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LEONHARD GERHARDS

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GROUP-THEORETICAL INVESTIGATIONS ON COMPUTERS II

by

Leonhard Gerhards

For solving group-theoretical problems often the structure of the maximal subgroups of a finite group G is of great importance. Although it is possible to determine the maximal subgroups of G by calculating the complete lattice $V(G)$ of all subgroups of G [4], [10], it seems to be profitable to develop an effective computational algorithm for determining only the maximal subgroups of G .

Making use of theoretical results of a paper of E. Altmann [0] the present paper – mainly written under computational aspects – contains a complete description of the principal methods of such a program for a finite group G containing a „Hall system $\{H_i / i \in I\}$ ” of subgroups H_i of G [0], [5].

The underlying group classes for the program are the class of finite solvable groups and the class of finite non solvable groups, which contain a chain of normal Hall groups [5].

In both cases the computational methods are based on results of the theory of factorizations of finite groups [1], [3]:

If G can be factorized by the groups H_i of a Hall system $\{H_i / i \in I\}$ of G and if for any subgroup U of G there exists a conjugate group U^* such that U^* is a factorization by the subgroups $U^* \cap H_i$ of H_i , then assuming the computational construction of the lattice $V(H_i)$ of all subgroups of H_i the lattice $V(G)$ of G can be determined by an iterative process constructing the maximal subgroups of maximal subgroups and their corresponding conjugate series.

The present paper consists of two central parts:

In section 1 we develop an effective algorithm for the determination of the maximal subgroups of a finite group G in the following cases:

- a) G is solvable
- b) G is non-solvable but contains a chain of normal subgroups.

In section 2 the algorithm will be extended to the calculation of the complete lattice $V(G)$ of all subgroups of G .

1. Determination of the maximal subgroups of a finite group G factorized by a Hall system

1.1 Preliminaries

1.1.1 Representation and multiplication of the elements of a finite group G

In the following any finite group G will be given abstractly by

- a) a system $\mathcal{A} = \{a_1, \dots, a_n\}$ of generating elements of G
- b) a system $R_1(\mathcal{A}) = e, \dots, R_k(\mathcal{A}) = e$ of defining relations.

If any element $a \in G$ can be represented uniquely by a „normal form” $a = a_1^{r_1} \cdot \dots \cdot a_n^{r_n}$ ($0 \leq r_i < |a_i|$, $a_i^0 = e$) we get a representation $\varphi(a)$ of $a \in G$ in form of the n -tuple $\langle r_1, \dots, r_n \rangle$. Assuming further that $\varphi(a_i a_j)$ can be calculated uniquely from $\varphi(a_i)$ and $\varphi(a_j)$ for all pairs $\{a_i, a_j\} \subseteq G$, multiplication in G is well defined. If such an algorithm of multiplication exists, the generating system $\alpha = \{a_1, \dots, a_n\}$ is called a „special generating system of G ”.

Basic programs for the multiplication of the elements of G are developed in [2], [6], [8].

1.1.2 Representation of subgroups of G

Let $|G| = p_1^{\alpha_1} \cdot \dots \cdot p_r^{\alpha_r}$ be the prime power decomposition of the order $|G|$ of G , $\{U\}$ the set of all subgroups $U \subseteq G$ and $\{S(U)\}$ the set of all systems $S(U)$ containing all cyclic subgroups of G of prime power order contained in U . Then we get a 1-1-correspondence $\{U\} \leftrightarrow \{S(U)\}$ between $\{U\}$ and $\{S(U)\}$:

$$(1.1) \quad G \supseteq U \leftrightarrow S(U) = \{ \langle z \rangle \subseteq G \mid \langle z \rangle \subseteq U, |\langle z \rangle| = p^\alpha, \alpha \geq 1, p \text{ prime} \}, \text{ and a system}$$

$$(1.2) \quad E(U) = \{z_1, \dots, z_m\} \quad (U \subseteq G, m = |S(U)|)$$

of generating elements of all cyclic subgroups of $S(U)$ forms a uniquely determined generating system of $U \subseteq G$.

To store the subgroups $U \subseteq G$ of G in the computer by „characteristic numbers”, the elements of $E(G)$ shall be listed. Then, if $E(U) = \{z_{i_1}, \dots, z_{i_q}\} \subseteq E(G)$ ($\{i_1, \dots, i_q\} \subseteq \{1, \dots, |E(G)|\}$) is a generating system of U by

$$(1.3) \quad K[U] = \sum_{j=1}^q 2^{i_j-1}$$

a dual number is defined, which uniquely corresponds to the subgroup U of G : $U \leftrightarrow K[U]$.

Using the Boolean operation of intersection “ \wedge ” we get

$$(1.4) \quad \begin{aligned} K[U] \wedge K[V] &= K[U \cap V] \\ U \subseteq V \leftrightarrow K[U] \wedge K[V] &= K[U] \end{aligned} \quad (U, V \subseteq G)$$

1.2 Factorization of G by a Hall system

1.2.1 Definition of a Hall system

A system $\mathcal{H} := \{H_i \mid i = 1, \dots, r\}$ of subgroups of G is called a Hall system of G , if

$$(1.5) \quad \begin{aligned} \text{a) } & G = H_1 \cdot \dots \cdot H_r \\ \text{b) } & H_i H_k = H_k H_i \quad (i, k = 1, \dots, r; i \neq k) \\ \text{c) } & (|H_i|, |H_k|) = 1 \\ \text{d) } & \{H_i \mid i = 1, \dots, r\} \text{ is conjugate to every system } \{H_i^* \mid i = 1, \dots, r\} \text{ of } G \text{ satisfying a) } \dots \text{ c) } \\ & (g H_i g^{-1} = H_i^* \text{ for some } g \in G). \end{aligned}$$

If G contains a Hall system \mathcal{H} , we also say that G is „factorized by the Hall system \mathcal{H} ”.

1.2.2 Sylow basis of a solvable group G

Let G be solvable of order $|G| = p_1^{\alpha_1} \cdot \dots \cdot p_r^{\alpha_r}$ (p_i prime number). Then G contains a system $\{P_i/i=1, \dots, r\}$ of p_i -Sylow subgroups of G satisfying (1.5), a), ..., d). $\{P_i/i=1, \dots, r\}$ is called a „Sylow basis of G ” [7].

