Astérisque

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Astérisque, tome 32-33 (1976), p. 29-41

<http://www.numdam.org/item?id=AST_1976__32-33__29_0>

Société Mathématique de France
Astérisque 32-33 (1976)

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To be invited to speak or write under this title is both an honour and a pleasure; and for me it is also in some sense a filial duty. For when I first began to study algebraic topology, I was deeply influenced by the work of Henri Cartan and his school. It is said that "imitation is the sincerest form of flattery"; I have published such flatteries upon the French school beginning almost twenty years ago; so you may believe that my expressions of respect are genuine, and are not staged to suit the occasion.

Of course, Henri Cartan did much work in algebraic topology and certainly I shall not be able to mention it all. For example, we have the two papers [1,2], which were very important for the development of our understanding of the cohomology of classifying spaces; and I can do no more than mention them.

I propose to jump right into the middle of things with the two papers [4,5]. Let me just remind you of the general construction given there; it is in the nature of things that I must explain matters which were then new, but are now familiar to most topologists. Suppose given a topological space X, say connected. By attacking cells to X, in a way which is now well-known, I can construct a space X' possessing a map $i : X \longrightarrow X'$ with the following properties.

(i) $i_* : \pi_r(X) \longrightarrow \pi_r(X')$ is iso for $r \le n$.

(ii) $\pi_r(X') = 0$ for $r > n$.

These properties characterise X' up to canonical equivalence. It is a convenient mnemonic to write $X(1,2,\ldots,n)$ for such a space X', to show that it has the same homotopy groups as X in dimensions $1,2,\ldots,n$.

By a construction which, again, has since become well-known, we may convert the map $X \longrightarrow X(1,2,\ldots,n)$ into a fibering, at the cost of replacing the space X by an equivalent space. Consider the fibre F. It comes provided with a map $j : F \longrightarrow X$, namely the inclusion of the fibre in the total space; and by considering the exact homotopy sequence of the fibering, we see that it has the following properties.

(i) $j_* : \pi_r(F) \longrightarrow \pi_r(X)$ is iso for $r > n$.

(ii) $\pi_r(F) = 0$ for $r \leq n$.

These properties characterise F, and it is a convenient mnemonic to write $X(n+1,\ldots,\infty)$ for such a space F, to show that it has the same homotopy groups as X in dimensions $n+1,\ldots,\infty$.

These constructions can be repeated. Let us take the space $X(1,\ldots,m)$ for some $m > n$ and perform on it the same constructions which we performed above on X. The base we obtain is easily seen to be a space $X(1,\ldots,n)$; the fibre may be called $X(n+1,\ldots,m)$, for it has the same homotopy groups as X in these dimensions. Alternatively, we could take the space $X(n+1,\ldots,\infty)$ and apply the same constructions with n replaced by m; we would obtain a base of the same type $X(n+1,\ldots,m)$.

In this way we get from X an extensive system of fiberings. The building-blocks for this system are the spaces $X(n)$, which have just one of the homotopy groups of X, all their other homotopy groups being zero. These then are Eilenberg-MacLane spaces; in the terminology of Eilenberg-MacLane, $X(n)$ is a space of type (π,n), where $\pi = \pi_n(X)$.

I have to make several points about this construction.

(i) It is certainly appropriate when we want to study maps from some other space W to X. For, by definition, fiberings are things which behave well when you map spaces into them, and we know what we get when we map a space W into an Eilenberg-MacLane space:

$$[W,X(n)] = H^n(W;\ \pi_n(X)).$$

In this way we get a very geometric approach to obstruction-theory. In order to calculate $[W,X]$, one should start from the cohomology groups of W with coefficients in the homotopy groups of X; then one must see what happens in the various fiberings of our system, and this gives something like a spectral sequence in the category of sets.

(ii) The construction gives one an understanding of the way in which one can build up the homotopy type of X, by knowing the homotopy groups $\pi_n(X)$ and the invariants which classify the various fiberings.

(iii) In this respect it was not entirely without precedent, for Postnikov had done something similar a little earlier in a semi-simplicial context, and there was also independent work by G.W. Whitehead.

(iv) But as compared with Postnikov's approach, the approach of Cartan-Serre was more geometric; it had the French virtues of clarity and lucidity; other topologists could understand it, and they could elaborate and manipulate the constructions to their own ends.

(v) Finally, it afforded a practical method of computing homotopy groups.

