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Structural Stability of Smooth Contracting Endomorphisms on Compact Manifolds

by

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§1. Introduction.

Many people including S. Smale [8] have been interested in the problem of finding structurally stable maps and classifying them. M. Shub in [7] studied expanding maps and Z. Nitecki in [6] increased the set to nonsingular endomorphisms. A singularity is a point where the derivative is not an isomorphism. Other mathematicians such as H. Whitney [9, 10, 11], J. Mather [5], R. Thorn and H. I. Levine [4] have studied maps between two manifolds which did have singularities and looked at the stability of such maps. In this paper we will use the structural stability of Smale because we are looking at maps from one manifold to itself. We will allow singularities, in fact, there are always singularities for contractions on a compact manifold.

In the paper M will always be a compact, C^∞, connected manifold without boundary and d will be a fixed metric on M. An endomorphism $f: M \to M$ is a contraction if for some λ, $0 < \lambda < 1$, $d(f(x),f(y)) \leq \lambda d(x,y)$ for all $x,y \in M$. By the compactness of M we see that the set of C^r contractions is an open subset of $C^r(M,M)$, the space of C^r maps from M to M with the C^r topology.

The endomorphism f is said to be topologically conjugate to another endomorphism g is there exists a homeomorphism h of M such that $h \cdot f = g \cdot h$. If f is in $C^r(M,M)$ then it is called C^r-structurally stable if there is a neighborhood N of f such that each g in N is topologically conjugate to f.

In [2] L. Block and I studied contracting endomorphisms on the circle and showed that the subset of all C^2-structurally stable contractions was open and dense in the C^2 topology. We also gave necessary and sufficient conditions for a C^2 contraction to be C^2 structurally stable. The major purpose of the present work is to extend those results to two dimensional manifolds.

Theorem 1. <u>The set of C^r-structurally stable contractions on any compact, connected, two dimensional, C^∞ manifold</u> M <u>without boundary is an open dense subset of all</u> C^r <u>contractions in the</u> C^r <u>topology for</u> $r \geq 12$.

The reason for taking $r \geq 12$ is found in the work of H. Whitney [9, 10, 11] who showed that for $r \geq 12$ the set of maps W is $C^r(M,M)$ which satisfy the following properties is open and dense in $C^r(M,M)$:

A. At each point x and f(x) there are coordinate charts such that f has one of the following normal forms:

1. regular $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x \\ y \end{pmatrix}$

2. fold $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x^2 \\ y \end{pmatrix}$

3. cusp $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} xy - x^3 \\ y \end{pmatrix}$

B. The images of folds intersect only pair-wise and transversally, whereas images of folds and cusps do not intersect.

H. Whitney also showed that f was in W if and only if given a neighborhood U of the identity in $C^0(M,M)$ there is a neighborhood V of f in $C^r(M,M)$ such that if $g \in V$ then there are two homeomorphisms h_1 and h_2 in U such that $f \cdot h_1 = h_2 \cdot g$.

We will call the maps in W Whitney maps.

J. Mather [5] extended these results to arbitrary dimensional manifolds by showing that there is an open dense set A of $C^\infty(M,M)$ such that if U is a neighborhood of the identity in $C^0(M,M)$ and $f \in A$ then there is a neighborhood U of f in $C^\infty(M,M)$ such that if $g \in V$ then there are two homeomorphisms h_1 and h_2 in U such that $f \cdot h_1 = h_2 \cdot g$. He calls such maps topologically stable.

Using Mather's results we will show the following:

<u>Theorem 5</u>. <u>On every</u> n-<u>dimensional compact</u> C^∞ <u>manifold</u> M <u>without boundary there is a</u> C^∞-<u>structurally stable contraction</u>.

Let us establish the following notation before we describe the C^r-structurally stable contracting endomorphisms on two dimensional manifolds M. We will use Σ_f for the set of singularities of an endomorphism f and x_f for the unique fixed point if f is a contraction. Two distinct points, $x,y \in M$ are said to be coincident under f if there exist non-negative integers, i and j, such that $f^i(x) = f^j(y)$.

Let K be the subset of all Whitney maps f which are contracting endomorphisms and which satisfy the following conditions:

1. The unique fixed point x_f of f is regular and is not coincident with any singularity.

2. A cusp point is not coincident with any other singularity.

3. For any set of three singularities, there is at most one subset of two elements which are coincident.

4. If $i \leq j$ and they are the smallest integers under which x and y, two singularities, are coincident; then

$$Df_x^i(T_x\Sigma_f) \oplus Df_y^j(T_y\Sigma_f) = T_{f^i(x)}M.$$

If $i = 0$ one has the added property that

$$Df_y^j(T_y\Sigma_f) \oplus \ker Df_x = T_xM.$$

From the possible forms of singularities it is clear that Σ_f is a one-dimensional manifold so that $T_y\Sigma_f$ is defined as its tangent space at y.

Theorem 2. K is an open dense subset of the C^r contractions on M.

Theorem 3. K is the set of all C^r-structurally stable contractions on M.

In the proof of the last theorem, one constructs a stratification S of M by using the singularities and distinguishing between cusps and folds. One then adds a finite number of images of the singularities and finally all the inverse images. These stratifications give information about the topological conjugacy classes.

Theorem 4. If $f,g \in K$ are topologically conjugate, then the conjugating homeomorphism h is a strata preserving map between $S(f)$ and $S(g)$.

§II. Transversality Results

For definitions and theorems covering transversality theory see Abraham and Robbin [1]. In the notation of Levine [4], the one jet $J^1(M,M)$ can be divided into three regular submanifolds S_0, S_1 and S_2 which correspond to jets having rank two, one and zero, respectively. Every Whitney endomorphism f has the property that its 1-extension $J^1(f): M \to J^1(M,M)$ is transverse to S_1. Since $J^1(f)$ is basically the derivative of f, it is C^{r-1} and $(J^1(f))^{-1}$ (S_1) is a C^{r-1} submanifold of M. Note that $\Sigma_f = (J^1(f))^{-1}(S_1)$, hence the singularity set for any f in W is a C^{r-1} submanifold.

This is also the setting for the transversal isotopy theorem (TIT) see [1]. This theorem says that given a neighborhood N of the inclusion map I in $C^{r-1}(\Sigma_f,M)$, there is a neighborhood A of f such that, if $g \in A$ there is an $h \in N$ sending Σ_f to Σ_g. In fact, h is a section over Σ_f in a total tubular neighborhood of Σ_f whose image is C^{r-2} flow isotopic to Σ_f.

One should be aware of the following two theorems which will be used many times in the lemmas of this section:

[1, pp. 46-47] Openness of Transversal Intersection (OTI): Let A, X, and Y be C^1 manifolds with X finite dimensional, $W \subset Y$ is a closed C^1 submanifold, $K \subset X$ a compact subset of X, and $\rho: A \to C^1(X,Y)$ a C^1 representation. Then the subset $A_{KW} \subset A$ defined by

$$A_{KW} = \{a \in A: \rho_a \pitchfork_x W \quad \text{for} \quad x \in K\}$$

is open.

[1, pp. 47-50] Transversal Density Theorem (TDT): Let A, X, Y be C^r manifolds, $\rho: A \to C^r(X,Y)$ a C^r representation, $W \subset Y$ a submanifold and $ev_\rho: A \times X \to Y$ the evaluation map. Define $A_W \subset A$ by

$$A_W = \{a \in A: \rho_a \pitchfork W\}.$$

Assume that

1. X has finite dimension n and W has finite codimension q in Y.

2. A and X are second countable.

3. $r > \max\{0, n-q\}$.

4. $ev_\rho \pitchfork W$.

Then A_W is residual (and hence dense) in A.

These basic transversality theorems will be used to prove Lemmas 1-5: Let $\mathcal{W}$ be the set of Whitney contracting endomorphisms.

Lemma 1: Let $K_1 = \{f \in \mathcal{W}$: The unique fixed point of f, x_f, is regular and coincident with no singularities$\}$, then K_1 is open and dense in $\mathcal{W}$.

Proof: (Openness) Let $f \in K_1$. Since x_f is a regular point and f a contraction, there is a compact neighborhood U of x_f on which f is a diffeomorphism and $f(U) \subset \operatorname{int} U$. Since $f(U)$ is a finite distance from ∂U and U is a finite distance from Σ_f; there is a neighborhood N_1 of f in $\mathcal{W}$ such that if $g \in N_1$, then g is a diffeomorphism on U and $g(U) \subset \operatorname{int} U$.

There is a positive integer n such that $f^n(M) \subset \text{int } U$, since f is a contraction. By the noncoincidence between x_f and singularities, there exists an $\varepsilon > 0$ such that the distance between x_f and $f^n(\Sigma_f)$ is greater than ε. There is a neighborhood N_2 of f in $\mathcal{W}$ such that, if $g \in N_2$ then $d(x_g, x_f) < \frac{\varepsilon}{2}$ and the distance between $g^n(\Sigma_g)$ and x_f is more than $\frac{\varepsilon}{2}$. Also, $g^n(M) \subset \text{int } U$.

To see that $N_1 \cap N_2$ is a subset of $\mathcal{K}_1$, note that if $g \in N_1 \cap N_2$; then $g^n(\Sigma_g) \subset \text{int } U$ and does not contain the fixed point x_g. Also g maps U diffeomorphically into int U. Thus no higher power of g can send an element of $g^n(\Sigma_g)$ to x_g, and there are no coincidences between x_g and elements of Σ_g.

(Density) If there is coincidence between x_f and some x in Σ_f, then the smallest integers for which this happens are of the form 0,j with j > 0. The proof will be by induction on j. Let $C_o = \{f \in \mathcal{W}: x_f \text{ is regular}\}$; and for each j > 0, $C_j = \{f \in \mathcal{W}: x_f$ is regular and there are no coincidences between x_f and points of Σ_f with integers 0 and j$\}$. Note that the proof of openness also shows that each of these sets is open.

Claim 1: C_o is dense in $\mathcal{W}$.

Proof: Let $x_f \in \Sigma_f$ and take a neighborhood U of x_f on which f has a normal form. If x_f is a cusp, let h be a C^∞ diffeomorphism which moves x_f, and is the identity outside of U. Note that hf has the same singularity set as f and each singularity retains its type. By taking h close to the identity, hf is in $\mathcal{W}$, and the fixed point of hf is still in U but is not x_f. Hence x_{hf} is not a cusp. The remaining possibility is that the fixed point is a fold. In this case if $T_{x_f}\Sigma_f \oplus Df_{x_f}(T\Sigma_f) \neq T_{x_f}M$, take a C^∞ vectorfield V, which is $(y,-x)$ in local coordinates at x_f. Let h_ε be its time ε diffeomorphism. By taking ε small, $h_\varepsilon f$ is a small C^r perturbation of f and has the same fixed point and singularity set. It does, however, rotate the image of Σ_f. Thus $T_{x_f}\Sigma_f \oplus D(h_\varepsilon f)_{x_f}(T\Sigma_f) = T_{x_f}M$. So suppose f satisfies this condition. Now shrink U, if necessary, to the extent that $f(U \cap \Sigma_f) \cap \Sigma_f = x_f$. Let $y \in U \cap \Sigma_f$; $y \neq x_f$. Look at the arc in $U \cap \Sigma_f$ connecting x_f and y and the arc in $U \cap f(\Sigma_f)$ connecting x_f and $f(y)$. These two arcs form some angle at x_f. Let v be the unit vector bisecting the angle. Let V be the C^∞ vectorfield that is constant at v in some neighborhood of x_f and zero outside some larger neighborhood. Let h_ε be the time ε diffeomorphism for V. If ε is small enough, there is a neighborhood U_1 of x_f such that no point in $U_1 \cap \Sigma_f$ is fixed under $h_\varepsilon \cdot f$. But since this is a small C^r perturbation, the fixed point for $h_\varepsilon \cdot f$ is in U_1. Thus the fixed point is regular and C_o is dense in $\mathcal{W}$.