1.2.3 Groups containing a chain of normal Hall groups

A subgroup chain

(1.6) $G = G_r \supset \dots \supset G_i \supset \dots \supset G_1 \supset G_0 = \langle e \rangle$

is called a „chain of normal Hall groups of G ”, if $G_i \triangleleft G$ and $(|G_i|, [G : G_i]) = 1$ ($i = 1, \dots, r$).

Since $P \subseteq G_i$ for all p -Sylow subgroups of G with $p/|G_i|$, G_i is a characteristic subgroup of G . Therefore, (1.6) is a chain of normal Hall groups of G , if and only if $G_i \triangleleft G_{i+1}$ and $(|G_i|, [G_{i+1} : G_i]) = 1$, ($i=1, \dots, r-1$). In the case that G contains a chain of normal Hall groups, G can be factorized by a Hall system [5] :

Theorem 1.1: Under the assumption that G contains a chain (1.6) of normal Hall groups we obtain:

- a) G contains a Hall system $\{H_i/ i = 1, \dots, r\}$ of subgroups of G such that:
 - α) $G_i = H_1 \cdot \dots \cdot H_i$ β) $H_i H_k = H_k H_i$
 - γ) $(|H_i|, |H_k|) = 1$ δ) $H_i \triangleleft H_i H_k$ ($i < k$)
 - e) $H_i \cong G_i/G_{i-1}$ ($i, k = 1, \dots, r, i \neq k$)
- b) Any two Hall systems of G are conjugate in G .

If G is not solvable, by the theorem of Feit-Thompson in a chain (1.6) of normal Hall groups of G at most one factor G_k/G_{k-1} is not solvable.

Under this assumption on G it can be shown that every Hall group H_i ($i \neq k$) of G can be substituted by a system of special Sylow subgroups of G [5]:

Theorem 1.2: Let G be a non solvable group with a chain (1.6) of normal Hall groups of G and let k be the only index such that G_k/G_{k-1} is not solvable. Then G contains a Hall system $\{H_i/i=1, \dots, r\}$ satisfying (1.7) and:

- a) For $i \neq k$ there exists a Sylow basis $P_{i,1}, \dots, P_{i,n_i}$ of H_i such that the „complete Hall system”
- (1.8) $P_{1,1}, \dots, P_{1,n_1}, \dots, P_{k-1,1}, \dots, P_{k-1,n_{k-1}}, H_k, P_{k+1,1}, \dots, P_{k+1,n_{k+1}}, \dots, P_{r,1}, \dots, P_{r,n_r}$
satisfies (1.7)
- b) Any two complete Hall systems of G are conjugate in G .

From Theorem 1.2 consequently follows that without regard to the non solvable part of G the development of a computational algorithm for a non solvable group G , which contains a chain of normal Hall groups, is similar to the case G being solvable.

1.3 Factorization of G [1], [3]

1.3.1 Let $G = H_1 H_2 = H_2 H_1$, $H_1 \cap H_2 = \langle e \rangle$ be a factorization of G by H_1, H_2 . Then for every $h_i \in H_i$ ($i=1,2$) we obtain a map $h_i^k : H_k \rightarrow H_k$ of H_k into H_k defined by :

(1.9)
$$\begin{aligned} h_1^2 h_2 &= H_1 h_2 h_1 \cap H_2 \quad \text{for all } h_2 \in H_2 \\ h_2^1 h_1 &= h_2 h_1 H_2 \cap H_1 \quad \text{for all } h_1 \in H_1 \end{aligned}$$

These mappings h_{1^k} together with the defining relations of the components H_i ($i=1,2$) of G define the structure of G . This is obvious, because multiplication in G is completely determined by the relation:

$$(1.10) \quad h_2 \circ h_1 = h_2^1 h_1 \circ h_1^2 h_2 ,$$

which is equivalent to (1.9).

1.3.2 By the theory of factorization [3] it follows that the set of mappings h_{1^k} forms a permutation subgroup $\Pi_{i,k}$ of the symmetric group $S_{|H_k|}$ of degree $|H_k|$. The set $N_i := \{h_i \in H_i / h_{1^k} h_k = h_k \text{ for all } h_k \in H_k\}$ is the maximal normal subgroup of G contained in H_i , which determines the homomorphism $\tau_{i,k} : H_i \rightarrow \Pi_{i,k}$, $H_i/N_i \cong \Pi_{i,k}$ [3]. An other important group for the theory of factorization is the „fix group“ $F_i := \{h_i \in H_i / h_k^2 h_i = h_i \text{ for all } h_k \in H_k\}$. Between F_i and the normalizer $N_G(H_k)$ of H_k in G we obtain the following relation [5] :

$$(1.11) \quad F_i = N_G(H_k) \cap H_i , \quad N_G(H_k) = F_i H_k = H_k F_i$$

1.3.3 If $G = H_1 \cdot \dots \cdot H_r$ is a factorization of G by a Hall system $\{H_i / i = 1, \dots, r\}$ of G , we can apply the theory of factorization to the subgroups $G_{i,k} := H_i H_k$ ($i \neq k$) of G . F_i^k may denote the fixgroup of $\Pi_{k,i}$ and N_i^k the maximal subgroup of $G_{i,k}$ contained in H_i .

Regarding the factorization $G = Q \cdot H_1$, $Q := H_1 \cdot \dots \cdot H_{i-1} H_{i+1} \cdot \dots \cdot H_r$, it follows

$$(1.12) \quad F_i = \bigcap_{\substack{k=1 \\ k \neq i}}^r F_i^k , \quad N_i = \bigcap_{\substack{k=1 \\ k \neq i}}^r N_i^k$$

and using (1.11):

$$(1.13) \quad \begin{aligned} N_G(Q) \cap H_i = F_i &= \bigcap_{\substack{k=1 \\ k \neq i}}^r F_i^k = \bigcap_{\substack{k=1 \\ k \neq i}}^r [N_{G_{i,k}}(H_k) \cap H_i] \\ &= \bigcap_{\substack{k=1 \\ k \neq i}}^r [N_G(H_k) \cap H_i] = \left[\bigcap_{\substack{k=1 \\ k \neq i}}^r N_G(H_k) \right] \cap H_i = \left[\bigcap_{k=1}^r N_G(H_k) \right] \cap H_i \end{aligned}$$

Since F_i consists of all elements of H_i normalizing all H_k , we get $f_i f_k f_i^{-1} f_k^{-1} \in H_i \cap H_k = \langle e \rangle$. Therefore, the system normalizer $F(\mathcal{H}) := \bigcap_{k=1}^r N_G(H_k)$ of G related to the Hall system $\mathcal{H} := \{H_i / i = 1, \dots, r\}$ of G

can be represented as the direct product of the $F_i : F(\mathcal{H}) = F_1 \times \dots \times F_r$.