Now this last point depends entirely on two premises. The minor premise is that one can calculate the homology of fiberings, using spectral sequences or, in the stable case, exact sequences. The major premise is that one knows the homology and cohomology of Eilenberg-MacLane spaces, as one does since the work of Cartan. Let us suppose we do know that; and let us suppose that we know the homotopy groups of X. Then we know the building-blocks $X(n)$ which enter; we know their homology and cohomology; if we also know the invariants which describe our fiberings, then we may hope to compute the homology and cohomology of these fiberings and so compute the homology and cohomology of X.

In practice, of course, we already know the homology and cohomology of X and we do not know its homotopy, so we run the method backwards. That is, we attempt an induction, computing at the n^{th} stage the homology and cohomology of $X(1,\ldots,n)$ or $X(n+1,\ldots,\infty)$, according to which variant of the method we use; and we obtain the homotopy groups of X as we go, by the formula

$$\begin{aligned}\pi_{n+1}(X) &\cong \pi_{n+1}(X(n+1,\ldots,\infty)) \\ &\cong H_{n+1}(X(n+1,\ldots,\infty)),\end{aligned}$$

or by the corresponding formula in the other case.

In fact, the paper [5] gives some pioneering calculations of this sort, including the calculation of $\pi_r(S^3)$ for $r \leq 8$. This method is still the preferred method of attack for suitable problems in homotopy-theory; sometimes the straight method I've discussed, sometimes with variations and elaborations due to later authors. To give an idea, let me cite some example, say the 2-primary components of the stable homotopy groups of spheres

$$\lim_{n\to\infty} \pi_{n+k}(S^n).$$

We now possess fairly reliable calculations, by essentially this method, for $k \leq 45$; and appropriately higher for primes other than 2.

The cohomology of Eilenberg-MacLane spaces is important for another reason besides the one I've mentioned so far, and that is its connection with cohomology operations. By a cohomology operation, I shall mean a natural transformation

$$\phi : H^n(\; ; \pi) \longrightarrow H^m(\; ; A)$$

for some fixed n, π, m and A. For example, the Steenrod square

$$Sq^i : H^n(\; ; Z_2) \longrightarrow H^{n+i}(\; ; Z_2)$$

is such a natural transformation, and so is the cyclic reduced power

$$P^k : H^n(\; ; Z_p) \longrightarrow H^{n+2k(p-1)}(\; ; Z_p).$$

Such operations are of course essential in many calculations, including calculations of the sort I have described. Now the set of natural transformations

$$\phi : H^n(\; ; \pi) \longrightarrow H^m(\; ; A)$$

is in (1-1) correspondence with

$$H^m(EM(\pi,n); A).$$

This follows from the fact that

$$H^n(\; ; \pi) \cong [\; , EM(\pi,n)];$$

it follows by what is now called Yoneda's lemma in category theory. Yoneda's lemma tells you about the natural transformations defined on any representable functor; but if you understand the special case of cohomology operations then you understand the general case also. In fact I think that in the historical development of this circle of ideas, two cases were particularly important; the case of cohomology operations, as studied by the French school, and the case of homotopy operations, as studied in America by, for example, Blakers and Massey. At that time the words "universal example" were used to indicate the basic idea of the Yoneda lemma.

In fact, in calculations one tends to use stable cohomology operations if one possibly can, rather than cohomology operations in the sense I have described. In terms of Eilenberg-MacLane spaces, one is concerned with

$$\lim_{\underset{n}{\longleftarrow}} H^{n+k}(EM(\pi,n); A);$$

an element of this limit corresponds to a sequence of cohomology operations

$$\phi_n : H^n(; \pi) \longrightarrow H^{n+k}(; A),$$

one for each n, and commuting with suspension or coboundary maps. For example, the Steenrod operations Sq^i and P^k are of this nature. If we take $\pi = A$, then we can compose such operations; if we take $\pi = A = Z_p$, we get an algebra of operations, usually called the mod p Steenrod algebra.