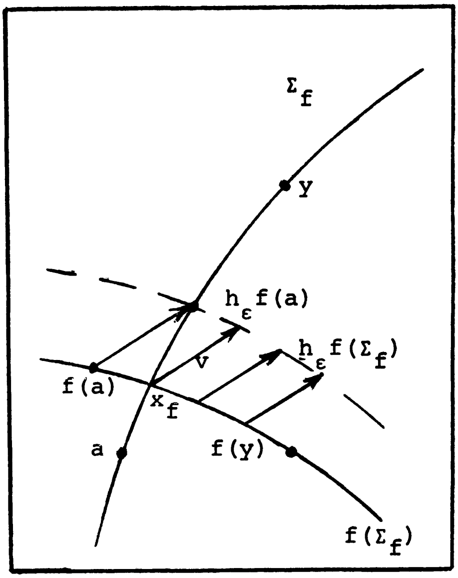

Figure 1

Claim 2: C_1 is dense in C_o.

Proof: Note that $f^{-1}(x_f) \cap \Sigma_f$ is a finite set because f is locally one to one on Σ_f. Thus only a finite number of perturbations will be needed. Suppose $f(x) = x_f$ with $x \in \Sigma_f$. The perturbation will consist of changing f on a compact neighborhood N of x which is contained in an open neighborhood where f has a normal form. From the normal forms, it is clear that there is only one point in N that is coincident with x_f. Thus $f(\partial N)$ is a finite distance away from x_f. Let V be a vectorfield that is zero on a neighborhood of $f(\partial N)$ and h_ε its time ε diffeomorphism. Then changing f to $h_\varepsilon \cdot f$ on N and keeping f on the complement of N gives a C^r perturbation of f. If x is a cusp, take a vectorfield V so that $V(x_f)$ is of unit length and in the opposite direction of the cusp. If x is a fold, take V so that $V(x_f)$ is of unit length and perpendicular to $f(\Sigma_f \cap N)$. Thus, in either case, the perturbed function has one less coincidence. Hence C_1 is dense in C_o.

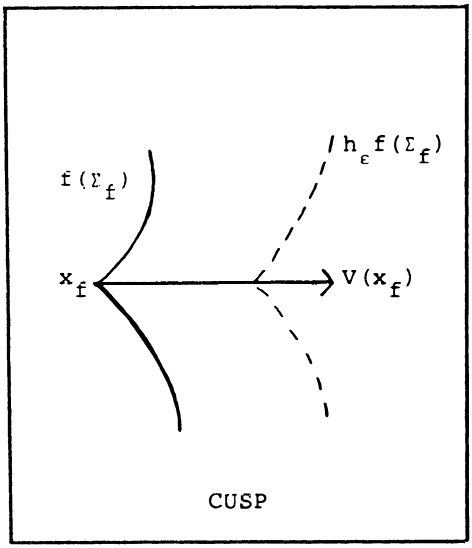

Figure 2

One is now ready for the induction step. Assume C_j is open and dense in $\mathcal{W}$. Thus if $x \in \Sigma_f$ is such that $f^i(x) \in \Sigma_f$ for $1 \leq i \leq j+1$, then $f^{j+1}(x)$ is at least as far from x_f as $f^j(\Sigma_f)$ is.

Hence there is an open set N about x, such that $f^{j+1}(N)$ is at least half as far from x_f as $f^j(\Sigma_f)$ is. Let A be the union of all these open sets. Then $\Sigma_f - A$ is a compact set on which f^{j+1} is locally one to one. Hence $f^{-(j+1)}(x_f) \cap \Sigma_f$ is a finite set. The perturbations of f are like those in the proof that C_1 is dense in C_0 with the role of x_f played by f(x) where $f^{j+1}(x) = x_f$. This is possible because $f^j(f(x))$ is equal to x_f after such perturbations and the orbit of f(x) consists entirely of regular points. Thus each C_j is open and dense in $\mathcal{W}$. Since K_1 is the intersection of a countable number of open dense sets, it is dense. Q.E.D

One should note that this lemma only uses the C^0 stability of the singularity set. The fact that it is C^1 stable will be used in the next lemma.

<u>Lemma 2</u>: <u>Let</u> $K_2 = \{f \in K_1$: <u>for any</u> x <u>and</u> $y \in \Sigma_f$, <u>a coincidence between them with integers</u> 0 <u>and</u> j <u>implies that neither</u> x <u>nor</u> y <u>is a cusp and</u> $Df_y^j(T_y\Sigma_f) \oplus T_x\Sigma_f = T_xM = Df_y^j(T_y\Sigma_f) \oplus \ker D_xf\}$. <u>Then</u> K_2 <u>is open and dense in</u> K_1.

<u>Proof</u>: (Openness) Let $f \in K_2$, then there is a compact neighborhood U of x_f and an integer n such that f is diffeomorphism on U, $f(U) \subset \text{int } U$, and $f^n(M) \subset \text{int } U$. Thus the points of coincidence that we are interested in can only happen with $j < n$. Note that f has only a finite number of cusps and their first n images are disjoint from Σ_f. Also note that $f^j(\Sigma_f)$ is compact and a finite distance from the set of cusps. From Whitney's stability theorem

(see the introduction) it is clear that if g is close to f then g will also have these properties.

Now $f^i|\Sigma_f \in C^1(\Sigma_f, M)$. Since $f^i|\Sigma_f$ is transverse to I on Σ_f, $f^i|\Sigma_f \times I$ is transverse to the diagonal Δ in $M \times M$. One can now apply the Openness of Transversal Intersection Theorem, because Σ_f is compact and Δ is closed. This theorem says that there are neighborhoods N_1 of f^i in $C^1(\Sigma_f, M)$ and N_2 of I in $C^1(\Sigma_f, M)$ such that if $\phi \times \psi \in N_1 \times N_2$, then ϕ is transverse to ψ. If g is a small enough perturbation of f, and h is the diffeomorphism close to I such that $h(\Sigma_f) = \Sigma_g$ given by TIT; then $g^i h \in N_1$ and $h \in N_2$. Hence $g^i h$ is transverse to h. This means that $g^i|\Sigma_g$ is transverse to Σ_g.

Since Σ_f is one dimensional, $f^i(\Sigma_f) \cap \Sigma_f$ is zero dimensional and, in fact, finite, since Σ_f is compact. If g is a perturbation of f; then, as noted, $g^i(\Sigma_g)$ is transverse to Σ_g and the coincidence points can be made arbitrarily close to those for f. So suppose $x, y \in \Sigma_g$ and $g^i(y) = x$. Then by continuity of the eigen directions, one obtains that $Dg^i_y(T_y\Sigma_g)$ and $\ker D_x g$ span $T_x M$. Hence K_2 is indeed open in K_1.

(Density) Let $C_j = \{f \in K_1$: for any x and $y \in \Sigma_f$; a coincidence between them with integers 0 and i, $i \leq j$, implies that neither x nor y is a cusp and $Df^i_y(T_y\Sigma_f)$ together with either $T_x\Sigma_f$ or $\ker D_x f$ span $T_x M\}$. Note that the openness of C_j in K_1 is proven above. Also note that $C_o = K_1$, so it is dense in K_1. One now proceeds by induction on j to show that each C_j is dense in K_1.

Suppose C_j is dense in K_1 and let $f \in C_j$. Let A be an open neighborhood of I in $\mathrm{Diff}^r(M)$ such that, for $h \in A$, $hf \in C_j$. Now consider the representation $\rho: A \to C^2(\Sigma_f, M)$ given by $\rho(h) = (hf)^{j+1}$. This representation is at least C^2, because the evaluation $ev_\rho: A \times \Sigma_f \to M$ can be thought of as first sending h to $\underbrace{(hf,\ldots,hf)}_{j+1}$ and then evaluating it j+1 times. Composition on the left is smooth and evaluation is C^2. Since the first three conditions stated in the Transversal Density Theorem are clearly satisfied, the only one of interest is the last. To check the last one, let $h \in A$ and $y \in \Sigma_f$. If $(hf)^{j+1}(y) \notin \Sigma_f$, then ev_ρ is transverse to Σ_f at (h,y). So suppose $(hf)^{j+1}(y) = x \in \Sigma_f$. If there is an integer i between 0 and j+1 such that $(hf)^i(y) = z \in \Sigma_f$, then the inductive hypothesis says that $D(hf)^i_y(T_y\Sigma_f)$ together with either $T_z\Sigma_{hf}$ or $\ker D_z hf$ span T_zM. Note that $\Sigma_{hf} = \Sigma_f$ and $\ker D_z hf = \ker D_z f$. Thus $D(hf)^{j+1}_y(T_y\Sigma_f) = D(hf)^{j+1-i}(T_z\Sigma_f)$, and by the inductive hypothesis $D(hf)^{j+1-i}(T_z\Sigma_f)$ together with either $T_x\Sigma_f$ or $\ker D_x f$ span T_xM. So we can suppose that the orbit of y under hf is regular between y and x. Take a smooth vectorfield V that is zero outside a small neighborhood of hf(y) and constant on some smaller neighborhood. Look at the curve of diffeomorphisms $\phi_t h$ through h where ϕ_t is the flow of V. Now ev_ρ acting on the curve $(\phi_t h, y)$ gives a curve in M at x which corresponds to some element in T_xM. Since the orbit of y is made up of regular points, any vector in T_xM can be realized by an appropriate choice of V. Hence ev_ρ is transverse to Σ_f at (h,y) and therefore it is transverse everywhere. Thus there is an open dense set of diffeomorphisms in A, such that if h is one of these, then $(hf)^{j+1}$ is transverse to Σ_f. In particular, there is one arbitrarily close to the identity. Since $\Sigma_{hf} = \Sigma_f$, we have been successful in perturbing f to an endomorphism that satisfies the first transversality condition.

Suppose $f \in C_j$ such that a coincidence between $x, y \in \Sigma_f$ with integers 0 and j+1 implies that $Df_y^{j+1}(T_y\Sigma_f) \oplus T_x\Sigma_f = T_xM$. Note that such maps are open in C_j. If $Df_y^{j+1}(T_y\Sigma_f) \oplus \ker D_xf \neq T_xM$, then the inductive hypothesis says that the orbit of y between y and x consists of regular points. Let V be the vectorfield which is zero outside a small neighborhood of f(y) and is $(x,y) \to (y,-x)$ in local coordinates at f(y). $D(\phi_t f)_y^{j+1}(T_y\Sigma_f)$ is basically rotated from $Df_y^{j+1}(T_y\Sigma_f)$ and $\ker D_x\phi_t f = \ker D_xf$. Thus taking t small gives a perturbation of f which satisfies the spanning condition. Since there are only a finite number of intersections of this form and the spanning condition is open, we can do a small perturbation and obtain the desired property.

