IZUMI KUBO

The motion of disks in a torus

Astérisque, tome 40 (1976), p. 113-116
<http://www.numdam.org/item?id=AST_1976__40__113_0>

Société Mathématique de France
Astérisque 40 (1976) p.113-116

THE MOTION OF DISKS IN A TORUS

Izumi KUBO
(Erlangen-Nürnberg Univ. , Nagoya Univ.)

The ergodicity of the dynamical systems of many particles in finite vessels is one of fundamental problems in classical statistical mechanics. In 1963, Ya. G. Sinai [8] announced a result for the problem :

Theorem 1.

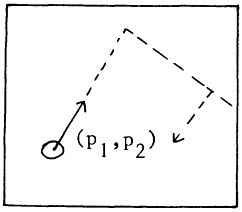
If the particles have central potentials of special type and the vessel is a rectangular box and if the particles are not so many, then the system is a K-system.

However, any proof has been not yet published except for the simplest model which is called a Sinai billiard. Let us observe a square billiard table such that at the boundary every billiard ball is reflected according the law of rigid collision.

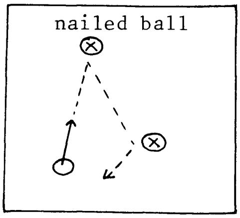
(A) Weyl billiard system is the dynamical system of the motion of a ball on the billiard table

Theorem 2. (Weyl)

The Weyl billiard system is ergodic if and only if the two components of the momentum are rationally independent.



Weyl billiard



Sinai billiard

(B) Sinai billiard system

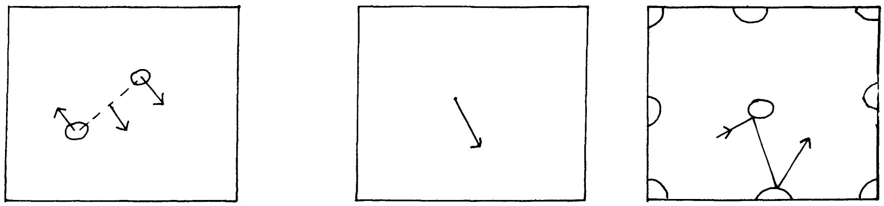
A simple example of Sinai billiard system is given by putting several convex obstructions on the billiard table, for example you may nail several balls.

Theorem 3. (i) (Sinai [9])
The Sinai billiard system is a K-system.
(ii) (Ornstein-Gallavotti [3], Kubo-Murata [7])
The Sinai billiard system is a Bernoulli flow.

(C) Two disks in a torus

The system of two rigid disks with the same mass in a torus is the simplest model of two particles system. The system can be resolved into the motion of the center of gravity and the relative motion.

Theorem 4. (Kubo [5])
(i) *The system is a factor flow of the product flow of a Weyl system and a Sinai billiard*
(ii) *The system is a not mixing flow with finite positive entropy. More over, it is ergodic iff the two components of the total momentum are rationally independent.*



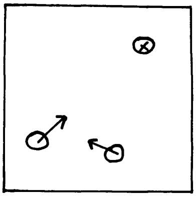
Problem

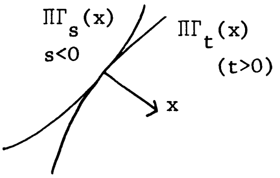
Show the ergodicity of the system of two rigid disks with different masses.

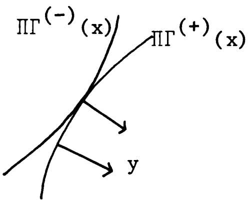
(D) Two disks in a torus with obstructions or two balls on the billiard table (with or without obstructions). In this model, you cannot apply the method in (C). However you may use the method of Sinai in (B), which is due to the idea of E. Hopf in the proof of the ergodicity of the geodesic flow on a compact riemannian manifold with negative curvature. Suppose that obstructions are convex.

Then the configuration space Q is a 4-dimensional domain with boundary, which is piecewise convex (but not strictly convex). The energy surface M is the unit tangent bundle of Q ; $M = \{x = (q,p) \; ; \; q \in Q \; , \; ||p|| = 1 \; , \; p \in \mathbb{R}^4\}$. Let us denote the flow by $\{S_t\}$, and let Π be the natural projection from M to Q ; $\Pi(q,p) = q$. Put $\Gamma_t(x) \equiv S_{-t}\, \Pi^{-1}\, \Pi S_t x$ for $x \in M$. Then $\Gamma_t(x)$ is a 3-dimensional surface in M and $\Pi(\Gamma_t(x))$ is also a 3-dimensional surface in Q , which is pieciwise convex. You can show the existence of the limit surfaces for almost every $x = (q,p) \in M$:

$$\Gamma^{(+)}(x) \equiv \lim_{t\to\infty} \Gamma_t(x) \quad \text{and} \quad \Gamma^{(-)}(x) \equiv \lim_{t\to-\infty} \Gamma_t(x)$$







