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A theorem on the monodromy of isolated singularities

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INTRODUCTION. - Let \( \pi : (X, x) \rightarrow (T, t) \) be a flat morphism between germs of smooth complex spaces, \((\Delta, t)\) its discriminant. Suppose that the fibre \((X^t, t) = \pi^{-1}(t)\) is a hypersurface with an isolated singularity at \(x\). Then \(\pi\) induces a fibre bundle on \(T - \Delta\) whose fibre \(M\) has the homotopy type of a bouquet of spheres of dimension \(r = \dim X - \dim T\) (see [4]); the associated representation \(\pi_1(T - \Delta, t) \rightarrow \text{Aut}(H^r(M, \mathbb{Z}))\) is called the monodromy of \(\pi\). By looking at the properties of a representation of \(\pi_1(T - \Delta, t)\) in the case that \(\pi\) is semiuniversal, we show an irreducibility property of such a representation. As a consequence we get a no-splitting principle for a hypersurface isolated singularity, that extends a known result for curves [3].

1. THE MONODROMY OF \(\pi\). - Let \(\pi : (\mathbb{C}^N, 0) \rightarrow (\mathbb{C}^n, 0)\) be a flat morphism whose fibre \((X_0, 0) = \pi^{-1}(0)\) is a hypersurface with an isolated singularity at \(0\). Denote by \((\Delta, 0)\) the discriminant of \(\pi\). Choose a small ball \(B\) around \(0\) in \(\mathbb{C}^N\). Then there exists a contractible open neighborhood \(U\) of \(0\) in \(\mathbb{C}^N\) such that \(\pi : B \cap \pi^{-1}(U) \rightarrow U\) is a smooth proper map and \(\pi\) is of maximal rank on \(\pi^{-1}(U - \Delta) \cap \partial B\) and on \(\pi^{-1}(U) \cap \partial B\). It follows that \(\pi : B \cap \pi^{-1}(U - \Delta) \rightarrow U - \Delta\) is a differentiable fibre bundle, whose fibre \(M\) is a compact differentiable \(2r\)-dimensional manifold with boundary, where \(r = N - n\); moreover since \(\pi\) is of maximal rank on \(\pi^{-1}(U) \cap \partial B\) we may suppose that the group of the bundle is made with diffeomorphisms of \(M\) which are the identity on \(\partial M\).

One knows (see [4]) that \(M\) is a parallelizable manifold which is homotopically equivalent to a bouquet of \(\mu\) spheres of dimension \(r\). In particular \(H^r(M, \mathbb{Z})\) is a free module of rank \(\mu\) over \(\mathbb{Z}\). Let \(p \in U - \Delta\) and identify \(M\) with \(\pi^{-1}(p) \cap B\). Then there is a homomorphism \(\pi_1(U - \Delta, p) \rightarrow \text{Aut} H^r(M, \mathbb{Z})\). Letting \(U\) vary, one deduces a homomorphism \(\sigma : \pi_1(\mathbb{C}^n - \Delta, 0) \rightarrow \text{Aut} H^r(M, \mathbb{Z})\), where \(\pi_1(\mathbb{C}^n - \Delta, 0)\) denotes the local fundamental group of \(\mathbb{C}^n - \Delta\) at \(0\).

(*) Remark that \(\pi_1(\mathbb{C}^n - \Delta, 0)\) is defined up to an inner automorphism \(\sigma\) that the indeterminacy of \(\sigma\) is just that of an inner automorphism of \(\pi_1(\mathbb{C}^n - \Delta, 0)\).
2. THE PRESENTATION OF $\pi_1(C^n - \Delta, 0)$. Let $(\Delta, 0) \subset (C^n, 0)$ be defined by an equation

$$w^m + a_1(z)w^{m-1} + \ldots + a_m(z) = 0$$

where $(w, z_1, \ldots, z_{n-1})$ are local coordinates on $(C^n, 0)$ and $a_i(0) = 0$ for $i = 1, \ldots, m$.