Let $f \in C_j$ and satisfy the two spanning conditions. If y is a cusp then $Df_y^{j+1}(T_y\Sigma_f) = 0$, so y must not be coincident with any singularity. To remove intersections between $f^{j+1}(y)$, $y \in \Sigma_f$, and a cusp, use perturbations similar to those used in removing coincidences between singularities and the fixed point. Thus one sees that each C_j is open and dense in K_1. Since $K_2 = \bigcap_{j=0}^{\infty} C_j$, K_2 is dense in K_1. Q.E.D

It should be noted that if $f \in K_2$, then f^i, for any i, is locally one to one on Σ_f; and $f^i|\{\text{folds}\}$ is an immersion.

<u>Lemma 3</u>: <u>Let</u> $K_3 = \{f \in K_2$: <u>If</u> $x,y,z \in \Sigma_f$, <u>then there do not exist integers</u> i,j <u>such that</u> $x = f^i(y) = f^j(z)\}$. K_3 <u>is open and dense in</u> K_2.

<u>Proof</u>: Let $K_{ij} = \{f \in K_2$: If $x,y,z \in \Sigma_f$, then $\{x\} \cap \{f^i(y)\} \cap \{f^j(z)\} = \phi\}$. Order the ordered pairs of non-negative integers (i,j), $i \leq j$, by $(a,b) < (c,d)$ if $b < d$ or $b = d$ and $a < c$. One now proceeds by induction to show that each K_{ij} is open and dense in K_2. Note that if $i = 0$, then $K_{ij} = K_2$.

(Openness) Consider the representation $\rho: C^2(M,M) \times C^2(M,M) \times C^2(M,M) \to C^2(\Sigma_f \times \Sigma_f \times \Sigma_f - U_{ij}, M \times M \times M)$ given by $(g_1,g_2,g_3) \to (g_1,g_2,g_3)\Big|_{\Sigma_f\times\Sigma_f\times\Sigma_f - U_{ij}}$, where $f \in K_{ij}$ and $U_{ij} = \phi$ if $i \neq j$. Here $U_{ii} = \Sigma_f \times V_{ii}$ where V_{ii} is a small neighborhood of the diagonal in $\Sigma_f \times \Sigma_f$. Note if V_{ii} is small enough, there is a neighborhood N of f such that if $g \in N$ and $h: \Sigma_f \to \Sigma_g$, the map given by TIT, then $g^i h \times g^i h: V_{ii} \to M \times M$, such that if $(x,y) \in V_{ii}$, then $g^i h(x) = g^i h(y) \Rightarrow x = y$. This follows from the local stability of Whitney maps and Lemma 2. Since $f \in K_{ij}$, $\rho(id,f^i,f^j) \overline{\pitchfork} \Delta_M$ where $\Delta_M = \{(x,x,x): x \in M\}$. Now by the openness of transversal intersection, there is a neighborhood $N_1 \times N_2 \times N_3$ of (id,f^i,f^j) in $C^2(M,M) \times C^2(M,M) \times C^2(M,M)$ such that if $(g_1,g_2,g_3) \in N_1 \times N_2 \times N_3$, then (g_1,g_2,g_3) also misses Δ_M on $\Sigma_f \times \Sigma_f \times \Sigma_f - U_{ij}$. Now if g is close enough to f and $h: \Sigma_f \to \Sigma_g$ is given by TIT, then $(h,g^i h,g^j h) \in N_1 \times N_2 \times N_3$ and hence $g \in K_{ij}$. Thus all the K_{ij} are open in K_2. Now since $f \in K_2$, there is a neighborhood N of f in K_2 and an integer n such that if g is in N, then $g^n(M)$ is contained in a neighborhood of x_g that gets mapped diffeomorphically into itself. Thus if $j \geq n$, there are clearly no coincidences of the type we are discussing; and K_{ij} contains N. Thus K_3 is open in K_2.

(Density) For $i = j$, by the inductive hypothesis the orbit of x between x and $f^i(x)$ consists of regular points if $x \in \Sigma_f$ and $f^i(x) \in \Sigma_f$. Since $f^i\colon \Sigma_f \to M$ is transverse to Σ_f, the number of such x_i is finite, and the TIT says that for g close to f there are corresponding points y_i close to the x_i. Suppose there are two points, say x_1 and x_2, such that $f^i(x_1) = f^i(x_2)$. Change f in a small neighborhood U of x_1 by composing f with ϕ_t. Here ϕ_t is a small time diffeomorphism coming from a vectorfield V that is zero outside a small neighborhood of $f(x_1)$. At $f(x_1)$, V should be in the direction which corresponds to the tangent space to Σ_f at $f^i(x_1)$. In other words, $Df_{f(x_1)}^{i-1}(V) \in T_{f^i(x_1)}\Sigma_f$. Note that $(f\phi_t)^i(\Sigma_f \cap U) \cap f^i(x_2) = \phi$. Thus the point in $\Sigma_f \cap U$ that goes to Σ_f under the perturbation does not go to $f^i(x_2)$. In this way, the number of such intersections can be reduced to zero.

Suppose $i \neq j$. Note that $f^{j-i}(\Sigma_f) \pitchfork \Sigma_f$, $f^i(\Sigma_f) \pitchfork \Sigma_f$, and $f^j(\Sigma_f) \pitchfork \Sigma_f$ because $f \in K_2$. Let $\{x_k\}$, $\{y_\ell\}$, and $\{z_m\}$ be the finite set of points in Σ_f that are mapped to Σ_f under f^{j-i}, f^i, and f^j respectively. By the inductive hypothesis, $\{f^{j-i}(x_k)\}$ and $\{f^i(y_\ell)\}$ are in one to one correspondence with $\{x_k\}$ and $\{y_\ell\}$ respectively. Suppose there is an x_s such that $f^{j-i}(x_s) = y_\ell$. Note that the orbit of y_ℓ is regular between y_ℓ and $f^i(y_\ell)$.

Thus one can do a perturbation of f in a neighborhood of y_ℓ so that $g^i(y_\ell) \notin \Sigma_f = \Sigma_g$. Since g is a small perturbation of f, the sets $\{x_k'\}$, $\{y_\ell'\}$, $\{z_m'\}$, and $\{g^{j-i}(x_k')\}$, $\{g^i(y_\ell')\}$, $\{g^j(z_m')\}$ are arbitrarily close to the corresponding sets for f. Thus if $f^{j-i}(x_k) \notin \{y_\ell\}$, then $g^{j-i}(x_k') \notin \{y_\ell'\}$; and g has at most one less point, namely, $x_s' = x_s$, such that $g^{j-i}(x_s') \notin \{y_\ell'\}$. In this way one can reduce the number of such coincidences to zero. In neighborhoods of y_ℓ where $f^i(y_\ell) = f^j(z_m)$, one can do similar perturbations so that $\Sigma_g = \Sigma_f$, $g^j(z_m) = f^j(z_m)$ but no singularity in the neighborhood goes to $g^j(z_m)$. Thus K_{ij} is dense in K_2. Since $K_3 = \bigcap K_{ij}$, the Baire Category Theorem says that K_3 is dense in K_2. Q.E.D.

Lemma 4: Let $K_4 = \{f \in K_3$: if x and y, two singularities, are coincident and if i and j are the smallest integers under which they collide, then $Df_x^i(T_x\Sigma_f) \oplus Df_y^j(T_y\Sigma_f) = T_{f^i(x)}M\}$. Then K_4 is open and dense in K_3 and hence in K_2.

Proof: Let $K_{ij} = \{f \in K_3$: if x and y are two singularities and $f^i(x) = f^j(y)$, then $Df_x^{i'}(T_x\Sigma_f) \oplus Df_y^{j'}(T_y\Sigma_f) = T_{f^{i'}(x)}M$ where i' and j' are the smallest integers under which x and y are coincident$\}$. One can also take $i \leq j$ and order the ordered pairs by $(a,b) < (c,d)$ if $b < d$ or $b = d$ and $a < c$. One now proceeds by induction to show that each K_{ij} is open and dense in K_2.

The first step is done because $K_{00} = K_{01} = K_3$. So assume all $K_{i'j'}$ are open and dense in K_2 for $(i',j') < (i,j)$. The inductive step breaks into two cases: first $i = j$, and second $i < j$.

Case 1: Let $i = j$. Since the Whitney maps satisfy this transversality condition with $i = j = 1$ (see introduction), K_{11} is open and dense in K_2. If $i > 1$, let $f \in K_3 \cap \bigcap_{(i',j')<(i,i)} K_{i'j'}$ and A be an open neighborhood of the identity in $\mathrm{Diff}^r(M)$, such that if $h \in A$ then $hf \in K_3 \cap \bigcap_{(i',j')<(i,i)} K_{i'j'}$. Consider the representation $\rho: A \to C^2(\Sigma_f \times \Sigma_f - \Delta, M \times M)$ given by $\rho(h) = ((hf)^i, (hf)^i) | \Sigma_f \times \Sigma_f - \Delta$ where Δ is the diagonal. The interesting question to check before applying the TDT is that $ev_\rho \pitchfork \Delta_M$, where Δ_M is the diagonal in $M \times M$. Let $(x,y) \in \Sigma_f \times \Sigma_f - \Delta$ such that $(hf)^i(x) = (hf)^i(y)$. Let i' be the smallest integer such that $(hf)^{i'}(x) = (hf)^{i'}(y)$. If $i' \neq i$, then $D(hf)^{i'}_x(T_x\Sigma_f) \oplus D(hf)^{i'}_y(T_y\Sigma_f) = T_{f^{i'}(x)}M$ since $hf \in K_{i'i'}$. Note also that $hf \in K_3$, and thus $f^{i'}(x)$ is regular and its orbit is regular. This tells us that the transversality property is passed along to give $D(hf)^i_x(T_x\Sigma_f) \oplus D(hf)^i_y(T_y\Sigma_f) = T_{f^i(x)}M$. Hence $ev_\rho \pitchfork \Delta_M$ at this point. So let us suppose $i' = i$. If the orbit of y does not consist of regular points, let z be the singularity. Note that there can be at most one singularity, and it must be less than i iterates from y. Let us say that $(hf)^k(y) = z$. Then $D(hf)^i_y(T_y\Sigma_f) = D(hf)^{i-k}_z(T_z\Sigma_f)$ and $D(hf)^{i-k}_z(T_z\Sigma_f) \oplus D(hf)^i_x(T_x\Sigma_f) = T_{(hf)^i(x)}M$, since $hf \in K_{i-k,i}$. So again $ev_\rho \pitchfork \Delta_M$. Now suppose the orbit of y is regular. Look at the curve of functions $\phi_t h$ through h, where ϕ_t is the flow of a smooth vectorfield that is zero outside a small neighborhood of $hf(y)$. $T(ev_\rho)_{(h,x,y)}$ sends the curve over to a vector of the form $(V,0)$ and since $D(hf)_{hf(y)}{}^{i-1}$ is an isomorphism, one gets every such vector. Thus $ev_\rho \pitchfork \Delta_M$, and there is an h arbitrarily close to the identity that $(hf)^i \times (hf)^i|_{\Sigma_f\times\Sigma_f - \Delta}$ is transverse to Δ_M. This says that K_{ii} is dense in K_2 and in every $K_{i'j'}$ where $(i',j') < (i,i)$.