1.4 Calculation of $\Pi_{i,k}$, F_i^k , N_i^k , computational comparson of products

1.4.1 Determination of $\Pi_{i,k}$

For the determination of $\Pi_{i,k}$ the elements of the components H_i ($i = 1, \dots, r$) of G may be numbered in the same sequence as they are generated by the generating process ((6)). Then, generating the subgroups $G_{i,k} = H_i H_k = H_k H_i$ ($i, k = 1, \dots, r; i < k$) one the one hand as a product of $H_i H_k$ on the other hand as a product of $H_k H_i$ we obtain by comparing the products:

$$(1.14) \quad h_i^{(l)} h_k^{(s)} = h_k^{(s, \nu)} h_i^{(l, \rho)} = h_i^{(l)} h_k^{(s)} \cdot h_k^{(s)} h_i^{(l)} \quad \begin{matrix} (s = 1, \dots, |H_k|) \\ (1 \leq l \leq |H_i|) \end{matrix}$$

From these relations we obtain the permutation $h_i^{(l)} \circ_{\text{of } H_k} h_i^{(l)}$ related to the element $h_i^{(l)} \in H_i : h_i^{(l)} \rightarrow h_i^{(l)} \circ_{(s)} = \binom{s}{s_0}$.

If ℓ runs from 1 to $|H_{i,k}|$ we get $\Pi_{i,k}$. Fixing s ($1 < s < |H_{i,k}|$) we similarly can determine for variable ℓ ($\ell=1, \dots, |H_{i,k}|$) the permutation $h_k^{(s)}i = \begin{pmatrix} \ell \\ \rho \end{pmatrix}$ related to $h_k^{(s)} \in H_{i,k}$, and if s runs from 1 to $|H_{i,k}|$ we get $\Pi_{k,i}$.

1.4.2 Determination of F_i^k and N_i^k

Let $G = H_1 \cdot \dots \cdot H_r$ be a factorization by a Hall system $\{H_i / i = 1, \dots, r\}$ and $E(H_j)$ ($j = i, k$) defined as in 1.1.2.

Then using the results of 1.3.2 by a fundamental well known generating process [6] the groups F_i^k and N_i^k can be determined:

$$(1.15) \quad \begin{aligned} F_i^k &= \langle z_i \rangle \text{ generated by all } z_i \in E(H_i), z_i z_k z_i^{-1} \in H_k \text{ for all } z_k \in E(H_k) \\ N_i^k &= \langle z_i \rangle \text{ generated by all } z_i \in E(H_i), z_i z_k z_i^{-1} \in H_i \text{ for all } z_k \in E(H_k). \end{aligned}$$

1.4.3 Comparison of products

Let $G = H_1 \cdot \dots \cdot H_r$ be a factorization of G by a Hall system $\{H_i / i = 1, \dots, r\}$ of G . Then, for proving the equality $U_i H_k = H_k U_i$, $U_i \subset H_i$ ($k \neq i$) we have to verify the invariance of $U_i P_k$ by applying the permutations $h_{k,i} \in \Pi_{k,i}$, $u_{i,k} \in \Pi_{i,k}$ to $U_i H_k$, respectively. Since H_k is invariant against all $h_{i,k} \in \Pi_{i,k}$ we only have to prove, whether U_i is invariant applying all $h_{k,i} \in \Pi_{k,i}$ to U_i . According to [[3], Theorem 2.2] it is sufficient to prove $h_k^{(\ell)} u_i \in U_i$ for all $u_i \in U_i$ and for all $h_k^{(\ell)}$ of a system $\{h_k^{(\ell)}\}$ of generating elements of H_k . Such comparisons of products will be used in the algorithm of determining the maximal subgroups of a finite group G . (cf. 1.7.3).

1.5 Determination of a Sylow basis of a solvable group G

1.5.1 The Sylow basis as a intersection of Sylow-complements [1], [3]

Let G be a solvable group of order $|G| = p_1^{\alpha_1} \cdot \dots \cdot p_r^{\alpha_r}$. Then, calculating a system $\{K_i / i = 1, \dots, r\}$ of p_i -Sylow complements K_i of G of order $|K_i| = \prod_{\substack{j=1 \\ j \neq i}}^r p_j^{\alpha_j}$, we get a Sylow basis of G by

$$\{P_i = \bigcap_{\substack{j=1 \\ j \neq i}}^r K_j / i = 1, \dots, r\}.$$

1.5.2 Determination of the system $\{K_i / i = 1, \dots, r\}$ of p_i -Sylow complements

Let $M_i = \{z_1, \dots, z_t\} \subseteq E(G)$ be the set of generating elements of all cyclic subgroups $\langle z_\ell \rangle \subseteq G$ ($\ell = 1, \dots, t$) of p_k -prime power order, where $p_k \neq p_i$.

If K_i is a p_i -Sylow complement of G , then $E(K_i) \subseteq M_i$, and conversely to every $z \in M_i$ there exists a p_i -Sylow complement containing z .

A p_i -Sylow complement K_i of G can successively be generated by the calculation of the subgroup chain

$$\langle e \rangle \subset U_1 \subset \dots \subset U_s = K_i \text{ with } U_1 = \langle z_1 \rangle, U_k = \langle U_{k-1}, z_{i_k} \rangle \text{ (} k=1, \dots, s \text{),}$$

where $1 < i_2 < \dots < i_k < \dots < i_s$ and i_k is the minimum of all $j \leq t$, such that

$$z_j \notin U_{k-1} \text{ and } p_i \nmid | \langle U_{k-1}, z_j \rangle |.$$

There exists an algorithm for determining $\langle U, g \rangle$ ($U \subseteq G$, $g \notin U$) described in [6].