The limit surfaces have the following properties :

(イ) $y \in \Gamma^{(+)}(x) \Longrightarrow \Gamma^{(+)}(y) = \Gamma^{(+)}(x)$, $z \in \Gamma^{(-)}(x) \Longrightarrow \Gamma^{(-)}(z) = \Gamma^{(-)}(x)$,

(ロ) $S_t\, \Gamma^{(+)}(x) = \Gamma^{(+)}(S_t x)$, $S_t\, \Gamma^{(-)}(x) = \Gamma^{(-)}(S_t x)$,

(ハ) $y \in \Gamma^{(+)}(x) \Longrightarrow \text{dis}\,(S_t x, S_t y) \to 0 \quad (t \to \infty)$

$z \in \Gamma^{(-)}(x) \Longrightarrow \text{dis}\,(S_t x, S_t z) \to 0 \quad (t \to -\infty)$

Since by (ハ) $f(S_t x) - f(S_t y) \to 0$ $(t \to \infty)$ for $y \in \Gamma^{(+)}(x)$ and $f(S_t x) - f(S_t z) \to 0$ $(t \to z)$ for $z \in \Gamma^{(-)}(x)$ hold for every continuous function f on M , the time averages

$$\bar{f}^{(+)}(x) = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(S_t x)dt \; , \quad \bar{f}^{(-)}(x) = \lim_{T\to-\infty} \frac{1}{T} \int_0^T f(S_t x)dt$$

satisfy the following properties

(ニ) $\bar{f}^{(+)}(y) = \bar{f}^{(+)}(x)$ for $y \in \Gamma^{(+)}(x)$, $\bar{f}^{(-)}(z) = \bar{f}^{(-)}(x)$ for $z \in \Gamma^{(-)}(x)$,

(ホ) $\bar{f}^{(+)}(x) = \bar{f}^{(-)}(x)$ for a.e.x. ,

(ヘ) $\bar{f}^{(+)}(S_t x) = \bar{f}^{(+)}(x)$, $\bar{f}^{(-)}(S_t x) = \bar{f}^{(-)}(x)$

Therefore, if you can check the property

(ト) $(\{\Gamma^{(+)}\}, \{\Gamma^{(-)}\}, \{S_t\})$ give measure theoretical coordinates (that is, you may use Fubini theorem in some sense), then you can see that the system is ergodic. However, I no proof of (ト) yet.

Theorem 5.

(i) *There exist an increasing partition* $\zeta^{(+)}$ *and a decreasing partition* $\zeta^{(-)}$ *such that* $S_t\,\zeta^{(+)} > \zeta^{(+)}$, $S_{-t}\,\zeta^{(-)} > \zeta^{(-)}$ *for* $t > 0$, $\bigvee_s S_s\,\zeta^{(+)} = \bigvee_s S_s\,\zeta^{(-)} = \varepsilon$, $\bigwedge_s S_s\,\zeta^{(+)} = \bigwedge_s S_s\,\zeta^{(-)} = \Pi(\{S_t\})$ *and* $h(\{S_t\}) = H(S_1\zeta^{(+)}|\zeta^{(+)}) = H(S_{-1}\,\zeta^{(-)}|\zeta^{(-)}) > 0$.

(ii) *If* (ト) *is true, then* $\Pi(\{S_t\})$ *is the trivial partition and hence the system is a K-system.*

(E) Many disks in a torus with or without obstructions

For the model, you can construct limit surfaces $\Gamma^{(+)}(x)$ and $\Gamma^{(-)}$ similar to (D) . Hence you can assert as same as theorem 5. However, for the case without obstructions, (ト) is not true, hence you have to use a similar trick to (C) .

(F) Disks with potentials

If particles have potentials of special type and the energy is sufficiently low, then such a dynamical system is a perturbation of a rigid system. By using the fact, you can see the similar results to the results in (B), (C) and (D). (cf. [4] [6] [7]) .

REFERENCES

[1] Bunimovich L.A. Matem. Sb. 94 (1974), 49-73

[2] Gallavotti G. Lecture Notes in Physics, 38 (1975), 236-295

[3] Gallavotti G. and Ornstein D., Comm. Math. Phys. 38, (1974), 83-101

[4] Kubo I., Lecture Notes in Physics, 39 (1975), 230-235

[5] _______, Lecture Notes (preparing)

[6] _______, (Perturbed Billiard Systems I. preprints)

[7] Kubo I. and Murata H. (Perturbed Billiard Systems II. preprints)

[8] Sinai Ya. G., Sov. Mat. Dokl. 4, n° 6 (1963), 1818-1822

[9] ___________, Russian Math. Surveys 25, (1970), 137-189.