To see the openness of K_{ii} in K_2, consider the representation ρ: $C^2(M,M) \times C^2(M,M) \to C^2(\Sigma_f \times \Sigma_f - V_{ii}, M \times M)$, given by $(g_1,g_2) \to (g_1,g_2)|_{\Sigma_f\times\Sigma_f - V_{ii}}$; where V_{ii} is as defined in the proof of Lemma 3 and f is in $K_{ii} \cap K_3 \cap \bigcap_{(i',j')<(i,i)} K_{i'j'}$. Now since $f \in K_{ii}$, $\rho(f^i,f^i) \pitchfork \Delta_M$ on $\Sigma_f \times \Sigma_f - V_{ii}$. By OTI, there is a neighborhood $N_1 \times N_2$ of (f^i,f^i) in $C^2(M,M) \times C^2(M,M)$ such that if $(g_1,g_2) \in N_1 \times N_2$, then $\rho(g_1,g_2) \pitchfork \Delta_M$ on $\Sigma_f \times \Sigma_f - V_{ii}$. If g is close enough to f and h: $\Sigma_f \to \Sigma_g$ is the map given by TIT, then $(g^ih,g^ih) \in N_1 \times N_2$. Also, if $(x,y) \in V_{ii}$ and $g^ih(x) = g^ih(y)$ then $x = y$. Thus K_{ii} is open in K_2.

<u>Case 2</u>: (Density) In this case $i < j$. The set of singular points $\{x_k\}$, such that $f^{j-i}(x_k) \in \Sigma_f$ is finite for $f \in K_3 \cap \bigcap_{(i',j')<(i,j)} K_{i'j'}$. In fact the orbit of such an x is regular except at $f^{j-i}(x)$. Note that $Df_x^{j-i}(T_x\Sigma_f) \oplus T_{f^{j-i}(x)}\Sigma_f = T_{f^{j-i}(x)}M$ and $Df_x^{j-i}(T_x\Sigma_f) \oplus \ker Df_{f^{j-i}(x)} = T_{f^{j-i}(x)}M$. Thus the normal form for the fold tells one that there are neighborhoods N_2 of x and N_1 of $f^{j-i}(x)$ such that, if $(a,b) \in N_1 \times N_2 \cap \Sigma_f \times \Sigma_f$ then $f^{j-i+1}(b) = f^1(a) \Rightarrow b = x$ and $a = f^{j-i}(x)$. Since the orbit of $f^{j-i}(x)$ is made up of regular points, one obtains that

$f^j(b) = f^i(a) \Rightarrow b = x$ and $a = f^{j-i}(x)$. In fact, the OTI tells us that if g is in a small neighborhood of f and $h\colon \Sigma_f \to \Sigma_g$ is given by TIT, then $g^j h(b) = g^i h(a) \Rightarrow h(b)$ is the unique point in $\Sigma_g \cap N_1$ such that $g^{j-i}(h(b)) \in \Sigma_g$. Find such neighborhoods in $M \times M$ for each x_k and let U be the union. Now consider the representation $\rho\colon A \to C^2(\Sigma_f \times \Sigma_f - U, M \times M)$ given by $h \to ((hf)^i, (hf)^j)|_{\Sigma_f \times \Sigma_f - U}$, where A, an open neighborhood of the identity in $\mathrm{Diff}^r(M)$, is such that if $h \in A$ then hf is close enough to f to satisfy the conditions in defining U and $hf \in K_3 \cap \bigcap_{(i',j')<(i,j)} K_{i'j'}$. The important condition to check before applying TDT is that $ev_\rho \pitchfork \Delta_M$. Suppose $(x,y) \in \Sigma_f \times \Sigma_f - \Delta$ such that $(hf)^i(x) = (hf)^j(y)$. If the orbit of x between x and $(hf)^i(x)$ contains a singularity z, then $D(hf)^i_x(T_x\Sigma_f) = D(hf)^k_z(T_z\Sigma_f)$ where $(hf)^{i-k}(x) = z$. By the inductive hypothesis $D(hf)^k_z(T_z\Sigma_f) \oplus D(hf)^j_y(T_y\Sigma_f) = T_{(hf)^j(y)}M$. Hence $ev_\rho \pitchfork \Delta_M$ at this point.

So suppose the orbit of x consists of regular points. If there is an integer ℓ between 0 and i such that $(hf)^{j-\ell}(y) = (hf)^{i-\ell}(x)$, then the inductive hypothesis says that $D(hf)^{j-\ell}_y(T_y\Sigma_f) \oplus D(hf)^{i-\ell}_x(T_x\Sigma_f) = T_{(hf)^{j-\ell}(y)}M$. But since the orbit of x consists of regular points, this is translated to $(hf)^j(y)$ and $ev_\rho \pitchfork \Delta_M$ at this point.

So suppose the smallest integers under which x and y are coincident are i and j. If the orbit of y contains a singularity, then one obtains $ev_\rho \pitchfork \Delta_M$ at this point just as when the orbit of x contained a singularity. Thus we can assume the orbits of x and y are both made up of regular points. Look at the curve $\phi_t h$ of diffeomorphisms through h where ϕ_t is the flow of a vector-field that is zero outside a small neighborhood of hf(y). $D(ev_\rho)_{(h,x,y)}$ sends this curve to a vector of the form (0,V). Since $(hf)^{j-1}$ is regular at hf(y), one can obtain all such vectors in this manner. Hence $ev_\rho \pitchfork \Delta_M$. Thus TDT says that there exists h arbitrarily close to the identity such that hf is in K_{ij} and that K_{ij} is dense in $K_3 \cap \bigcap_{(i',j')<(i,j)} K_{i'j'}$ and hence in K_2.

(Openness) To see the openness of K_{ij} consider the representation $\rho\colon C^2(M,M) \times C^2(M,M) \to C^2(\Sigma_f \times \Sigma_f - U, M \times M)$ given by $(g_1,g_2) \to (g_1,g_2)|_{\Sigma_f\times\Sigma_f - U}$ where $f \in K_3 \cap \bigcap_{(i',j')<(i,j)} K_{i'j'}$ and U is as above. Since $(f^i,f^j) \pitchfork \Delta_M$, the OTI says that there is a neighborhood $N_1 \times N_2$ of (f^i,f^j) such that if $(g_1,g_2) \in N_1 \times N_2$, then $(g_1,g_2) \pitchfork \Delta_M$. Now if g is close enough to f; then $(g^i h, g^j h) \in N_1 \times N_2$, where $h\colon \Sigma_f \to \Sigma_g$ is given by TIT. By the definition of U, the points $(x,y) \in U \cap \Sigma_f \times \Sigma_f$ such that $g^i h(x) = g^j h(y)$ are also coincident in the form $g^{j-i}h(y) = x$. Hence $g \in K_{ij}$, and K_{ij} is open in $K_3 \cap \bigcap_{(i',j')<(i,j)} K_{i'j'}$ and thus in K_2.

Now $K_4 = K_3 \cap \bigcap_{(i,j)} K_{ij}$. Thus K_4 is dense in K_2. To see that K_4 is open in K_2, let $f \in K_4$. There is an integer n such that $f^n(M)$ is in a neighborhood A of x_f on which f is a diffeomorphism. Then $f^n(M) - f^{n+1}(M) = F$ is a fundamental domain of x_f. There is an integer J such that $f^J(F)$ contains an iterate of every singularity. Thus this set expresses all of the different types of intersections between singularities. That is, any intersection in $f^{J+n+1}(M)$ of iterates of singularities is an iterate of such an intersection in $f^J(F)$, and if the one in $f^J(F)$ is transverse, the one in $f^{J+n+1}(M)$ is also transverse. Thus a neighborhood of f in K_4 is just a finite intersection of neighborhoods of f in K_{ij} with $j \leq J+n+1$. Hence K_4 is open and dense in K_2. Q.E.D.

It should be noted that the endomorphisms in K_4 have the property that if $x,y \in \Sigma_f$, $f^i(x) = f^j(y)$, and $x \neq f^{j-i}(y)$ then $Df^i_x(T_x\Sigma_f) \oplus Df^j_y(T_y\Sigma_f) = T_{f^i(x)}M$. This is because $f^{i'}(x)$ is regular and its orbit is also regular. i' and j' are the smallest integers under which x and y are coincident. This also shows that a cusp x is coincident with no other singularity, because $Df^i_x(T_x\Sigma_f) = 0$.

We are now ready to prove Lemma 5 which can be done with a finite number of perturbations.

<u>Lemma 5</u>: <u>Let</u> $K_5 = \{f \in K_4$: <u>for any set of three singularities there is at most one subset of two elements which are coincident</u>$\}$. K_5 <u>is open and dense in</u> K_4 <u>and hence in</u> K_2.

<u>Proof</u>: Order the set of triples $\{(i,j,k)\} \in \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} : 0 \leq i \leq j \leq k\}$ by $(a,b,c) < (a',b',c')$ if $c < c'$ or $c = c'$ and $b < b'$; or $c = c'$, $b = b'$, and $a < a'$. Let $K_{(i,j,k)} = \{f \in K_4$: if x, y and z are different singularities of f and $(i',j',k') \leq (i,j,k)$ then $\{f^{i'}(x)\} \cap \{f^{j'}(y)\} \cap \{f^{k'}(z)\} = \phi\}$. We will prove by induction that each $K_{(i,j,k)}$ is open and dense in $K_{(i',j',k')}$ if

$(i',j',k') < (i,j,k)$. Note that Lemma 3 shows that $K_{(0,j,k)}$ is open and dense in K_2, which shows that the first step is finished. So suppose (i,j,k) is an arbitrary triple. There are three sets $\{(a_\ell,b_\ell)\}$, $\{(c_\ell,d_\ell)\}$, and $\{(e_\ell,f_\ell)\}$ of points in $\Sigma_f \times \Sigma_f - \Delta$ which are the points of intersection between f^i and f^j, f^i and f^k, and f^j and f^k respectively. Lemma 4 says that each of these sets is finite and that the intersections are transverse, except when $f^{j-i}(b_\ell) = a_\ell$, $f^{k-i}(d_\ell) = c_\ell$, or $f^{k-j}(f_\ell) = e_\ell$. Thus a small perturbation will keep the number of such points the same and their position and k iterates arbitrarily close. We will do a