Consider a nice stratification of $\Delta$, say a stratification verifying Whitney's conditions. By the curve selection lemma one can see that there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$ the hypersurface $||z|| = \varepsilon$ is transversal to each stratum.

It follows that, if $U_\varepsilon = \{(w, z) \mid ||z|| < \varepsilon\}$ then for $0 < \varepsilon' < \varepsilon < \varepsilon_0$ the inclusion $U_\varepsilon - \Delta \to U_\varepsilon - \Delta$ is a homotopy equivalence. Moreover let

$$\eta(\varepsilon) = \sup_{U_\varepsilon \cap \Delta} |w|,$$

$$U_\varepsilon, \eta(\varepsilon) = \{(w, z) \in U_\varepsilon \mid |w| < \eta(\varepsilon)\}.$$ Then the inclusion $U_\varepsilon \cap \Delta \to U_\varepsilon$ is a homotopy equivalence. Since $a_i(0) = 0$ for $i = 1, \ldots, m$ one has that $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$, so that $U_\varepsilon, \eta(\varepsilon)$ is an arbitrary small neighborhood of $0$ in $C^n$. In particular if $0 < \varepsilon < \varepsilon_0$ and $p \in U_\varepsilon - \Delta$, then $\pi_1(C^n - \Delta, 0) \to \pi_1(U_\varepsilon - \Delta, p)$ is an isomorphism.

NOTATIONS. - $U = U_\varepsilon$; $V = U \cap \{w = 0\}$; $\varphi : U \to V$ the projection. Fix $|w_0| >> \varepsilon$.

For $z \in V$, $L_z$ is the straight line $\varphi^{-1}(z)$ and $p_z = (w_0, z) \in L_{z_0}$.

Finally let $\Gamma$ denote the discriminant of $\varphi : U \cap \Delta \to V$, $\tilde{\Gamma} = \varphi^{-1}(\Gamma)$.

Suppose $z_0 \in V - \Gamma$. Then one has a diagram:

$$0 \to \pi_1(L_{z_0} - \Delta) \to \pi_1(U - \Delta U \tilde{\Gamma}) \xrightarrow{\beta} \pi_1(V - \Gamma) \to 0$$

$$\downarrow \alpha$$

$$\pi_1(U - \Delta)$$

where the base point is always $p_{z_0}$, $j$ and $\alpha$ are induced by inclusions, $\beta$ by $\varphi$ and $\gamma$ by the map $z \mapsto (w_0, z)$.

Remark that $\varphi : U - \Delta U \tilde{\Gamma} \to V - \Gamma$ is a fibre bundle with fibre $L_{z_0} - \Delta$ and that $z \mapsto (w_0, z)$ induces a cross section of such a bundle. So from the homotopy sequence of a fibre bundle we get

1) the horizontal line is exact and $\beta \circ \gamma$ is the identity on $\pi_1(V - \Gamma)$.

Moreover one has obviously
2) $\alpha$ is surjective

3) $\alpha \circ Y$ is the null homomorphism

Consider the sequence $0 \to \pi_1(V - \Gamma) \to \pi_1(U - \Delta \cup \sim) \to \pi_1(U - \Delta) \to 0$.

This must not be exact at $\pi_1(U - \Delta \cup \sim)$. Nevertheless one has

4) $\ker \alpha$ is generated by the conjugated of $\operatorname{Im} Y$.

**Proof.** - Obviously for $v \in \operatorname{Im} Y$ and $b \in \pi_1(U - \Delta \cup \sim)$ one has $b^{-1}vb \in \ker \alpha$.

Let $b \in \ker \alpha$. Then $b = \lambda c$ with $c \in \pi_2(U - \Delta, U - \Delta \cup \sim)$. Let us represent $c$ by a map $\varepsilon : [0, 1] \times [0, 1] \to U - \Delta$ which is transversal to $\sim$. Then $\varepsilon^{-1}(\sim)$ is a finite set of points, let say $p_1, \ldots, p_s$. The following picture shows that $b$ is equivalent in $\pi_1(U - \Delta \cup \sim)$ to a product $w_1 \ldots w_s$ of simple loops around $\sim$.

