oplus C$, we cannot find a real linear endomorphism of $C \oplus C$ taking one set onto the other. It follows, looking at derivatives at the origin, that we cannot have differential stability of S^1 invariant maps transverse to $0 \in C$. Of course, this argument is based on Whitney's "cross ratio" examples.

For the remainder of this paper we shall give a brief sketch of some of the main ideas used in the proof of the stability theorem.

Let V and W be finite dimensional linear G-spaces and suppose that G does not act trivially on any proper subspace of V . We let $P_G(V,W)$ and $C_G^\infty(V,W)$ respectively denote the sets of equivariant polynomial and C^∞ maps from V to W . We also let $P_G(V)$ and $C_G^\infty(V)$ denote the sets of real valued G-invariant polynomial and C^∞ maps on V respectively.

We say that a set $\{F_1,\ldots\ldots,F_k\} \subset P_G(V,W)$ is a minimal set of generators for $P_G(V,W)$ at zero (an "MSG") if it is a minimal set of generators for the $P_G(V)$-module of fractions

$$\{P/q : P \in P_G(V,W), q \in P_G(V), q(0) \neq 0\} .$$

The number of elements in an MSG depends only on the given representations of G on V and W . If G acts trivially on the vector space T , then an MSG for $P_G(V,W)$ is also an MSG for $P_G(V \times T,W)$.

If $\{F_1,\ldots,F_k\}$ is an MSG for $P_G(V,W)$ it is a straightforward consequence of the Malgrange division theorem that $\{F_1,\ldots,F_k\}$ generates the $C_G^\infty(V)$-module $C_G^\infty(V,W)$ - at least on some neighbourhood of zero. Combining this with the previous remark, we find that if $f \in C_G^\infty(V \times T, W)$, there exist $q_1, \ldots\ldots, q_k \in C_G^\infty(V \times T)$, such that on some open neighbourhood of $V \times \{0\}$ in $V \times T$ we have

$$f(x,t) = \sum_{j=1}^{k} q_j(x,t) F_j(x) .$$

Let $Q(f): V \times T \longrightarrow R^k$ denote the map defined by

$$Q(f)(x,t) = (q_1(x,t), \ldots, q_k(x,t)) .$$

Although $Q(f)$ need not be uniquely determined by f and the choice of MSG, the map

$$\gamma(f): T \longrightarrow R^k$$

defined by $\gamma(f)(t) = Q(f)(0,t)$ is uniquely determined. The map $\gamma(f)$ will be used in our local definition of G-transversality.

Let $F: V \times R^k \longrightarrow W$ denote the polynomial defined by

$$F(x,t) = \sum_{j=1}^{k} t_j F_j(x) .$$

For $t \in R^k$, we let $X(t) \subset V$ denote the algebraic variety $\{x \in V : F(x,t) = 0\}$. Let $X \subset V \times R^k$ denote the zero set of F. Our definition of transversality and description of local models for transversal intersection rely on a careful study of equisingularity properties of the family $\{X(t) : t \in R^k\}$ (rather, germs of this family along V). The type of equisingularity that we study is a generalization of "Whitney equisingularity" (see [10]).

We prove that we may find a decreasing family $A_1 \supseteq \ldots \supseteq A_k$ of real algebraic subsets of R^k satisfying

a) Codimension $(A_j) \geq j$,

b) $A_j \setminus A_{j+1}$ is a, possibly empty, semi-algebraic manifold of codimension j.

c) $\{X(t) : t \in R^k\}$ is Whitney equisingular "transverse" to $A_j \setminus A_{j+1}$ and Whitney equisingular over $R^k \setminus A_1$.

As far as condition c) goes, the family $\{X(t) : t \in R^k\}$, although not equisingular on $A_j \setminus A_{j+1}$, will be equisingular if we take as new parameter transversal j-dimensional germs to $A_j \setminus A_{j+1}$. This is most easily seen when $j = k$. A_k will then consist of a single point, the origin. Since there is only one germ transverse to the origin, we have equisingularity with this new parametrization trivially. The complete description of c) requires a number of technicalities and we shall give full details elsewhere ([4]).

The fact that the varieties A_j are algebraic, allows us to choose a unique "minimal" family which we call a "fundamental equisingularity sequence" (for $P_G(V,W)$).

We may now state our definition of G-transversality.

DEFINITION

Let $f \in C_G^\infty(V \times T, W)$. We say f is G-transversal to $0 \in W$ at $(0,0) \in V \times T$ if the map

$$\alpha(f): T \longrightarrow R^k$$

is transversal to a fundamental equisingularity sequence for $P_G(V,W)$ at $0 \in T$.

Remark. "Transversal" to a fundamental equisingularity sequence $A_1 \supseteq \ldots \supseteq A_k$ means transversal to each A_j, where A_j is given a minimal Whitney stratification.

Most of the main properties of G-transversality follow straightforwardly from the above definition. For example, local models for G-transversal intersection are easily obtained by studying the map $\tilde{Q}(f): V \times T \longrightarrow V \times R^k$, defined by

$$\tilde{Q}(f)(x,t) = (x, Q(f)(x,t)) .$$

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