1.6 Construction of a chain of normal Hall groups for a non-solvable group G and the determination of a complete Hall system of G

In the following let G be a non-solvable group of order $|G| = p_1^{\alpha_1} \cdot \dots \cdot p_r^{\alpha_r}$.

1.6.1 The lattice of normal Hall groups of G

The set of normal Hall groups of G forms a complete and distributive lattice $\mathfrak{L}(G)$. Any two subgroups series of $\mathfrak{L}(G)$ have isomorphic refinements. Therefore, a complete Hall system of G belonging to a chain of normal Hall groups, which cannot be refined, is up to an inner automorphism of G uniquely determined [O].

1.6.2 Construction of special minimal Hall groups

For any prime number $p_i / |G|$ we get a minimal group $N_{(i)} \in \mathfrak{L}(G)$ such that $p_i / |N_{(i)}|$.

To construct $N_{(i)}$ let $Z_i := \langle z_{i,1}, \dots, z_{i,s_i} \rangle$ be the subgroup of G generated by the generating elements $z_{i,k}$ of all cyclic p_i -subgroups $\langle z_{i,k} \rangle$ of prime power order of G ($i = 1, \dots, r$).

We set $M_i^1 := \{p_i / p_i / |Z_i|\}$ and define inductively:

$$(1.16) \quad M_i^{k+1} := M_i^k \cup \bigcup_{p_j \in M_i^k} M_j^1 \quad (i = 1, \dots, r)$$

Then to any M_i^k ($i = 1, \dots, r$) there uniquely corresponds a vector

$$(\beta_{i1}^k, \dots, \beta_{ir}^k), \quad \text{where} \quad \begin{cases} \beta_{ij}^k = 1, & \text{if } p_j \in M_i^k \\ \beta_{ij}^k = 0, & \text{if } p_j \notin M_i^k \end{cases}$$

and these vectors together form a matrix (β_{ij}^k) ($1 < i, j < r$).

Using the Boolean operations for addition:

$$\begin{aligned} 0 + 0 &= 0 \\ a + b &= 1, \quad \text{if at least one term of the sum is } \neq 0 \end{aligned}$$

from (1.16) we get by matrix multiplication:

$$(1.17) \quad (\beta_{ij}^k)^k = (\beta_{ij}^1)^k \quad (i, j = 1, \dots, r; 1 < k < r - 1)$$

If $(\beta_{ij}^1)^k = (\beta_{ij}^1)^{k+1}$, then $|N_{(i)}| = p_1^{\beta_{i1}^k \alpha_1} \cdot \dots \cdot p_r^{\beta_{ir}^k \alpha_r}$ and $N_{(i)}$ can be generated by the generating elements of all cyclic p_j -subgroups of prime power order of G, where p_j is running through the set of all prime numbers, the exponent β_{ij}^k of which is equal to 1.

1.6.3 Determination of a chain of normal Hall groups of G

Because $N \in \mathfrak{L}(G)$ is uniquely determined by the set of prime numbers dividing $|N|$, we get $N = \prod_{p_i / |N|} N_{(i)}$. Using

fundamental program systems described in [6] it is possible to determine the lattice $\mathfrak{L}(G)$ and consequently an appropriate chain of normal Hall groups.

1.6.4 Determination of a complete Hall system of G

In the following let G be a finite non solvable group with a chain (1.6) of normal Hall groups. Without loss of

generality for our investigations we can consider the chain $G = G_3 \supset G_2 \supset G_1 \supset G_0 = \langle e \rangle$, where G_2/G_1 is not solvable and G_2/G_1 does not contain a normal Hall group.

If $|G_1| = p_1^{\alpha_1} \cdot \dots \cdot p_t^{\alpha_t}$, by the method described in 1.5 it is possible to determine a p_i -Sylow complement

Q_i ($i = 1, \dots, t$) and $F = \bigcap_{i=1}^t Q_i$ is a subgroup of G such that G is a splitting extension of G_1 by F :

$$G_1 \triangleleft G = F \cdot G_1, \quad F \cap G_1 = \langle e \rangle, \quad F \cong G/G_1$$

Further, the system

$$\{P_i = \bigcap_{\substack{j=1 \\ j \neq i}}^t (Q_j \cap G_1) / i = 1, \dots, t\}$$

is a Sylow basis of G_1 such that $P_i \triangleleft FP_i$ ($i = 1, \dots, t$). $H = F \cap G_2$ is a subgroup of G , which represents the factor group G_2/G_1 in G_1 . Since $H \cong G_2/G_1$, by assumption H does not contain a normal Hall group, and we still have to construct a complete Hall system for the chain $F \supset H \supset \langle e \rangle$:

If P_j is a p_j -Sylow subgroup of H with P_j/H , we get by [12] , IV, Exerc.:

$$N_F(P_j)/N_H(P_j) \cong F/H,$$

which means that $N_H(P_j)$ is a normal Hall group of $N_F(P_j)$. By the theorem of Feit-Thompson H has even order. In the case $p_j = 2$ it follows that $N_H(P_j)/P_j$ is of odd order and consequently solvable. Therefore, $N_H(P_j)$ and moreover $N_F(P_j)$ are solvable. Calculating a subgroup L of $N_F(P_j)$ such that $N_F(P_j) = L \cdot N_H(P_j)$ and applying the method of 1.5 to L we obtain a Sylow basis P_{t+1}, \dots, P_{t+s} of L . Then the system $P_1, \dots, P_t, H, P_{t+1}, \dots, P_{t+s}$ is a complete Hall system of G as desired in Theorem 1.2.

1.7 Algorithm for the determination of the maximal subgroups of a solvable group G

In this section let G be a solvable group of order $|G| = p_1^{\alpha_1} \cdot \dots \cdot p_r^{\alpha_r}$. Then $G = P_1 \cdot \dots \cdot P_r$, where $\{P_i / i = 1, \dots, r\}$ is a Sylow basis of G .

