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Preliminary notes for the talk "Traces of analytic solutions of the heat equation"

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PRELIMINARY NOTES FOR THE TALK "TRACES OF ANALYTIC SOLUTIONS OF THE HEAT EQUATION"

by N. Aronszajn

Preface

In this paper necessary preliminaries are given for understanding the talk "Traces of analytic solutions of the heat equation." Since the talk is restricted to one hour, it is hoped that the audience will have had a chance to look over these preparatory notes.

No proofs are given here. Some of the proofs in Chapter I and all the proofs for Chapter II will be published in a forthcoming monograph on polyharmonic functions which is referred to in this paper as PHF.

Contents.

CHAPTER I.
GENERAL ANALYTIC FUNCTIONS.

§1. Notations.

The notations used here are chosen in order to abbreviate the formulas in this paper. It should be noted that they differ in many respects from notation of other authors working in related branches of mathematics.

$\mathbb{R}^n$ is considered as the real subspace of $\mathbb{C}^n$ (the imaginary subspace is $i\mathbb{R}^n$). For a vector $x \in \mathbb{C}^n$, $x = (x_1, x_2, \cdots, x_n)$, we put

$$|x| = \sum_{k=1}^{n} (|x_k|^2)^{1/2}, \quad \|x\| = \max_{k=1,\cdots,n} |x_k| .$$

$$x^2 = \sum_{k=1}^{n} x_k^2$$

For two vectors x and y in $\mathbb{C}^n$ we put

$$(x, y) = \Sigma x_k \overline{y_k}, \qquad (xy) = \Sigma x_k y_k$$

For $x \in \mathbb{C}^n$ and α a complex number we write

$$x^\alpha = (x^2)^{\alpha/2}, \quad x|^\alpha = \prod_{k=1}^{n} x_k^\alpha$$

If $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{C}^n$, we write also $x|^\alpha = \prod_{k=1}^{n} x_k^{\alpha_k}$.

For nonnegative integers we adopt a notation which in one respect is not consistent with that for general complex numbers. To avoid any misunderstanding, we will reserve the small letters j, k, ℓ, m, n, p, q, for nonnegative integers or finite systems of such (unless otherwise indicated). If k is a system, $k = \{k_j\}$ we write

$$|k| = \sum_j |k_j| .$$

We maintain the notation $k|^\alpha$ as for general complex systems and write

$$k! = \prod_j (k_j!) .$$

We will denote by $B^n_R(x^0)$ the open ball $[x \in \mathbb{R}^n:\ |x-x^0| < R]$; its boundary will be denoted by $S^{n-1}_R(x^0)$. If $x^0 = 0$ we will just write B^n_R and S^{n-1}_R. We will use the same notation in $\mathbb{C}^n$ which will be identified with $\mathbb{R}^{2n}$.

§2. Entire functions.

We will adopt the following definitions of exponential order and type for entire functions $f(z)$ of n complex variables, $z = (z_1, \cdots, z_n)$.

$f(z)$ is of exponential order ρ, $0 \leqq \rho < \infty$ if for every $\varepsilon > 0$ there exists a constant C_ε such that

$$|f(z)| \leqq C_\varepsilon\, e^{\|z\|^{\rho+\varepsilon}} \quad \text{for all } z \in \mathbb{C}^n. \tag{1}$$

The smallest ρ of this kind is called the least exponential order of f.

$f(z)$ is of exponential order ρ and type τ, $0 < \rho < \infty$, $0 \leqq \tau < \infty$, if for every $\varepsilon > 0$ there exists a constant C'_ε such that

$$|f(z)| \leqq C'_\varepsilon\, e^{(\tau+\varepsilon)\|z\|^\rho} \quad \text{for all } z \in \mathbb{C}^n. \tag{2}$$

If ρ is the least order of f then the smallest τ of this kind (if it exists) is called the least type of f.

Every entire function will be considered as of exponential order ∞ and type 0.

Proposition I. *If f is of order ρ (with type τ) then for every compact $K \subset \mathbb{C}^n$ and every $\varepsilon > 0$, there exists a constant $C_{K,\varepsilon}$ such that*

$$|f(z)| \leqq C_{K,\varepsilon}\, e^{\|z-x\|^{\rho+\varepsilon}}$$

or

$$|f(z)| \leqq C_{K,\varepsilon}\, e^{(\tau+\varepsilon)\|z-x\|^\rho}$$

for every $x \in K$, $z \in \mathbb{C}^n$.