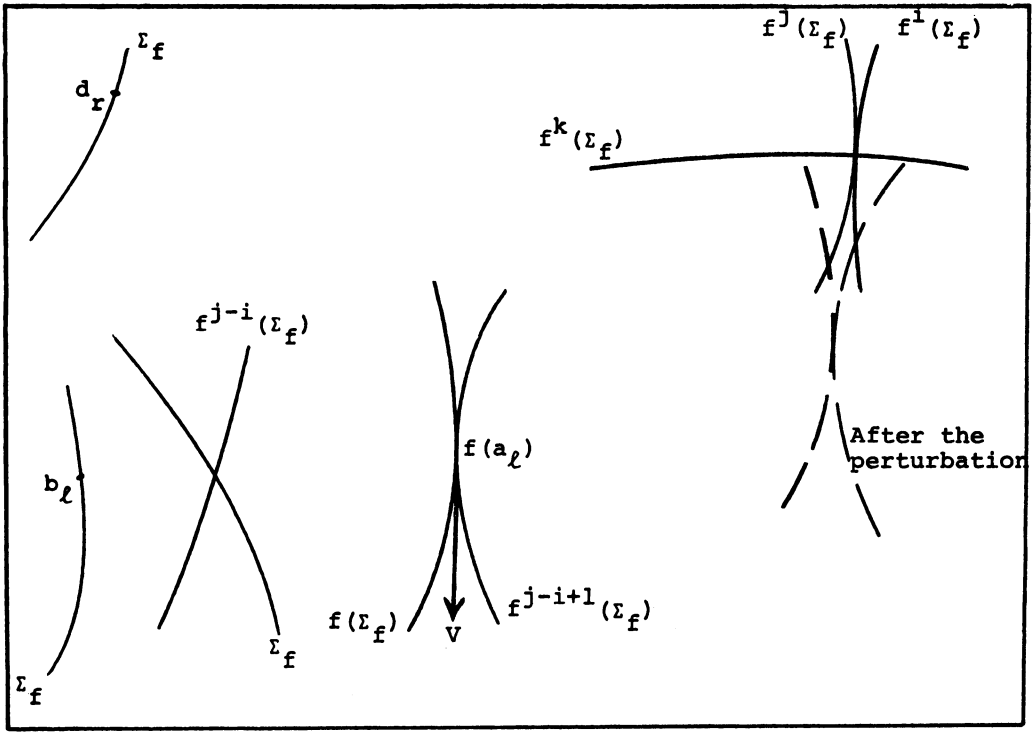

Figure 3

finite number of perturbations to obtain that no a_ℓ is a c_r. Suppose that some $a_\ell = c_r$. By the inductive hypothesis, either $f^{j-i}(b_\ell) \neq a_\ell$ or $f^{k-i}(d_r) \neq c_r$ and the orbit of a_ℓ is regular. Suppose $f^{k-i}(d_r) \neq c_r$. If $f^{j-i}(b_\ell) = a_\ell$, let V be the vectorfield that is zero outside of a small neighborhood of $f(a_\ell)$ and $V(f(a_\ell)) \in Df_{a_\ell}(T_{a_\ell}\Sigma_f)$. Let $h = \phi_\varepsilon$ for some small ε. Now consider the perturbation of f which is f outside of a small neighborhood of a_ℓ and hf on the neighborhood.

Note that the perturbed map has $f^j(b_\ell) = f^i(a_\ell)$, but $f^i(a_\ell) \notin f^k(\Sigma_f)$. There is, however, a singularity very close to a_ℓ that does go to $f^k(\Sigma_f)$. In this way, one can decrease the number of a_ℓ that equal c_r. Note that this perturbation also works in the case where $f^{k-i}(d_r) = c_r$. If $f^{j-i}(b_\ell) \neq a_\ell$ and $f^{k-i}(d_r) \neq c_r$, we use the same type of perturbation, except that $V(f(a_\ell))$ is perpendicular to $Df_{a_\ell}(T_{a_\ell}\Sigma_f)$. Under this perturbation $f^k(d_r) = f^j(b_\ell)$, but $f^i(a_\ell) \neq f^k(d_r)$ and neither does any point in a neighborhood of a_ℓ in Σ_f. In this case, we have also reduced the number of points where $a_\ell = c_r$. Thus in a finite number of steps, we change f so that it is in $K_{(i,j,k)}$. Hence $K_{(i,j,k)}$ is dense in all other $K_{(i',j',k')}$ with $(i',j',k') < (i,j,k)$ if $i \neq k$. If $i = k$ this type of perturbation can also be used to reduce the number of triple intersections to zero.

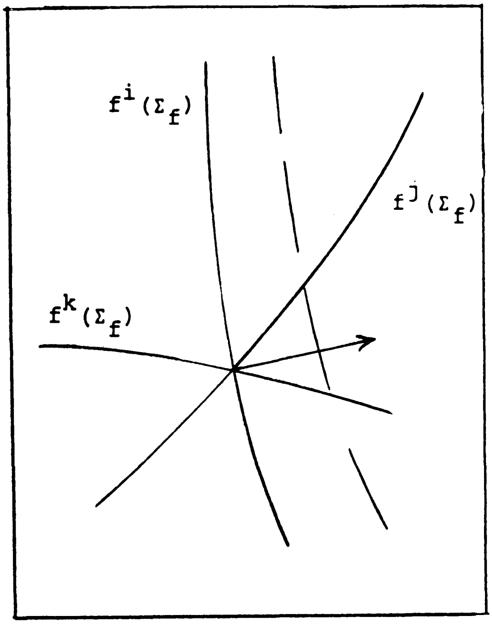

Figure 4

To see the openness of $K_{(i,j,k)}$, consider the representation $\rho: C^2(M,M) \times C^2(M,M) \times C^2(M,M) \to C^2(\Sigma_f \times \Sigma_f \times \Sigma_f - U_{(i,j,k)}, M \times M \times M)$, given by restricting (g_1,g_2,g_3) to $(g_1,g_2,g_3)|_{\Sigma_f\times\Sigma_f\times\Sigma_f - U_{ijk}}$. Here $f \in K_{(i,j,k)}$ and

$$U_{ijk} = \phi \text{ if } i \neq j \neq k$$

$$U_{ijk} = V_{ii} \times \Sigma_f \text{ if } i = j \neq k$$

$$U_{ijk} = \Sigma_f \times V_{jj} \text{ if } i \neq j = k$$

$$U_{ijk} = \{(x,y,z): (x,y),\ (x,z),\text{ or } (y,z) \in V_{ii}\} \text{ if } i = j = k,$$

where V_{ii} was defined in Lemma 3 as an open neighborhood of Δ in $\Sigma_f \times \Sigma_f$. Now since $(f^i,f^j,f^k) \pitchfork \Delta_M$ on $\Sigma_f \times \Sigma_f \times \Sigma_f - U_{ijk}$, there is a neighborhood $N_1 \times N_2 \times N_3$ such that; if $(g_1,g_2,g_3) \in N_1 \times N_2 \times N_3$, then $(g_1,g_2,g_3) \pitchfork \Delta_M$ on $\Sigma_f \times \Sigma_f \times \Sigma_f - U_{ijk}$. Thus if g is close to f and $h: \Sigma_f \to \Sigma_g$ is given by TIT, then $(g^ih,g^jh,g^kh) \in N_1 \times N_2 \times N_3$. Hence $(g^ih,g^jh,g^kh) \pitchfork \Delta_M$, which means that there are no triple intersections from $\Sigma_f \times \Sigma_f \times \Sigma_f = U_{ijk}$. But by the definition of U_{ijk}, there can be no triple coincidences from U_{ijk} either. Hence K_{ijk} is open in each $K_{i'j'k'}$ where $(i',j',k') < (i,j,k)$.

As in Lemma 4, there is an integer n such that $f^n(M) - f^{n+1}(M)$ is a fundamental domain F; and there is an integer J such that $f^J(F)$ contains an image of each singularity. Thus if there are no triple intersections with $k \leq n + J$, there will be no triple coincidences. This finiteness property tells us that K_5 is open in K_4 and hence in K_2. The density of K_5 follows from the Baire Category Theorem since $K_5 = \bigcap_{(i,j,k)} K_{ijk}$. Q.E.D.

These lemmas combine to give the following theorem:

Theorem 2: K is an open dense subset of C^r contractions on M.

§III. Stratifications and Density of C^r-Structurally Stable Contractions.

In the first part of this section, subdivisions of M are constructed and shown to be stratifications. These stratifications are then used to show necessary and sufficient conditions for a contraction to be C^r-structurally stable and also to give many topological invariants of the topological conjugacy class. The last part deals with the problem of generalizing these results to higher dimensions.

Definition: A stratification of M is a finite collection of connected disjoint submanifolds without boundary $\{L_i\}$ such that (1) $\bigcup_i L_i = M$ and (2) if $\bar{L}_i \cap L_j \neq \phi$, then $\bar{L}_i \supset L_j$ and $\dim L_j < \dim L_i$.

When one has a stratification S and an endomorphism f, there are three basic operations that can be performed to give different subdivisions of M. In certain cases these subdivisions are stratifications. The first new subdivision is indicated by f(S). To obtain the stratum of f(S) that contains x, let P be the set of all points y such that there is a one to one correspondence between $f^{-1}(x) \cap L_i$ and $f^{-1}(y) \cap L_i$ for each stratum L_i in S. The connected component of P that contains x is the desired set.

The second operation is indicated by intersection. To find the stratum of $S \cap f(S)$ which contains a given point x, take the connected component of the set of points that belong to exactly the same strata as x. The strata of the third subdivision, $f^{-1}(S)$, are the connected components of the inverse images of the strata in S.

Let S_1 be the stratification of M using the singularities of an endomorphism f in K as follows: the zero dimensional strata are the cusps, the one dimensional strata are the connected components of $\Sigma_f - \{x \in \Sigma_f : x \text{ is a cusp}\}$, and the two dimensional strata are the connected components of $M - \Sigma_f$. From the normal forms, it is clear that S_1 is a stratification of M.

<u>Proposition 1</u>: <u>For</u> f $\in$ K, <u>each of the subdivisions of</u> M <u>in the following sequence is a stratification</u>:

$$
\begin{aligned}
&S_1 \\
&S_2 = f(S_1) \\
&S_3 = S_1 \cap S_2 \\
&\vdots \\
&S_{2n} = f(S_{2n-1}) \\
&S_{2n+1} = S_{2n} \cap S_1 .
\end{aligned}
$$

<u>Proof</u>: Since S_1 is a stratification, one can proceed with the inductive step and show that S_i is a stratification. Suppose i is even. Then the zero dimensional strata of S_i are f of the zero dimensional strata of S_{i-1} plus the first coincidences with

integers $\frac{i}{2}$ and $0 < j \leq \frac{i}{2}$. Since there are only a finite number of such coincidences, there are only a finite number of zero dimensional strata in S_i. The one dimensional strata are f of the one dimensional strata in S_{i-1}, which may be subdivided because of a new coincidence between singularities. There are only a finite number of one dimensional strata, and the closure of any one of them only adds at most two points which are zero dimensional strata. The two dimensional strata are the connected components of M minus the one and zero dimensional strata. There are only a finite number of such sets and they satisfy the conditions to make S_i a stratification.

So suppose i is odd, then $S_i = S_1 \cap S_{i-1}$. The zero dimensional strata are the zero dimensional strata of S_1 and S_{i-1} plus the points on Σ_f which are images of other singularities under j iterates of f where $0 < j \leq \frac{i-1}{2}$. The set of such coincidences is finite, and hence there are only a finite number of zero dimensional strata. The one dimensional strata are the one dimensional strata for S_1 and S_{i-1} with some subdivision because of the coincidences. Since there are only a finite number of subdivisions, there are only a finite number of one dimensional strata and the closure of any one adds at most two zero dimensional strata. The two dimensional strata are the connected components of M minus the one and zero dimensional strata. Just as in the case where i was even; there are only a finite number of such strata, and they satisfy the necessary conditions to make S_i a stratification. Q.E.D.