$w_i$ simple means that it is composed of an arc $\tau$ from $p_i$ to a point near a regular point $\sim$ of $\sim$, a small circle around $\sim$ and then back with $\tau^{-1}$. One can construct a cylinder in $U - \Delta \cup \sim$ whose boundaries are two circles, one being that of $w_i$, the other being in $V - \Gamma$. Choose an arc $\tilde{\tau}$ in $V - \Gamma$ from $p_i$ to that circle, and call $v_1$ the resulting simple loop; if $\alpha_1$ is defined so that it follows $\tilde{\tau}$ and then a path along the cylinder from one circle to the other and then $\tau^{-1}$, one realizes that $w_i = \alpha_1 v_1 \alpha_1^{-1}$. So $b$ is a product of elements, each of which conjugated to an element of $\operatorname{Im} Y$.

**Corollary.** - $\ker \alpha \cap \ker \beta$ is the minimal normal subgroup $N$ of $\ker \beta$ that contains the elements of the form $bvb^{-1}v^{-1}$ with $b \in \ker \beta$, $v \in \operatorname{Im} Y$.

**Proof.** - Let $b \in \pi_1(U - \Delta \cup \sim)$, $v \in \operatorname{Im} Y$.

Then $bvb^{-1} = (b, \gamma \beta(b^{-1})(\gamma \beta(b) \cdot \gamma \beta(b^{-1}))(\gamma \beta(b) \cdot b^{-1})) = b \cdot \gamma \beta(b^{-1})^{-1}$

where $\gamma \beta \in \operatorname{Im} \gamma$, $\beta \in \ker \beta$. So if $b \in \ker \alpha$, one has from 4) and this
remark that $b = b_1 v_1 b_1^{-1} ... b_s v_s b_s^{-1}$ with $b_i \in \ker \beta$, $v_i \in \Im \gamma$.

Moreover $b \in \ker \beta$ implies $\beta(v_1 ... v_s) = 1$ and hence $v_1 ... v_s = 1$, since $\beta$ is injective on $\Im \gamma$. Let $n_i = b_i v_i b_i^{-1} v_i^{-1} \in \mathbb{N}$; then $b = n_1 v_1 ... n_s v_s = v_s ... v_1 n_1 v_1 ... n_s v_s$. From this and the remark that $v \in \Im \gamma$, $n \in \mathbb{N}$ implies that $v^{-1} n v \in \mathbb{N}$ deduces $b \in \mathbb{N}$.

Let $R$ be a straight line in $V$ s. th. $\pi_1(R - \Gamma) \to \pi_1(V - \Gamma)$ is surjective and let $H$ denote $\varphi^{-1}(R)$. From the preceding result we know that $\pi_1(H - \Delta)$ and $\pi_1(U - \Delta)$ have the same generators with the same relations, hence they are isomorphic. This is the local version of a theorem of Zariski - Van Kampen on the presentation of the fundamental group of $P_n$ minus a hypersurface; compare the article of Cheniot [2] in this volume. Now $H = \mathbb{C} \times E$ where $E = \{z \in \mathbb{C} \mid |z| < \varepsilon\}$ and $\Delta = \Delta \cup H$ is defined by some equation $w^m + b_1(z)w^{m-1} + ... + b_m(z) = 0$, with the $b_i$ holomorphic on $E$. Suppose that $\pi$ is versel. Then [see [1], [6]] one may suppose that $\Delta_1$ has the following properties:

i) it is irreducible

ii) it has only cusps or ordinary double points as singularities, with distinct images on $E$.

iii) it is flat on the $z$-direction, i.e. $z$ is a transversal parameter at any point of $\Delta_1$.

Let $\sigma: \tilde{\Delta}_1 \to \Delta_1$ be the normalization of $\Delta_1$ and consider $\tau = \varphi \circ \sigma: \tilde{\Delta}_1 \to E$. Then $\tau$ is only ramified at points that correspond to cusps of $\Delta_1$, and the ramification index at those points is two.