THEOREM III. Let $f(z) \equiv f(z';z'')$ be analytic in $G \subset \mathbb{C}^n$ and let the set $A \subset G \cap \mathbb{R}^n$ be such that at every $x \in A$ the restriction of f to z'' is an entire function. Then if A is of positive Lebesgue measure in $\mathbb{R}^n$ the restriction of f to z'' at every point $x \in G$, is entire.[1]

As a consequence, under assumptions of the theorem the full analytic continuation of $f(z)$ has for domain $G' \times \mathbb{C}^{n-m}$ where G' is in general a covering manifold over $\mathbb{C}^m$.

§3. A new type of Cauchy formula; Almansi expansion.

Consider the ball $B^n_R \subset \mathbb{R}^n$ and the corresponding harmonicity cell $\tilde{B}^n_R \subset \mathbb{C}^n$ (the general definition of harmonicity cells will be given in Chapter II, §1.). $\tilde{B}^n_R$ can be defined as the set of all $z = x + iy$, x and y in $\mathbb{R}^n$, such that

$$x^2 + y^2 + 2\sqrt{x^2y^2-(xy)^2} < R^2 .$$

$\tilde{B}^n_R$ is a circled convex domain and the extreme points of the closure $\bar{\tilde{B}}^n_R$ form a real analytic n-dimensional manifold Γ^n_R consisting of all points $z = \xi\theta$ with $\theta \in S^{n-1}_1$ and $\xi \in S^1_R \subset \mathbb{C}^1$.

THEOREM I. Let $f(z)$ be analytic regular in G. If $\bar{\tilde{B}}^n_R \subset G$ then for $x \in \tilde{B}^n_R$ we have

$$f(x) = \frac{1}{2\pi i \omega_n} \int_{S^1_R} \int_{S^{n-1}_1} \frac{\xi^{n-1}}{(\xi\theta - x)^n} f(\xi\theta)\, d\sigma(\theta)\, d\xi \,, \tag{1}$$

where ω_n is the area of S^{n-1}_1, $d\sigma(\theta)$ is the area element on S^{n-1}_1. Another form of this formula in terms of exterior differential forms is

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[1] The proof is given in PHF by using analytic capacities introduced there. A somewhat weaker version of the theorem can be obtained by using results of P. Lelong [see Ann. Inst. Fourier, XI (1961) pp. 515-562].

Proposition. II. Let $x \in \mathbb{C}^n$. An entire function $f(z)$ is of exponential order ρ (type τ) if and only if for $\forall\ \epsilon > 0$, $\exists C_{x,\epsilon}$ such that

$$|D^k f(x)|^{1/|k|} \leq C_{x,\epsilon}\ (k!)^{1/|k|}\ |k|^{-1/(\rho+\epsilon)}$$

$$\text{or}\quad |D^k f(x)|^{1/|k|} \leq C_{x,\epsilon}\ (k!)^{1/|k|}\ |k|^{-1/\rho}\ (e\rho\,(\tau+\epsilon))^{1/\rho}$$

for every derivative D^k.[1] If for $C_{x,\epsilon}$ we choose the smallest constant satisfying this relation, then for fixed ϵ, as function of x, $C_{x,\epsilon}$ is locally bounded on $\mathbb{C}^n$.

Remark. Our definition of exponential order and type does not correspond to the classical definitions and was introduced in order to simplify the relations between exponential and Laplacian order and type. The classical order and type are more akin to the least order and type. The exact connection between classical and our terminilogy is as follows:

Classical Terminology		Terminology of this paper
Order ρ maximal type	$\Longleftrightarrow$	Least order ρ without type
Order ρ type τ	$\Longleftrightarrow$	Least order ρ with least type τ
Order ρ of minimal type	$\Longleftrightarrow$	Least order ρ with type 0.

The next theorem deals with general entire functions.