It should be noted that this proposition is a simple consequence of the lemmas proved in §II as is the next proposition.

Proposition 2: If $f \in K$ then there is a positive integer N such that for any integer $n \geq N$, each subdivision in the following sequence is a stratification of M:

$$\psi_1 = f^{-1}(S_{2n+1}) \cap S_{2n+1}$$
$$\psi_2 = f^{-1}(\psi_1) \cap \psi_1$$
$$\vdots$$
$$\psi_n = f^{-1}(\psi_{n-1}) \cap \psi_{n-1}.$$

Also,

$$\psi_N = \psi_{N+1},$$

Proof: As in Lemma 4, there is an integer m such that $f^m(M) - f^{m+1}(M) = F$, a fundamental domain; and there is an integer J such that $f^J(F)$ contains an image of every singularity. So let $N = m + J$. If $n \geq N$, the difference between S_{2n+1} and S_{2n+3} is in $f^N(M)$, where S_{2n+3} is a refinement of S_{2n+1}. The new strata in S_{2n+3} are images of strata in S_{2n+1}.

From the normal forms and the fact that M is compact, it is clear that f is finite to one. Thus f^{-1} of any zero dimensional strata is a finite number of points. There are several local pictures that should be studied at this point. First, if x is a regular point, f^{-1} of a neighborhood of $f(x)$ in a neighborhood of x has the same subdivisions as the neighborhood of $f(x)$. This is because f is a diffeomorphism in a neighborhood of x. If x is a cusp; then, from the normal form, we get the following picture:

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} xy - x^3 \\ y \end{pmatrix}$$

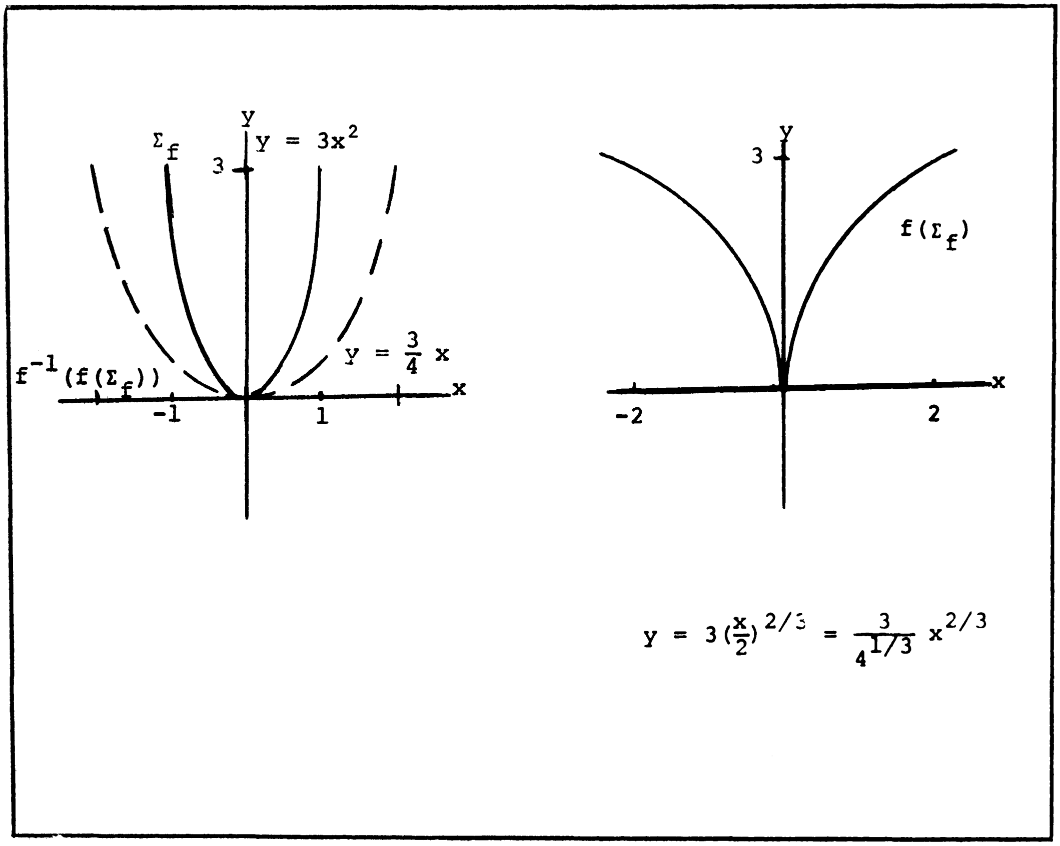

Figure 5

Note that in this local picture there are four two dimensional strata, four one dimensional strata and one zero dimensional stratum. It is important to see that f is one to one on the closure of each of the two dimensional stratum.

If x is a fold point, then it is either a zero dimensional stratum or it is on a one dimensional stratum. If it is on a one dimensional stratum and f(x) is also on a one dimensional stratum, then f^{-1} adds nothing to the local picture at x. If f(x) is a zero dimensional stratum, then x and some other singularity are coincident at f(x). Since this is the first time they are coincident, the intersection is transverse. Thus f^{-1} introduces a curve transversal to Σ_f at x. Actually, x becomes a zero dimensional stratum; and the curves break up into four one dimensional strata. In looking at the local picture for an arbitrary ψ_i, f(x) would not have to be the coincident point between x and some other singularity. The other possibility is that f(x) is the inverse image of the point of coincidence between x and some other singularity. But the picture at f(x) would still be two curves intersecting transversely and hence f^{-1} would introduce the same picture at x.

If x is a fold which is a zero dimensional stratum, then there are two curves passing transversely through x which locally form four one dimensional strata. The local picture at f(x) is two curves that are tangent at f(x) giving four one dimensional strata and one zero dimensional stratum. Now each of the two one dimensional strata that are not first images of Σ_f have two inverse images near x. This gives six two dimensional, six one dimensional and one zero dimensional strata.

Thus $f^{-1}(\psi_i)$ has a finite number of zero and one dimensional strata, and the closure of the one dimensional strata add at most two points which are zero dimensional strata. Since the two dimensional strata are the connected components of the complement of the union of the zero and one dimensional strata, $f^{-1}(\psi_i)$ is a stratification of M. $f^{-1}(\psi_i) \cap \psi_i$ is a refinement of $f^{-1}(\psi_i)$, which one obtains by adding the zero and one dimensional strata of S_{2n+1} in $f^n(M)$ whose images are not strata of S_{2n+1}. Thus each ψ_i is a stratification of M.

To see that $\psi_N = \psi_{N+1}$, one can think of obtaining the ψ_i by adding inverse images to the stratification of $f^N(M)$ given by S_{2n+1}. A point x will be a zero dimensional strata for ψ_i if $f^j(x)$, $j \leq i$, is a zero dimensional stratum. Let $f^j(x)$ be a zero dimensional stratum. Then there is an integer k, $0 < k \leq N$, such that $f^k(x) \in f^N(M)$ is a zero dimensional stratum of S_{2n+1}. Hence x is a zero dimensional strata of ψ_N as well as ψ_{N+1}. The same argument shows that the one dimensional strata of ψ_N and ψ_{N+1} are the same. Hence $\psi_N = \psi_{N+1}$. Q.E.D.

Let us improve the notion slightly before continuing. Let $f \in K$ and m the smallest integer such that f is a homeomorphism on $f^m(M)$ and $f^m(M) \cap \Sigma_f = \phi$. Let $f^m(M) - f^{m+1}(M) = F$ and J be the smallest integer such that $f^J(F)$ contains an image of each singularity. Let $N = m + J$ and $S(f) = \psi_N$ starting with $\psi_1 = f^{-1}(S_{2N+3}) \cap S_{2N+3}$. There are several important properties of $S(f)$ that should be noted.

Lemma 6: Let L be a stratum of $S(f)$ outside of $f^N(M)$, then $f(L)$ is a stratum and f is a covering map from L to $f(L)$.

Proof: Since L is outside $f^N(M)$, L is a connected component of $f^{-1}(L')$ for some stratum L'. From the normal forms, one sees that f maps L locally diffeomorphically into L'; thus $f(L)$ is open and connected in L'. If it is also closed in L', then $f(L) = L'$.

To see that $f(L)$ is closed in L' let $x \in L' \cap \overline{f(L)} - f(L)$, and $\{y_i\}$ be a sequence of points in L such that $\{f(y_i)\} \to x$. Let z be a limit point of $\{y_i\}$. By continuity, $f(z) = x$; but $z \in f^{-1}(L')$ and not in L. This is impossible, so $f(L) = L'$.

One now wants to show that the cardinality of f^{-1}, card f^{-1}, is locally constant on L' with $f|L$. Since f is a local diffeomorphism onto L', card f^{-1} cannot locally decrease. So suppose there is a point x where card f^{-1} increases. That is, there is a sequence $\{y_i\} \to x$ such that card $f^{-1}(x) <$ card $f^{-1}(y_i)$. Take neighborhoods of each point of $f^{-1}(x)$ on which f is a diffeomorphism. Outside of these neighborhoods, there is a set of points $\{z_i\}$ such that $f(z_i) = y_i$. Let b be a limit point of $\{z_i\}$. By continuity, $f(b) = x$; but b is not one of the inverse images of x. This contradiction shows that f^{-1} is locally constant on L'. Since L' is path connected, each point has the same number of inverse images. Hence f is a covering map. Q.E.D.

One would expect that if two maps in K were close then the stratifications of M that they produce should be close in some sense. This is the content of the next proposition.

Proposition 3: Let $f \in K$ and U be a neighborhood of the identity in $C^0(M,M)$. Then there is a neighborhood N of f in K such that if $g \in N$ then there exists a homeomorphism $h \in U$ that sends strata of $S(f)$ to those of $S(g)$.

Proof: The first step is to construct an open neighborhood of the union of the zero and one dimensional strata of $S(f)$. For the zero dimensional strata, L_i, pick open sets V_i whose closures are disjoint and for which there are diffeomorphisms $\phi_i: V_i \to R^2$ with $\phi_i(L_i) = 0$ which give the normal local picture depending on the type of zero dimensional strata. Let U_i be ϕ_i^{-1} of the open unit disk in R^2. Since the one dimensional strata are C^r submanifolds, they have tubular neighborhoods which can be taken to be disjoint. The union of the U_i and the tubular neighborhoods give us an open set containing the zero and one dimensional strata of $S(f)$. The open set we want is obtained from this one by shrinking the tubular neighborhoods if necessary so that if x is on a one dimensional stratum outside of U_i then the fiber of the tubular neighborhood through x is outside of $\phi_i^{-1}(B(\frac{1}{2}))$ where $B(\frac{1}{2})$ is the open ball of radius $\frac{1}{2}$ centered at the origin in R^2.