**LEMMA.** - The permutation group of the Riemann surface $\Delta_1$ is the full group of permutations of $r$ elements.

**Proof.** - It is sufficient to remark that a) it is transitive (because of the irreducibility of $\Delta_1$) and b) it is generated by transpositions (because $\Delta_1$ is simply ramified over distinct points).

Let $\gamma$ be a simple loop in $L_{z_0} - \Delta$ from $z_0$ turning positively around some element of $L_{z_0} \cap \Delta$. Its image in $\pi_1(U - \Delta)$ will be called a geometric generator.
A THEOREM ON THE MONODROMY...

THEOREM 1. - i) Let \( \gamma_1, \gamma_2 \) be geometric generators; there exists \( \delta \in \pi_1(U - \Delta) \) s. th. \( \gamma_1 \delta = \delta \cdot \gamma_2 \).

ii) Suppose that \( \Delta \) is not smooth, and let \( \gamma \) be a geometric generator; there exists a geometric generator \( \gamma' \) s. th. \( \gamma \gamma' \gamma = \gamma' \gamma \gamma' \).

Proof. - i) Remark first that any two loops around the same \( z_i \) are conjugated in \( \pi_1(L_{z_0} - \Delta) \) and hence also in \( \pi_1(U - \Delta) \). Then the preceding lemma assures that if \( z_i, z_j \in L_{z_0} \cap \Delta \) and \( \gamma \) is a loop around \( z_i \), then \( \gamma \) is equivalent in \( \pi_1(U - \Delta) \) to some loop around \( z_j \).

ii) Since \( \Delta \) is not smooth, \( \Delta \) must be ramified somewhere so that \( \Delta \) has at least a cusp. Suppose that \( z_0 \) is near that ramification point and that \( \gamma \) is a loop around a point near the corresponding cusp. Then the loop \( \gamma' \) that goes like \( \gamma \) until that cusp and then links the other point near the cusp, satisfies the required relation. In general one knows from i) that \( \gamma \) is conjugated to an element. Then it remains only to remark that each conjugated to a geometric generator is a geometric generator.

3. PICARD-LEFSCHETZ THEORY. - This theory describes the monodromy of a semiuniversal deformation in the following way (see [5]): the fibre over each simple point of \( \Delta \) has just an isolated singularity of the type \( \Sigma x_1^2 = 0 \); hence to each geometric generator \( \gamma \), there is associated a vanishing cycle \( e \in H^r_r(M, \mathbb{Z}) \) uniquely determined up to the sign by \( \gamma \). The action of \( \gamma \) on \( H^r_r(M, \mathbb{Z}) \) is given by the Picard-Lefschetz formula

\[
(*) \quad h \to h + (-1)^s \langle e, h \rangle e, \quad h \in H^r_r(M, \mathbb{Z})
\]

where \( s = (r + 1)(r + 2)/2 \) and \( \langle , \rangle : H^r_r(M, \mathbb{Z}) \times H^r_r(M, \mathbb{Z}) \to \mathbb{Z} \) is the cap product. Moreover \( \langle e, e \rangle \) is zero if \( r \) is odd and \( 2(-1)^s r(r+1)/2 \) if \( r \) is even.
Let \( \{z_1, \ldots, z_m\} = L_z \cap \Delta \) and choose simple loops \( \gamma_1, \ldots, \gamma_m \) in such a way that \( \gamma_i \) turns positively around \( z_i \) and \( \gamma_i, \gamma_j \) don't intersect outside \( z_0 \).

Call \( e_i \) the vanishing cycle associated to \( \gamma_i \). Then \( \gamma_1, \ldots, \gamma_m \) generate freely \( \pi_1(L_{z_0} - \Delta) \) and \( e_1, \ldots, e_m \) are a base of \( H_1(M, \mathbb{Z}) \) over \( \mathbb{Z} \).

**Theorem 2.** Let \( I, J \) be a partition of \( \{1, \ldots, m\} \). There exist \( i \in I \) and \( j \in J \) s.th. \( \langle e_i, e_j \rangle \neq 0 \).