1.7.1 Basic theorems

Without proofs we write up the basic theorems, which will be used for the development of the computational algorithm for determining all maximal subgroups of G :

Theorem 1.3: Let U be a subgroup of the solvable group G , $|G| = p_1^{\alpha_1} \cdot \dots \cdot p_r^{\alpha_r}$, $|U| = p_1^{\beta_1} \cdot \dots \cdot p_r^{\beta_r}$ ($0 < \beta_i < \alpha_i; i = 1, \dots, r$). Then to any Sylow basis $P_1(U), \dots, P_r(U)$ of U there corresponds a Sylow basis P_1, \dots, P_r of G such that $P_i(U) = P_i \cap U$ ($i = 1, \dots, r$) ([7], p 666) . .

Theorem 1.4: Every maximal subgroup M of G has prime power index in G ([7], p. 164) .

Theorem 1.5: Let M be a maximal subgroup of G . Then:

- a) If $M \triangleleft G$, then $|G/M| = p$ (p prime number)
- b) If $M \not\triangleleft G$, $[G : M] = p^n$, and if G'_p is a p -Sylow complement of G such that $G'_p \subseteq M$, then $N_G(G'_p) \subseteq M$ ([7], p. 734) .

1.7.2 Consequences of Theorems 1.3.–1.5 for the development of the algorithm

A) Let M be a maximal subgroup of G . Since any two Sylow bases are conjugate in G , it follows from Theorem 1.3 that there exists a conjugate maximal subgroup M^* of G such that from the factorization $G = P_1 \cdot \dots \cdot P_r$

we obtain a factorization $M^* = (M^* \cap P_1) \cdot \dots \cdot (M^* \cap P_r)$ for M^* .

B) By Theorem 1.4, however, only one term $M^* \cap P_i$ is different from $P_i : M^* \cap P_k = P_k$ ($k \neq i$), hence $M^* = P_1 \cdot \dots \cdot (M^* \cap P_i) \cdot \dots \cdot P_r$.

C) $G'_p = P_1 \cdot \dots \cdot P_{i-1} P_{i+1} \cdot \dots \cdot P_r$ is a p_i -Sylow complement contained in M^* .

If $M \not\trianglelefteq G$, by Theorem 1.5 we get $N_G(G'_p) \subseteq M^*$, hence $F_i = N_G(G'_p) \cap P_i \subseteq M^* \cap P_i$, where F_i is the i -th component of the system normalizer $F = F_1 \times \dots \times F_r$ of G (1.3.3). F_i depends only on the factorization $G = P_1 \cdot \dots \cdot P_r$ of G . In the case $M \triangleleft G$ we get

$$G/M = P_1 \cdot \dots \cdot P_r / P_1 \cdot \dots \cdot P_{i-1} (M \cap P_i) P_{i+1} \cdot \dots \cdot P_r \cong P_i/M \cap P_i.$$

This means that $M \cap P_i$ is a maximal subgroup of P_i with $[P_i : (M \cap P_i)] = p_i$, $(M \cap P_i) P_k = P_k (M \cap P_i)$ ($k = 1, \dots, r, k \neq i$).

1.7.3 The algorithm

By the following algorithm it is possible to calculate all maximal subgroups of the solvable group $G = P_1 \cdot \dots \cdot P_r$ of order $|G| = p_1^{\alpha_1} \cdot \dots \cdot p_r^{\alpha_r}$ factorized by the p_i -Sylow groups of a Sylow basis $\{P_i / i = 1, \dots, r\}$ of G . We assume that the lattice $V(P_i)$ of all subgroups of P_i ($i = 1, \dots, r$) has been determined by one of the methods described in [2], [4], [10].

Let us fix the index i ($1 \leq i \leq r$). Since $|P_i| = p_i^{\alpha_i}$, $V(P_i)$ consists of $\alpha_i + 1$ layers. The s -th layer Σ_s of $V(P_i)$ contains only those subgroups of P_i having order p_i^s ($0 \leq s \leq \alpha_i$).

The groups $Q_{0,1}, \dots, Q_{0,s_0}$ of the layer Σ_{α_i-1} of $V(P_i)$, which satisfy

$$(1.18) \quad Q_{0,j} P_k = P_k Q_{0,j} \quad (j=1, \dots, s_0, k=1, \dots, r, k \neq i) \quad *)$$

are the components of the normal maximal subgroups

$$M_{0,j} = P_1 \cdot \dots \cdot P_{i-1} Q_{0,j} P_{i+1} \cdot \dots \cdot P_r \quad (j = 1, \dots, s_0)$$

of G , for which $P_i \triangleleft M_{0,j}$.

For the further iterative procedure only the groups of the set

$$S_0(P_i) := \{H \in \bigcup_{\nu=2}^{\alpha_i} \Sigma_{\alpha_i-\nu} / F_1 \subseteq H\} \quad **)$$

are relevant.

Now we define inductively that part of the algorithm, by which non-normal maximal subgroups of G are defined:

If $S_t(P_i) \neq \emptyset$ ($1 \leq t \leq \alpha_i$), where

$$S_t(P_i) = \{H \in S_{t-1}(P_i) / H \not\subseteq Q_{t-1,j} \text{ and } |H| \neq p^{\alpha_i-t} \quad (j = 1, \dots, s_{t-1})\}$$

we pick up all subgroups $Q_{t,1}, \dots, Q_{t,s_t} \in S_t(P_i)$, which belong to the $(\alpha_i - t - 1)$ -th layer of $V(P_i)$ and satisfy the relations:

$$Q_{t,j} P_k = P_k Q_{t,j} \quad (j = 1, \dots, s_t, k = 1, \dots, r, k \neq i).$$

*) For proving the relations 1.18 see (1.4.3).

***) $\bigcup_{\nu=2}^{\alpha_i} \Sigma_{\alpha_i-\nu}$ is equal to the union of all groups $H \in V(P_i)$ contained in the layers $\Sigma_{\alpha_i-\nu}$ ($\nu = 2, \dots, \alpha_i$) of $V(P_i)$.

Then the groups

$$M_{t,j} = P_1 \cdot \dots \cdot P_{i-1} Q_{t,j} P_{i+1} \cdot \dots \cdot P_r \quad (j = 1, \dots, s_t)$$

are non-normal subgroups of G , for which $P_1, \dots, P_{i-1}, M_{t,j} \cap P_i, P_{i+1}, \dots, P_r$ is a Sylow basis.