Now in each $\phi_i(U_i)$ let $\{x_1,\ldots,x_n\}$ be the intersection of the circle of radius $\frac{1}{2}$ with the images under ϕ_i of the one dimensional strata in $S(f)$. Since we have the normal picture in $\phi_i(U_i)$ there will be one and only one such point for each one dimensional strata whose closure contains the origin. Let a be the minimum distance between the x_i's. Now if $0 < \alpha < a$ then the set of points inside $B(\frac{1}{2})$ that are a distance α from the zero and one dimensional strata forms a finite number of curves $\{c_1,\ldots,c_n\}$.

In fact there are as many such curves as there are x_i's. The set of points in $B(1) \setminus B(\frac{1}{2})$ which are a distance α from the one dimensional strata is a finite number of curves $\{\ell_1,\ldots,\ell_{2n}\}$ the number being twice the number of x_i's. Two of these combine with each c_i to give n curves. By choosing α small enough the ℓ_j will be in the tubular neighborhood of the one dimensional strata and will be contained in the image of some section.

Let $\{x_1',x_2',\ldots,x_n'\}$ be the intersection of $B(\frac{3}{4})$ with the one dimensional strata in $\phi_i(U_i)$. By choosing α even smaller if necessary we can assume that the fiber through x_i' intersects two of the ℓ_j's and this part of the fiber stays in $B(1) \setminus B(\frac{1}{2})$. Now the part of the fibers through the x_i''s connecting the ℓ_j's union the part of the ℓ_j's from these intersections to the c_i's union the c_i's gives the boundary of an open set containing the origin which is homeomorphic to a disk. If we take out the images of the strata we get n open sets each homeomorphic to a disk.

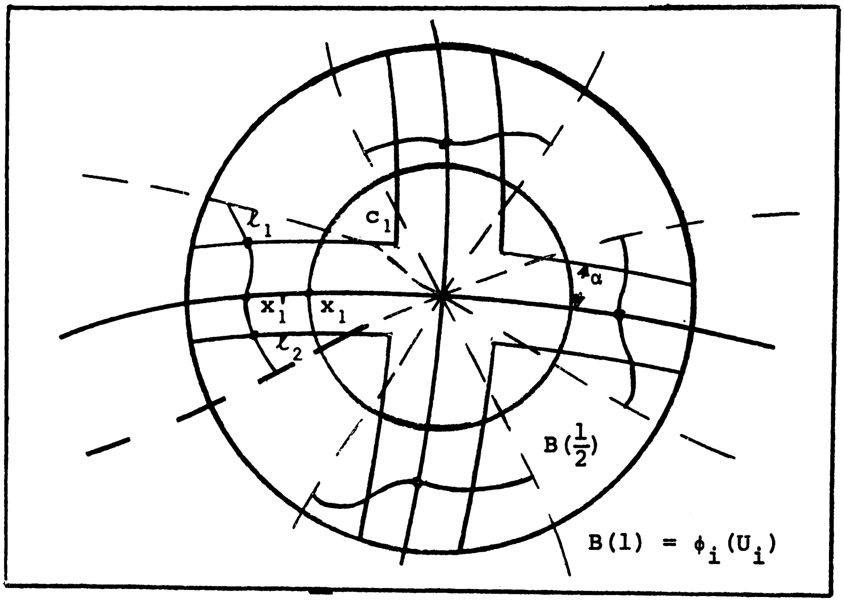

Figure 6

Before continuing with the proof of this proposition let us consider the following lemma which establishes some of the properties of the maps near f.

Lemma 7: *Let $f \varepsilon \mathcal{K}$ then there is a neighborhood N of f such that if $g \varepsilon N$ then*

1. *$S(g)$ has the same number of zero and one dimensional strata as $S(f)$.*
2. *g has the same normal structures on each U_i as f does.*
3. *each ϕ_i maps the zero and one dimensional strata of $S(g)$ in U_i into the α neighborhood of those for $S(f)$.*
4. *outside of $\bigcup_i \phi_i^{-1}(B(\frac{1}{2}))$ the one dimensional strata of $S(g)$ are in the tubular neighborhoods and are images of sections.*

Proof: In defining $S(f)$ the smallest integer m such that $f^m(M)$ contained no singularities and f was a homeomorphism on $f^m(M)$ was found. Since Σ_f was defined by a transversal intersection, and Σ_f is a finite distance, say ε, from $f^m(M)$; there is a neighborhood N_1 of f such that if $g \varepsilon N_1$, then Σ_g is within $\frac{\varepsilon}{2}$ of Σ_f and $g^m(M)$ is within $\frac{\varepsilon}{2}$ of $f^m(M)$. Thus $g^m(M) \cap \Sigma_g = \phi$.

Now since Σ_f is a finite distance from $f^m(M)$, f is a local diffeomorphism on any open neighborhood of $f^m(M)$ that does not intersect Σ_f. In fact, by choosing a small open neighborhood of $f^m(M)$, f is a diffeomorphism. Now by using the openness of diffeomorphisms on compact manifolds we see that there is a neighborhood N_2 of f such that if $g \in N_2$ then g is a diffeomorphism on a fixed manifold that contains $f^m(M)$. By shrinking N_2 if necessary we can make sure that $g^m(M)$ is contained in the fixed manifold. Thus g is a homeomorphism on $g^m(M)$ and m satisfies the two conditions for every map in a neighborhood of f.

We now want to make sure m is the smallest integer that will work. If $f^{m-1}(M) \cap \Sigma_f \neq \phi$ then the interior of $f^{m-1}(M)$ contains a singularity x because an intersection of the boundary of $f^{m-1}(M)$ and Σ_f would be transverse. Now by taking a small neighborhood N_3 of f we can guarantee that if $g \in N_3$ then $g^{m-1}(M)$ contains a fixed neighborhood of x and Σ_g also has a point in this neighborhood. Thus m-1 will not work if $f^{m-1}(M) \cap \Sigma_f \neq \phi$. So suppose $f^{m-1}(M) \cap \Sigma_f = \phi$ but f is not a homeomorphism on $f^{m-1}(M)$. The only way for this to happen is for f to fail to be one to one. In fact f must send two interior points to the same point. For suppose the intersection is the image of one interior point with a boundary point. Then since f is a local diffeomorphism and there are interior points in every neighborhood of the boundary points we could find two interior points that have the same image. If the intersection was between two boundary points, this would be the first coincidence between two fold and thus be transversal. Hence in this case we can again find two interior points which have the same image. Now pick disjoint open sets about each of these points whose closures are in the interior of $f^{m-1}(M)$. Now there is a neighborhood N_4 of f such that if $g \in N_4$ then the two closed neighborhoods are in $g^{m-1}(M)$ and the images of the two sets intersect. Thus m is indeed locally constant.

The other number that was used in defining $S(f)$ was J, the smallest integer such that $f^J(f^{m+1}(M) - f^m(M))$ contained an image of each singularity. Notice that if g is close to f then Σ_f is homeomorphic to Σ_g and the number of zero dimensional strata of Σ_f and Σ_g are the same because they come from transversal intersections. We can, therefore, also assume they are of the same type and pointwise close. Since their images must also be close the boundary of $f^m(M)$ and $g^m(M)$ must be made up of corresponding one dimensional strata. Thus the boundary of each $f^J(f^{m+1}(M) - f^m(M))$ corresponds with that of $g^J(g^{m+1}(M), g^m(M))$. Thus all other strata have images in the interior of $f^J(f^{m+1}(M) - f^m(M))$. So by C^0 stability the corresponding strata in Σ_g have images in the interior of $g^J(g^{m+1}(M) - g^m(M))$. Also, since J was the smallest integer for some strata under f it must also be the smallest for the corresponding strata for g. Hence J is also locally constant.

As we have noticed the subdivision of Σ_f and Σ_g corresponded in both number and type. Since $S(f)$ and $S(g)$ are arrived at by taking the same number of forward iterates and then all the inverse iterates, we see that the number and type of zero and one dimensional strata in $S(f)$ and $S(g)$ are the same.

Parts 2 and 3 of this lemma now follow easily from the C^0 stability of the normal forms while 4 is a result of the higher stability of the one dimensional strata Q.E.D.

Now let us return to the proof of Proposition 3.

To define the homeomorphism h, let h be the identity outside of the union of the tubular neighborhoods and the U_i. On the tubular neighborhoods between two x_i', the new one dimensional stratum is a section. Thus we can reparameterize the fibers so it is the zero section. This reparameterization can be viewed as a homeomorphism of this part of the tubular neighborhood that takes the old zero section to the new one. We can choose the reparameterization so that the homeomorphism is the identity outside of any fixed open set that contains the two sections and the parts of the fibers between them. Defining it this way we see that the homeomorphism will extend to the identity. Now since in $B(1) - B(\frac{1}{2})$ the new zero section is within α of the old, we can take the reparameterization to be the identity on each of the ℓ_i. Let D_i be the closed disk in U_i bounded by the C_i, the parts of the fibers through the x_i''s connecting the ℓ_i's and the parts of the ℓ_i's connecting these intersections and the C_i. On U_i outside of D_i union the parts of the tubular neighborhoods where h is already defined we defined the map to be the identity. On the closed set we use the definition we already have on the fiber through x_i', and the identity on the ℓ_i's and C_i's. Since the part of the stratum connecting x_i' and the origin is homeomorphic to a straight line and the part of the stratum for g connecting the image of x_i' under the homeomorphism and the zero dimensional stratum for g in U_i is also homeomorphic to a straight line, we send the one to the other. We now fill in the rest any way we want. This can be done because we have defined a homeomorphism from the boundary of a set that is homeomorphic to a disk to the boundary of another set that is homeomorphic to a disk. It is clear that this gives us a homeomorphism that sends strata to strata.

By taking the diameter of each U_i less than ε and the arclength of each fiber in the tubular neighborhoods less than ε, the homeomorphism will move each point at most ε. Thus the homeomorphisms can be taken to be in any neighborhood of the identity. Q.E.D.

We are now ready to see the density of C^r-structurally stable contractions.

Theorem 1: The set of C^r-structurally stable contractions on any compact two dimensional, C^∞ manifold M without boundary is an open dense subset of all C^r contractions in the C^r topology for $r \geq 12$.

Proof: The openness of the set is clear from the definition. Density will be shown by proving that every endomorphism f in K is C^r-structurally stable. If U is a small neighborhood of f in K and $g \varepsilon U$ then $g^m(M) - g^{m+1}(M) = G$ is a fundamental domain and $g^J(G)$ contains an image of each singularity. Here m and J are the integers used to define F and $S(f)$. From the last proposition there is a homeomorphism h close to the identity which sends the strata of $S(f)$ to the strata of $S(g)$. Although h does not have to be a topological conjugacy, being close to the identity gives it another property that looks like a strata conjugacy. That is, if L and f(L) are strata of $S(f)$; then h(L) and h(f(L)) are strata of $S(g)$ and $gh(L) = hf(L)$ as sets. Thus if L is a point stratum, then h is a conjugacy at this point.