As for the calculation of the cohomology of Eilenberg-MacLane spaces, the history is well known. Eilenberg and MacLane realised the importance of the project; they introduced the bar construction; and they calculated the groups $H_{n+k}(EM(\pi,n);\ Z)$ for $k \leq 5$, except possibly for the case $k = 5,\ n = 2$. But, in fact, the only incompleteness of their work in that case was the following: their theological preconceptions forced them to describe the group $H_{n+k}(EM(\pi,n);Z)$ as a functor of π, and if k is large compared with n this functor is awkward to describe, owing to higher-order non-additivity phenomena. However one can take the following position: it is the business of the group $H_{n+k}(EM(\pi,n);Z)$ to be a functor, it has no choice, so let us leave it to look after its business and let us look after ours,which is to prove something useful about it. And this was brilliantly done by Serre, who calculated $H^*(EM(\pi,n);Z_2)$ without any restrictions on the dimensions but with coefficients mod 2. His methods were particular to the prime 2, and at that time did not extend to calculations with coefficients mod p (although such an extension became possible later.) Finally, we know, the problem was completely solved by Cartan; his results are summarised in [6,7] and fully set out in the Séminaire Cartan [10].

I have to make several points about this work.

(i) It was conclusive. Mathematicians love a conclusive theorem. After Cartan's work nobody needed to work any more on the homology of Eilenberg-MacLane spaces; the task was to apply the information which Cartan had obtained.

Iii) For the enthusiast it is perhaps tempting to analyse the algebraic style of Cartan's proof, and relate it to his earlier work. For example, the strategy has something in common with his work on homological algebra [3,9]; one gives rules prescribing some class of algebraic "constructions"; although there are many different "constructions" legitimate under the rules, one shows that they all yield the same invariants; one then makes calculations using the most convenient construction. Again, one can find precedents in Cartan's earlier work for the idea of making an algebraic analogue of a fibering. But on the grounds of space I will not elaborate these points.

(iii) One corollary of this work was that it allowed Cartan to give a new and simple derivation [8] of the relations between cohomology operations which had been found by Adem. Let me recall the principle. Suppose, for example, that we know that the space of natural transformations

$$\theta : H^4(\ ; Z_p) \longrightarrow H^{2p}(\ ; Z_p)$$

is Z_p, generated by the Steenrod operation P^2. We observe that the composite P^1P^1 is another such natural transformation; we wish to know what multiple of P^2 it is. It is sufficient to try one suitable special case, say

$$x_1x_2 \in H^4(CP^\infty \times CP^\infty; Z_p)$$

(where x_1 and x_2 come from the 2-dimensional generators for the cohomology of the two factors.) We find

$$P^1P^1(x_1x_2) = 2x_1^px_2^p$$

$$P^2(x_1x_2) = x_1^px_2^p$$

and we deduce

$$P^1P^1 = 2P^2.$$

Let me repeat. Other methods allow you to construct cohomology operations; for example, Steenrod constructed the operations Sq^i and P^k by using permutations, without benefit of the Eilenberg-MacLane spaces. But the approach of Cartan-Serre is the only method which allows you to be sure that you have constructed all possible operations. Other methods allow you to prove relations between operations; for example, Adem proved $Sq^2Sq^2 = Sq^3Sq^1$, etc., by using Steenrod's permutation method. The method of Cartan-Serre is the only method which allows you to be sure you have obtained all possible relations. These points become relevant when we seek to replace ordinary cohomology by a generalised cohomology theory.

Let me go on to some of the later development of the subject. It follows from Cartan's work that one obtains a complete grasp of the structure of the mod p Steenrod algebra; and this was later put in a very elegant form by John Milnor, using the dual of the Steenrod algebra, that is

$$\varinjlim_{n} H_{n+k}(EM(Z_p,n);Z_p).$$

I gave a variant of the Cartan-Serre method so far as it applies to computing stable homotopy groups. Under suitable hypotheses on X and Y, there is a spectral sequence

$$Ext_A^{s,t}(\tilde{H}^*(Y;Z_p), \tilde{H}^*(X;Z_p)) \Longrightarrow {}_p\{X,Y\}_{t-s}.$$

On the left, A means the mod p Steenrod algebra, and Ext_A is the usual functor of homological algebra [9]. On the right, $\{X,Y\}$ means the group of stable homotopy classes:

$$\{X,Y\}_r = \lim_{n\to\infty} [S^{n+r}X, S^nY].$$

The subscript p instructs one to take the p-component of the group to which it is attached.

John Milnor used this spectral sequence to good effect in calculating cobordism groups, for example, in calculating $\pi_*(MU)$.

Other authors, such as Massey and Peterson, arrived at a similar spectral sequence for computing unstable homotopy groups, but of course it involves complicating the definition of Ext, and it tends to involve hypotheses on X and Y which are not always verified.