$S_{t+1}(P_i)$ will be obtained from $S_t(P_i)$ by eliminating all groups of $S_t(P_i)$, which are subgroups of the $Q_{t,j}$ ($j = 1, \dots, s_t$) or satisfy a special order relation:

$$S_{t+1}(P_i) = \{H \in S_t(P_i) / H \subseteq Q_{t,j} \text{ and } |H| \neq p^{\alpha_i - t - 1} \quad (j = 1, \dots, s_t)\}$$

If $S_m(P_i) = \emptyset$ ($1 < m < \alpha_i$) all maximal subgroups M of G , for which $P_1, \dots, P_{i-1}, M \cap P_i, P_{i+1}, \dots, P_r$ is a Sylow basis, are determined.

Repeating this method for each $i \in \{1, \dots, r\}$ we get the set \mathcal{R} of all maximal subgroups M of G , for which $M \cap P_1, \dots, M \cap P_r$ is a Sylow basis.

From the theory of Sylow systems finally follows that the set \mathcal{C} of all maximal subgroups of G will be obtained by the application of special inner automorphisms $\tau(g_i)$ of G on all elements

$$K \in \mathcal{R} : \tau(g_i)K := g_i^{-1}Kg_i,$$

where g_i are the representatives of the coset decomposition $G = Fg_1; \dots; Fg_t$ ($g_1 = e$) of G by the system normalizer $F = \bigcap_{j=1}^r N_G(P_j)$ of G .

1.8 The algorithm for non-solvable groups containing a chain of normal Hall groups

1.8.1 Basic theorems

Let G be a finite non solvable group, which contains a chain $G = G_r \supset \dots \supset G_i \supset \dots \supset G_1 \supset \langle e \rangle$ of normal Hall groups. Then, similar to the theorems of section 1.7 in [0] the following fundamental results are proved:

Theorem 1.6: Let $G = G_r \supset \dots \supset G_i \supset \dots \supset G_1 \supset \langle e \rangle$ be a chain of normal Hall groups of G and M a maximal subgroup of G . Then $[G : M] / |G_i/G_{i-1}|$ for some $i \in \{1, \dots, r\}$. Additionally, if G_i/G_{i-1} is solvable, $[G : M]$ is a prime power. ([0], Theorem II, 1.4).

Theorem 1.7: Let $G = G_r \supset \dots \supset G_i \supset \dots \supset G_1 \supset \langle e \rangle$ be a chain of normal Hall groups of G and U a subgroup of G with a chain $U = U_r \supseteq \dots \supseteq U_i \supseteq \dots \supseteq U_1 \supseteq \langle e \rangle$ of normal Hall groups $U_i = U \cap G_i$ ($i = 1, \dots, r$) of U . Then to every complete Hall system $\mathcal{K}(U)$ of U there exists a complete Hall system $\mathcal{K}(G)$ of G such that the elements of $\mathcal{K}(U)$ can be obtained by intersecting the elements of $\mathcal{K}(G)$ with U . ([0], Theorem II, 1.11).

Theorem 1.8: Let $G = G_r \supset \dots \supset G_i \supset \dots \supset G_1 \supset \langle e \rangle$ be a chain of normal Hall groups of G . With regard to this chain let further

$$P_{1,1}, \dots, P_{1,n_1}, \dots, P_{k-1,1}, \dots, P_{k-1,n_{k-1}}, H_k, P_{k+1,1}, \dots, P_{k+1,n_{k+1}}, \dots, P_{r,1}, \dots, P_{r,n_r}$$

be a complete Hall system of G , where H_k is the non solvable part of the Hall system. Then:

- If M is a maximal subgroup of G such that $[G : M] \not\propto |H_k|$, then $[G : M] = p^\alpha$ for some prime number p .
- If $M \triangleleft G$, $[G : M] \propto |H_k|$, then $[G : M] = p$.
- If $M \not\triangleleft G$, $[G : M] \not\propto |H_k|$, then there exists a maximal subgroup M^* of G , which is conjugate to M , such that

$$P_{1,1} \cap M^*, \dots, P_{1,n_1} \cap M^*, \dots, P_{k-1,1} \cap M^*, \dots, P_{k-1,n_{k-1}} \cap M^*, H_k, P_{k+1,1} \cap M^*, \dots, P_{k+1,n_{k+1}} \cap M^*, \dots, P_{r,1} \cap M^*, \dots, P_{r,n_r} \cap M^*$$

is a complete Hall system of M^* with:

$$\begin{aligned} P_{j,i} \cap M^* &= P_{j,i}, \text{ if } P \not\leq P_{j,i} \\ P_{r,s} \cap M^* &= P_{r,s}^* \supseteq F_{r,s}, \text{ if } P / |P_{r,s}| \\ F_{r,s} &= P_{r,s} \cap N_G(H_k) \bigcap_{j,i \neq r,s} N_G(P_{j,i}). \end{aligned} \quad ([0], \text{Theorem II. 1.13})$$

1.8.2 Some remarks about the proofs of Theorem 1.6 – Theorem 1.8

Theorem 1.6 follows trivially for $n > 2$, if it is true for $n = 2$. Therefore, let $G = F \cdot N$, $N \triangleleft G$, $(N|, |F|) = 1$ be a splitting extension of G and let further M be a maximal subgroup of G . Then, if $M \supseteq N$, M/N is maximal in G/N and Theorem 1.6 follows from Theorem 1.4. If $M \not\supseteq N$, we get $G = M \cdot N$, hence $|G| = |M| \cdot |N| / |M \cap N|$, which yields $[G : M] = |N| / |M \cap N|$. In the case N being solvable for an appropriate prime number p N contains a non-trivial characteristic p -subgroup C . For $C \not\subseteq M$ we get $G = M \cdot C$ and for $C \subseteq M$ the result follows by induction.