Let L be a one dimensional strata that is in the boundary of $f^N(M)$. Define another map from L to h(L) by $g^{-1}\cdot h\cdot f$. This can be done because f(L) is a stratum of $S(f)$ and g is a diffeomorphism from h(L) to hf(L). Note that this new map is also close to the identity and would agree with h on $\bar{L} - L$. There is an open neighborhood A of L which contains no other one dimensional strata and no zero dimensional strata. The closure of A contains $\bar{L} - L$ and also L, but these are the only zero and one dimensional strata it contains. Using h on the boundary of A and the new map on L, one can construct a new homeomorphism on A to h(A) that is strata preserving. Now construct similar homeomorphisms on corresponding neighborhoods of each one dimensional strata in the boundary of $f^N(M)$. Note that the new strata preserving homeomorphism H is a conjugacy on the boundary of $f^N(M)$. Then change H on $f^{J+i}(F)$ to g^iHF^{-i} where f^{-i} is taken in $f^J(F)$. Also send x_f to x_g. One should note that the new map K is a conjugacy on $f^N(M)$.

To define the conjugacy outside of $f^N(M)$, we will send the strata of $S(f)$ to the strata of $S(g)$ that have already been identified by K. To get the desired map, remember that f^i and g^i are close and are covering maps on a given stratum L. They also have their images in $f^J(F)$ and $g^J(G)$ for some $i \leq N$. Since K is close to the identity and sends $f^i(L)$ to $g^i(K(L))$, there is a unique lift close to the identity sending L to K(L). It should be noted that this lift is independent of i as long as $f^i(L) \subset f^N(M)$, because K is a conjugacy on $f^N(M)$. Doing this on each stratum gives a new map of M that is one to one, onto, close to the identity, and preserves strata. In fact, it is a homeomorphism on each stratum and satisfies the appropriate commutative diagram to be a topological conjugacy. The only thing that needs to be checked is that it is continuous where two different strata come together. To see this one should look at the different local pictures as in Proposition 2. Since the map is arbitrarily close to the identity, it sends local strata to local strata correctly. Since f is one to one on the closure of every local stratum, the map is indeed continuous and hence a homeomorphism. Q.E.D.

Although this is the basic result it can also be considered as the first part of the next theorem, which gives necessary and sufficient conditions for a contraction to be C^r-structurally stable.

<u>Theorem 3</u>: *K* <u>is precisely the</u> C^r-<u>structurally stable contractions on</u> M.

<u>Proof</u>: From the proof of Theorem 1, we know that every endomorphism in *K* is C^r-structurally stable. The C^r endomorphism of M, which are stable using two different homeomorphisms, are the Whitney endomorphisms. Thus the C^r-structurally stable contractions must also be Whitney endomorphisms.

Suppose g is a C^r-structurally stable contraction on M. Since *K* is dense in the set of contractions, there is an f ε *K* such that f and g are topologically conjugate. If h is a topological conjugacy, then h(x) is regular, a fold, or a cusp according to whether x is respectively regular, a fold, or a cusp. This is because a topological conjugacy must preserve the number of points that go to a given point in a neighborhood of x. If x is regular, it is one to one; at a fold, it is two to one; and at a cusp, three to one. The fixed points and all orbits are also preserved by h. Thus, x and y are coincident under f if h(x) and h(y) are coincident under g. This establishes that g satisfies the following three conditions:

1. The fixed point x_g of g is regular and is not coincident with any singularity.

2. A cusp point is not coincident with any other singularity.

3. For any set of three singularities there is at most one subset of two elements which are coincident.

It also shows that the folds for g have the same number and type of intersections. Since g is C^r-structurally stable, there is a neighborhood of g which also satisfies these conditions. Indeed, if g did not satisfy one of the transversality conditions, an arbitrarily small perturbation could change the number of intersections of a given type, which is a contradiction. Hence $g \in K$.

Q.E.D.

Using methods very similar to the ones used in this proof, one can prove the following:

<u>Theorem 4</u>: <u>If</u> f <u>and</u> g <u>are</u> <u>two</u> C^r-<u>structurally</u> <u>stable</u> <u>contractions</u> <u>which</u> <u>are</u> <u>topologically</u> <u>conjugate</u>, <u>then</u> <u>a</u> <u>conjugating</u> <u>homeomorphism</u> h <u>is</u> <u>strata</u> <u>preserving</u> <u>between</u> $S(f)$ <u>and</u> $S(g)$.

<u>Proof</u>: Theorem 3 says that f and g ε K and hence S(f) and S(g) can be defined. In the proof of the last theorem, it was pointed out that h must send singularities to singularities; hence h is a homeomorphism from Σ_f to Σ_g. The zero dimensional strata of Σ_f are cusps and folds which are coincident with other folds. As was pointed out in the last theorem, this finite set of special points also has to be preserved by h. Since the one dimensional strata of S(f) in Σ_f are the connected components of Σ_f minus the finite set of special points, h must preserve these strata. Since h preserves orbits, $h(f^i(M)) = g^i(M)$. If $f^i(M)$ contains no singularities, then neither does $g^i(M)$. Also if f is one to one on $f^i(M)$, then g is also one to one on $g^i(M)$. Thus if m is the smallest integer for which $f^m(M) - f^{m+1}(M) = F$ is a fundamental domain, then it is also the smallest integer for which $g^m(M) - g^{m+1}(M) = G$ is a fundamental domain. The smallest integer J such that $f^J(F)$ contains an image of each singularity also holds for g and, in fact, h sends $f^J(F)$ to $f^J(G)$. Thus the integers used to define S(f) and S(g) are the same.

The zero and one dimensional strata of S(f) and S(g) are obtained from Σ_f and Σ_g respectively by taking N iterates and then the inverse images. Since h preserves orbits, the images of strata in Σ_f must go to images of the corresponding strata in Σ_g and similarly for all inverse images. It is this orbit preserving ability of h that guarantees that the zero and one dimensional strata of S(f) go to zero and one dimensional strata of S(g). The two dimensional strata are the connected components of the complement of the zero and one dimensional strata. Hence h must

must preserve these strata also, and h is strata preserving between $S(f)$ and $S(g)$. Q.E.D.

It should be noted that this theorem and its proof give many invariants of the topological conjugacy class of an endomorphism in K, some of which are the numbers m, J, and N, and the numbers of circles of singularities, of cusps, and folds that are coincident with other folds. Since f is a covering map from one stratum to another, its covering number is also an invariant. It seems quite reasonable that the topological conjugacy classes could be characterized by using these invariants.

To begin studying the n dimensional case, we shall show that there are structurally stable contractions on every n dimensional manifold. This will be done for C^∞ contractions with the help of Mather's topological stability of maps.

<u>Theorem 5</u>: <u>On every</u> n <u>dimensional compact</u> C^∞ <u>manifold</u> M <u>without boundary</u>, <u>there is a</u> C^∞-<u>structurally stable contraction</u>.

<u>Proof</u>: Start with a topologically stable map $f: M \to R^n$ [see 5]. Since such maps have many regular values, let y be one of them and let $x \in f^{-1}(y)$. Let U be an open neighborhood of x such that f is a diffeomorphism on $\bar{U}$, and h be a diffeomorphism from R^n into U sending y to x. By taking h to be a strong contraction, one can make sure that $hf = g$ is a contraction. Since y is a regular value, x is a regular value as well as the fixed point of g. Because g maps U diffeomorphically inside itself and contains the first image of every singularity, we see that x is coincident

with no singularities. Also $g(M) - g^2(M) = G$ is a fundamental domain. There is an integer J such that $g^J(G)$ contains an image of every singularity. Now consider the map g^{J+1}. It has the property that $g^{J+1}(M) - (g^{J+1})^2(M)$ is a fundamental domain and contains an image of each of the singularities. Let H be a topologically stable map which is close to g^{J+1}. Since the singularity set for H is close to the one for g^{J+L}, $H(M) - H^2(M)$ is a fundamental domain that contains an image of each singularity. In fact, H and any contraction F close to H map $\bar{U}$ diffeomorphically inside itself. Since $F(M) - F^2(M)$ as well as $H(M) - H^2(M)$ contain the first image of each singularity, $H^2(M)$ and $F^2(M)$ are contained in the interior of $H(M)$ and $F(M)$ respectively. There also exist two homeomorphisms h_1 and h_2 of M such that $Fh_1 = h_2H$ and the homeomorphisms are arbitrarily close. Since $H^2(M)$ is contained in the interior of $H(M)$, there is an open neighborhood V of $H(M)$ in U such that $\overline{H(V)}$, which is a neighborhood of $H^2(M)$, contains no first images of singularities. Since h_1 and h_2 can be made arbitrarily close, a simple isotopy in V gives a new homeomorphism h_3 of M which is h_1 outside of V and h_2 on $H(M)$. Change h_3 on $H(V) - H^2(M)$ to be Fh_3H^{-1} where H^{-1} is taken in V. Note this agrees with h_2 on $\partial H(V)$. Now iterate this map inwards to x_H and send x_H to x_F. This homeomorphism h_4 is a conjugacy everywhere except on $H^{-1}(H(V)) - V$. Since $\overline{H(V)}$ consists entirely of regular values, H on the connected components of $H^{-1}(\overline{H(V)})$ is a covering map and F is a covering map from h_1 of the connected components to $h_2\overline{H(V)} = h_4\overline{H(V)}$. One can take all of the homeomorphisms close to the identity so that there is a unique lift of h_4. Note that the lift agrees with h_1 on the boundaries of the connected components of $H^{-1}(H(V)) - V$. This is because h_4 agrees with h_2 on the boundary of $H(V)$. Using these lifts on $H^{-1}(H(V))$ gives the topological conjugacy. Hence H is topologically conjugate to F and is, in fact, C^∞-structurally stable. Q.E.D.

It should be noted that this proof gives sufficient conditions for a C^∞ contraction to be C^∞-structurally stable.

Corollary 1: *If f is a C^∞ contraction on M which is topologically stable and $f(M) - f^2(M)$ is a fundamental domain which contains no singularities but does have an image of each singularity, then f is C^∞-structurally stable.*

These are certainly not all of the structurally stable contractions. It is even reasonable to conjecture that the structurally stable contractions are dense in the set of all contractions as is true in the one and two dimensional cases.

BIBLIOGRAPHY

1. R. Abraham, J. Robbin and A. Kelley. Transversal Mappings and Flows, Benjamin, New York, 1967.

2. L. Block and J. Franke. "A classification of the structurally stable contracting endomorphism of S^1", Proc. Amer. Math. Soc., 36(1972), 597-602.

3. J. Franke. Contracting Endormorphisms on Compact Manifolds, Thesis, Northwestern University, 1973.

4. H. I. Levine. "Singularities of differential mappings", Lecture Notes in Mathematics, 192, Springer, 1971.

5. J. Mather. Notes on Topological Stability, Harvard University, 1970

6. Z. Nitecki. "Nonsingular endomorphisms of the circle", Proc. Symposium Pure Math., Vol. 14, (1970), 203-220.

7. M. Shub "Endomorphisms of compact differentiable manifolds", Amer. J. Math., 91(1969), 175-199.

8. S. Smale. "Differentiable dynamical systems", Bull. Amer. Math. Soc., 73(1967), 747-817.

9. C. T. C. Wall. "Lectures on C^∞-stability and classification", Lecture Notes in Mathematics, 192, Springer, 1971.

10. H. Whitney. "On singularities of mappings of Euclidean spaces, I, mappings of the plane into the plane", Ann. of Math., 62(1955), 374-410.

11. __________ "Singularities of mappings of Euclidean spaces", Symp. Int. de Top. Alg., Mexico, (1958), 285-301.

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