**Proof.** From formula (*) one gets that the images \( \overline{\gamma}_1, \overline{\gamma}_j \) of \( \gamma_i, \gamma_j \) in \( \text{Aut} \ H_1(M, \mathbb{Z}) \) commute if \( \langle e_i, e_j \rangle = 0 \). Suppose that this happens for all \( i \in I \) and \( j \in J \). Fix \( i_1 \in I \), \( j_1 \in J \); because of theorem 1 one can write 
\[ \gamma_{i_1} \delta = \delta \gamma_{j_1} \] 
where \( \delta \in \pi_1(U - \Delta) \) and hence \( \delta \) is a product of \( \gamma_i, \gamma_j \). Since each \( \overline{\gamma}_i \) commutes with each \( \overline{\gamma}_j \) one can write \( \delta = \delta_1 \delta_2 \) where \( \delta_1 \) is a product of \( \overline{\gamma}_i \) and \( \delta_2 \) a product of \( \overline{\gamma}_j \). So one has 
\[ \overline{\gamma}_j^{-1} \gamma_{i_1}^{-1} \delta_1 \gamma_j \overline{\gamma}_i = \overline{\gamma}_j \gamma_i \] 
and hence \( \gamma_{i_1} = \gamma_{j_1} \). This equality, with the help of formula (*) and the fact that \( e_{i_1} \neq e_{j_1} \), gives \( \langle e_{i_1}, h \rangle = 0 \) for all \( h \in H_1(M, \mathbb{Z}) \). This cannot happen.

In fact theorem 1 says that there exists a geometric generator \( \gamma' \) such that \( \gamma_{i_1} \gamma', \gamma_{j_1} = \gamma', \gamma_{i_1} \gamma' \); if \( e' \) denotes the vanishing cycle associated to \( \gamma' \) one sees from formula (*) that this relation is equivalent to \( \langle e', e_{i_1} \rangle = \pm 1 \) and this concludes the proof.

**Corollary.** The set of points where \( \Delta \) is locally reducible is contained in the set of points where \( \Delta \) has smaller multiplicity than at the origin.

**Proof.** Let \( t' \in \Delta \) where \( \Delta \) has irreducible components \( \Delta_1, \ldots, \Delta_s \), \( s \geq 2 \). Then \( \pi^{-1}(t') \) has \( s \) singular points \( x_1, \ldots, x_s \) s.th. the multiplicity \( m_i \) of \( \Delta_i \) at \( t' \) is the number of vanishing cycles at \( x_i \), \( i = 1, \ldots, s \).

Choose \( L_{z_0} \) near \( t' \) and define \( \{z_1, \ldots, z_m\} = \Delta \cap L_{z_0} \), \( I = \{1, \ldots, m\} \), 
\[ I_\alpha = \{i \in I \mid z_i \in \Delta_\alpha\} \] 
\( \alpha = 1, \ldots, s \). Suppose \( \sum m_i = m \);

it follows that \( (I_\alpha) \) is a partition of \( I \). Let \( e_i \) denote the vanishing cycle at \( z_i \). One has \( \langle e_i, e_j \rangle = 0 \) for \( i \in I_\alpha \), \( j \in I_\beta \) and \( \alpha \neq \beta \); in fact...
$e_1, e_j$ have representative cycles lying in disjoint balls around $x_1, x_j$ respectively. This cannot happen because of the theorem 2, so that the multiplicity $\sum_{\alpha=1}^s m_{\alpha}$ of $\Delta$ at $t'$ must be less than $m$.

Remark. - This result can be expressed in the following way: if a deformed fibre of a hypersurface isolated singularity has more than one singularity, then the direct sum module of vanishing cycles at those singularities is a proper submodule of that of vanishing cycles at the original singularity.

BIBLIOGRAPHIE


[2] D. CHENIOT "Théorèmes de Zariski et de Van Kampen sur $\pi_1(\mathbb{P}_{\mathbb{C}})$" this volume.