If G is a solvable group of order $|G| = p_1^{\alpha_1} \cdot \dots \cdot p_r^{\alpha_r}$, A a subgroup of its automorphism group such that $(|A|, |G|) = 1$ and if further π is a set of prime numbers dividing $|G|$, then it can be shown that any A -allowable π -subgroup U of G is contained in an A -allowable π -Hall group of G . Using this result under the same assumptions on G it follows that to every A -allowable Sylow system $P_1(U), \dots, P_r(U)$ of an A -allowable subgroup U of G there corresponds an A -allowable Sylow system P_1, \dots, P_r of G such that $P_i \cap U = P_i(U)$ ($i = 1, \dots, r$). Making use of this result Theorem 1.7 follows immediately.

Let M be a maximal A -allowable subgroup of the solvable group G . Then, if $M \triangleleft G$ and A induces the identity on G/M , G/M is cyclic of prime order. In the case that $M \triangleleft G$ and A does not induce the identity on G/M or $M \not\triangleleft G$, we obtain $[G : M] = p^\alpha$ ($\alpha > 1$) for some prime number p and $[N_G(G'_p)]^A \subseteq M$ for a p -complement G'_p of G contained in M . From this Theorem 1.8 follows consequently.

1.8.3. Computational consequences

From the point of view of computation Theorem 1.8 shows, that the algorithm in the case of a non solvable group G , which contains a complete Hall system $P_1, \dots, P_t, H, P_{t+1}, \dots, P_{t+s}$, with the exception of the non solvable part H of G is exactly the same as in the case G being solvable.

Proceeding from a generating system $E(H)$ (cf. 1.1.2) and using a subgroup chain

$$\langle e \rangle = K_0 \subset K_1 \subset \dots \subset K_s = H \text{ where } K_i = \langle K_{i-1}, z_{t_i} \rangle, z_{t_i} \in E(H) \quad (i = 1, \dots, s)$$

in [6], 1 it is pointed out, that making coarser this subgroup chain it is possible to find an appropriate generating system of H , the elements of which can be multiplied most effectively.

A complete Hall system $P_1, \dots, P_t, H, P_{t+1}, \dots, P_{t+s}$ can be determined by the method developed in 1.6.4. Since H is not solvable, it is necessary to calculate $V(H)$ by an algorithm described in [4], [6], [10].

By Theorem 1.6, Theorem 1.7 and the property of conjugateness of a Hall system it follows that in the case M being maximal in G with $[G : M] / |H|$ there exists a conjugate subgroup M^* of M in G such that $M^* = P_1 \cdot \dots \cdot P_t Q P_{t+1} \cdot \dots \cdot P_s$, $Q \subset H$. But if $M^* \triangleleft G$, Q is not generally maximal in H . Therefore, the normality of M^* must be proved separately.

If $Q_{j,1}, \dots, Q_{j,s(j)}$ are the subgroups of the j -th layer of $V(H)$ such that $M_{j,k}^* = P_1 \cdot \dots \cdot P_{t-1} Q_{j,k} P_{t+1} \cdot \dots \cdot P_{t+s}$ ($k = 1, \dots, s(j)$) are maximal in G , then only subgroups of the lower layers of $V(H)$, which are not contained in the $Q_{j,k}$'s, can generate further maximal subgroups of G .

Taking notice only of this property the algorithm developed for p -Sylow groups P can also be used in the case H being a non-solvable group of the Hall system of G .

2. Extension of the method for determining the maximal subgroups of a finite group G to an algorithm for the determination of the complete lattice V(G) of G

2.1 The case of a solvable group G

2.1.1 The semi-lattice T(P₁, . . . , P_r) as the underlying structure of V(G)

Let {P_i / i = 1, . . . , r} be a Sylow basis of a solvable group G of order |G| = p₁^{α₁} · . . . · p_r^{α_r}. Then, having calculated the semi-lattice T(P₁, . . . , P_r) of all subgroups U of G, for which U ∩ P₁, . . . , U ∩ P_r is a Sylow basis, by Theorem 1.3 and by making use the properties of a Hall system we obtain the complete lattice V(G) of G from T(P₁, . . . , P_r) by the application of special inner automorphisms τ(g_i) (i = 1, . . . , s) to the elements of T(P₁, . . . , P_r) where g_i are the representatives of the coset decomposition of G by the System normalizer F of the Sylow basis {P_i / i = 1, . . . , r}.

2.1.2 Construction of T(P₁, . . . , P_r)

To construct T(P₁, . . . , P_r) of G, it is necessary to determine the maximal subgroups M of G with M ∩ P₁, . . . , M ∩ P_r being a Sylow basis of M and for every M similar the maximal subgroups M' with M' ∩ P₁, . . . , M' ∩ P_r being a Sylow basis of M', a.s.o. .

Assuming that U is a subgroup of G such that U ∩ P₁, . . . , U ∩ P_r is a Sylow basis of U, a maximal subgroup V of U with Sylow basis

$$V \cap (U \cap P_1) = V \cap P_1, \dots, V \cap (U \cap P_r) = V \cap P_r$$

can be determined, if it is possible to calculate the permutation groups Π_{i,k}(U) related to the factorization U = (U ∩ P₁) · . . . · (U ∩ P_r) of U.

2.1.3 Calculation of the permutation group Π_{i,k}(U)

Let Ω_i = {a_{i,1}, . . . , a_{i,t(i)}} be a generating system of P_i ([6],1) and {a_{i,1}k, . . . , a_{i,t(i)}k} the set of permutations of Π_{i,k} related to Ω_i.

Then, if p₁^(k) are the elements of a generating system {p₁^(k)} of U ∩ P₁ and if

$$p_1^{(k)} = a_{i,1}^{c_1} \cdot \dots \cdot a_{i,t(i)}^{c_{t(i)}}$$

is the representation of p₁^(k) as a word of the a_{i,j}'s (j = 1, . . . , t(i)), we obtain the following permutations

p₁^(k) of P_k :

$$(2.1) \quad \begin{aligned} p_1^{(k)k} &= (a_{i,1}k)^{c_1} \cdot \dots \cdot (a_{i,t(i)k})^{c_{t(i)}} && (i > k) \\ p_1^{(k)k} &= (a_{i,t(i)k})^{c_{t(i)}} \cdot \dots \cdot (a_{i,1}k)^{c_1} && (i < k) \end{aligned}$$

The restrictions of these permutations on U ∩ P_k are the required generating elements of the groups Π_{i,k}(U). In this way we are able to construct Π_{i,k}(U) for every U ∈ T(P₁, . . . , P_r).