For the case $X = Y = S^0$ the stable spectral sequence has been explored by various authors, most intensively perhaps for the prime $p = 2$, of which I will now speak. If we discount the line $t = s$, where the behaviour is known, it is found that the non-zero part of this spectral sequence is confined to a region $t \geq 3s - \varepsilon$ (approximately). Moreover, in a region $3s - \varepsilon \leq t \leq 6s - \eta$ (approximately) the behaviour of the E_2 term $\mathrm{Ext}_A^{s,t}(Z_2, Z_2)$ is systematic, so that one might in principle give an account of it. Unfortunately, the behaviour of the spectral sequence in this region is a bit like an Elizabethan drama, full of action, in which the business of each character is to kill at least one other character, so that at the end of the play one has a stage strewn with corpses and only one actor left alive (namely the one who has to speak the last few lines.) Almost all the elements kill each other by differentials.

On this topic I must make two points.

(i) So far as I know the literature does not contain a formal statement and a rigourous proof of the assertions in the previous paragraph. From one point of view this is a deficiency which it would be desirable to rectify.

(ii) From another point of view, the assertions reveal the Adams spectral sequence as an inefficient method for computing homotopy groups. One reason for this, when you come to think about it, is that there is a limited amount of information built into the Steenrod algebra. For example, as Bott says, the Steenrod algebra doesn't know about the solution of the "Hopf invariant one problem"; nobody told it. This makes it reasonable to try and replace ordinary cohomology with some other theory which is better informed; ideally we would like to use the most powerful generalised cohomology theory for which we can actually carry out calculations.

Now of course a great step in this direction was taken by Novikov, who pointed out and exploited the many virtues of the theory MU^*, complex cobordism.

Actually if you want to study p-primary phenomena, it is sufficient to localise MU and then take one summand of it; in this way we arrive at the Brown-Peterson spectrum BP. Novikov did not have a good grasp of the algebra of operations on BP-cohomology, but this was provided later by Quillen.

The Princeton team of Miller, Ravanel and Wilson - using ideas coming from Morava - have recently calculated

$$\mathrm{Ext}^{2*}_{BP_*(BP)}(\pi_*(BP), \pi_*(BP)).$$

It seems that at the moment such studies afford our best hope of making some sort of systematic sense of some part of p-primary homotopy theory.

In my original lecture, I finished with a section designed to entertain the experts by telling them something they had not previously heard. This concerned a problem about operations on the K-theory of torsion-free spaces, which on the face of it does not yield to the method of the "universal example", but can be made to do so by dualising. Apart from tending to show the continued liveliness of the basic ideas, this section did not contribute to the survey implied by my title; and since it is to be published elsewhere, I omit it here.

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References

[1] H. Cartan. Notions d'algèbre différentielle; application aux groupes de Lie et aux variétés où opère un groupe de Lie.
Colloque de topologie, Bruxelles 1950, pp15-27.

[2] H. Cartan. La transgression dans un groupe de Lie et dans un espace fibré principal.
Colloque de topologie, Bruxelles 1950, pp57-71.

[3] H. Cartan, Extension du théorème des "chaînes de syzygies".
Univ.Roma. Ist Naz.Alta Mat.Rend.Mat.e Appl. 11 (1952) 156-166.

[4] H. Cartan and J-P. Serre. Espaces fibrés et groupes d'homotopie. I. Constructions générales.
C.R. Acad.Sci. Paris 234 (1952) 288-290.

[5] H. Cartan and J-P. Serre. Espaces fibrés et groupes d'homotopie. II. Applications.
C.R.Acad.Sci. Paris 234 (1952) 393-395.

[6] H. Cartan. Sur les groupes d'Eilenberg-MacLane. I. Méthode des constructions.
Proc.Nat.Acad. USA 40 (1954) 467-471.

[7] H. Cartan. Sur les groupes d'Eilenberg-MacLane II.
Proc.Nat.Acad.Sci. USA 40 (1954) 704-707.

[8] H. Cartan. Sur l'itération des opérations de Steenrod.
Comment.Math.Helv. 29 (1955) 40-58.

[9] H. Cartan and S. Eilenberg. Homological Algebra.
Princeton U.P. 1956.

[10] H. Cartan. Algèbres d'Eilenberg-MacLane et homotopie.
Séminaire Cartan 7, 1954/55, Secrétariat Mathématique 1955.

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