2.2 The case of a non solvable group G containing a chain of normal Hall groups

Let P₁, . . . , P_tH₀P_{t+1}, . . . , P_{t+s} be a complete Hall system of the non solvable group G related to a chain of normal Hall groups.

If U ∈ T(P₁, . . . , P_tH₀P_{t+1}, . . . , P_{t+s}), then we use an appropriate generating system for H₀ to calculate Π_{0,k}(U) from Π_{0,k} (k = 1, . . . , t + s).

2.2.1 Determination of an appropriate generating system of H_0

Outgoing from a generating system $E(H_0)$ of H_0 (cf. 1.1.2), such a generating system for H_0 can be determined by an appropriate subgroup chain:

$\langle e \rangle = K_0 \subset \dots \subset K_j \subset \dots \subset K_s = H_0$ where $K_j = \langle K_{j-1}, h_j \rangle$, $h_j \in E(H_0)$, $[K_j : K_{j-1}] = r_j$, $(j=1, \dots, s)$.

If $R_j := \{\alpha_j^{(\nu)} / \nu = 0, \dots, r_j-1\}$ is a system of representatives of a right coset decomposition of K_j by K_{j-1} ($j = 1, \dots, s$), $\alpha_j^{(0)} = e$, every element $h_0 \in H_0$ has a unique representation in the form

$$(2.2) \quad h_0 = \alpha_1^{(\lambda_1)} \cdot \alpha_2^{(\lambda_2)} \cdot \dots \cdot \alpha_s^{(\lambda_s)}$$

and if the relations

$$(2.3) \quad \begin{matrix} \alpha_k^{(\lambda_k)} \alpha_\ell^{(\lambda_\ell)} = \alpha_1^{(\mu_1)} \cdot \dots \cdot \alpha_k^{(\mu_k)} & \left[\begin{matrix} \mu_\nu = f(\nu, k, \ell, \lambda_k, \lambda_\ell) \\ \nu = 1, \dots, k \end{matrix} \right] \\ (k=1, \dots, s, \ell=1, \dots, k, \lambda_k=0, \dots, r_{k-1}, \lambda_\ell=0, \dots, r_{\ell-1}) \end{matrix}$$

are known, the system $\alpha = \bigcup_{j=1}^s R_j$ is a generating system of H_0 of the desired form ([6], 1.3.4).

2.2.2 Computational Reduction for the calculation of $\Pi_{0,k}$

Since $P_k \triangleleft H_0 \cdot P_k$ ($k = 1, \dots, t$) and $H_0 \triangleleft H_0 \cdot P_k$ ($k = t+1, \dots, t+s$), the elements of $\Pi_{0,k}$ ($k = 1, \dots, t$) are automorphisms of P_k and $\Pi_{0,k}$ ($k = t+1, \dots, t+s$) only consists of the identical permutation of P_k . Therefore, the operation of $\Pi_{0,k}$ on P_k ($k = 1, \dots, t$) is already uniquely determined by the operation of the elements $\alpha_j^{(\lambda_j)}$ (k related to the elements $\alpha_j^{(\lambda_j)}$ of the generating system of H_0) applied to a generating system of P_k ([3], Theorem 2.2). Similar conditions are valid for $\Pi_{k,0}$. But using these reductions the calculation of $\Pi_{0,k}(U)$ from $\Pi_{0,k}$ for $U \in T(P_1, \dots, P_t, H_0, P_{t+1}, \dots, P_{t+s})$ is a time-saving procedure in the computational program.

References

[0] Altmann, E.:
Über ein Verfahren zur Berechnung der maximalen Untergruppen einer endlichen Gruppe mit einer Hall'schen Normalverteilerkette,
Berichte der Gesellschaft für Mathematik und Datenverarbeitung, Bonn, 19, (1969)

[1] Altmann, E.:
Anwendung der Theorie der Faktorisierungen,
Forschungsbericht des Landes Nordrhein-Westfalen, No. 1902, (1968)

[2] Geller, E.:
Ein Programm zur Bestimmung der Automorphismengruppen endlicher auflösbarer Gruppen,
Berichte der Gesellschaft für Mathematik und Datenverarbeitung, Bonn, 51, (1971)

[3] Gerhards, L. a. Altmann, E.:
A computational method for determining the automorphism group of a finite solvable group,
(Proc. Conf. on Comp. Algebra, Oxford, August 1967), Pergamon Press, New York, 61-74, (1969)

[4] Gerhards, L. a. Lindenberg, W.:
Ein Verfahren zur Berechnung des vollständigen Untergruppenverbandes endlicher Gruppen auf Dualmaschinen,
Num. Math. 7, 1-10, (1965)

[5] Gerhards, L.:
On the construction of the automorphism group of a finite group,
will be published in Cahiers Mathématiques, Montpellier

[6] Gerhards, L.:
Group theoretical investigations on computers,
will be published in Proceedings of the conference „Utilisation des calculateurs en mathématiques pures“,
Limoges, (1975)

- [7] Huppert, B.:
Endliche Gruppen I,
Berlin, (1967)
- [8] Jürgensen, H.:
Calculation with the elements of a finite group given by generators and defining relations,
(Proc. Conf. on Comput. Algebra, Oxford, Aug. (1967)), Pergamon, New York, 47–57, (1969)
- [9] Lindenberg, W.:
Über eine Darstellung von Gruppenelementen in digitalen Rechenautomaten,
Num. Math. 4, 151–153, (1962)
- [10] Lindenberg, W. a. Gerhards, L.:
Combinatorial construction by computer of the set of all subgroups of a finite group by composition
of partial sets of its subgroups,
(Proc. Conf. on Comput. Algebra, Oxford, Aug. (1967)), Pergamon, New York, 75–82, (1969)
- [11] Redei, L.:
Die Anwendungen des schiefen Produktes in der Gruppentheorie,
J. reine u. angew. Math. 188, 201–228, (1950)
- [12] Zassenhaus, H.:
The theory of groups,
Celsea Publ. Comp. (1958)

Leonhard GERHARDS
Gesellschaft für Mathematik
und Datenverarbeitung
D-5205 ST. AUGUSTIN 1
Schloss Birlinghoven
Postfach 1240 (R.F.A.)