Let $f(z)$ be analytic in $G \subset \mathbb{C}^n$. Divide the variables $z_1, \cdots, z_n$ into two groups $z' = (z_1, \cdots, z_m)$ and $z'' = (z_{m+1}, \cdots, z_n)$. Thus $z = (z';z'')$ (similarly, $x = (x';x'')$ etc.). Consider $x \in G$. In a small neighborhood consider the function $f(x', z'')$ as function of z'' for fixed x'. This function will be called the restriction of f at x to z''.

1. For $k = (k_1, \cdots, k_n)$ $D^k = \dfrac{\partial^{|k|}}{\partial z_1^{k_1} \cdots \partial z_n^{k_n}}$.

$$(1') \qquad f(x) = \frac{1}{2\pi i\omega_n} \int_{\Gamma_R^n} \frac{f(z)}{(z-x)^n} \, dz_1 \wedge \cdots \wedge dz_n \,. \quad ^{1.}$$

Remark 1. 1° The manifold Γ_R^n covers twice its geometric image in $\mathbb{C}^n$; 2° for n odd we determine the value of $(z-x)^{-n} = ((z-x)^2)^{-n/2}$ for $z = \zeta\theta \in \Gamma_R^n$, $x \in \tilde{B}_R^n$ by choosing the positive value when $\zeta = R$ (i.e. $z = R\theta \in S_R^{n-1} \subset \mathbb{R}^n$ and $x \in B_R^n \subset \mathbb{R}^n$. It is then easy to prove that by analytic continuation $(z-x)^{-n}$ becomes a single-valued function on $\Gamma_R^n \times \tilde{B}_R^n$ and there is no ambiguity in (1) or (1').

As an immediate consequence of Theorem I we obtain

THEOREM II. (Almansi development). *If f is regular in $\tilde{B}_R^n + x^0$, $x^0 \in \mathbb{C}^n$ there exists a unique development, the Almansi development at x^0*

$$(2) \qquad f(x) = \sum_{k=0}^{\infty} (x-x^0)^{2k} h_k(x^0;x),$$

where $h_k(x^0;x)$ is a harmonic function in the complex variables x. Formula (2) is valid for $x-x^0 \in \tilde{B}_R^n$, the series in (2) converging absolutely and uniformly on compacts. The Almansi coefficients $h_k(x^0;x)$ can be obtained for any R_1, $0 < R_1 < R$ and $x \in \tilde{B}_{R_1}^n$ by the formula

$$(3) \qquad h_k(x^0;x) = \frac{1}{2\pi i\omega_n} \int_{\Gamma_{R_1}^n} \frac{z^2-(x-x^0)^2}{z^{2k+2}} \, \frac{f(z+x^0)}{(z-x+x^0)^n} \, dz_1 \wedge \cdots \wedge dz_n \,.$$

By using the Laplace operator in complex variables, $\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$ we obtain the useful formula

$$(4) \qquad \Delta^k f(x^0) = 4^k k! \, \frac{\Gamma(k+(n/2))}{\Gamma(n/2)} \, h_k(x^0;x^0) \quad \text{for } k = 0, 1, 2, \cdots,$$

and any x^0 in the domain of regularity of f .

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1. A short proof can be given by using exterior differential forms and calculus of residues in $\mathbb{C}^n$ (see J. Leray, Bull. Soc. Math. de France, 87(1959), pp. 81-180). In PHF a longer but more elementary proof leading to (1) is given.

§4. Analytic functionals.

For a domain $G \subset \mathbb{C}^n$ we denote by $\mathcal{H}(G)$ the class of all analytic regular functions in G with the Montel topology of uniform convergence on compacts. The class $\mathcal{H}'(G)$ of continuous linear functionals on $\mathcal{H}(G)$, commonly called analytic functionals was well studied in recent years and we will stress here only a few needed facts.

If the domains G_k satisfy $G_k \subset G_{k+1}$ and $\bigcup_k G_k = G$ then $\bigcup_k \mathcal{H}'(G_k) = \mathcal{H}'(G)$. We will be especially interested in the case when G is a polycylinder

$$P^n_r(x^0) = [z \in \mathbb{C}^n;\ |z_k - x^0_k| < r_k],\ r = (r_1, \cdots, r_n)\ .$$

In this case the analytic functionals in $\mathcal{H}'(P^n_r(x^0))$ can be identified with the class of all analytic functions regular on the closed polycylinder $\bar{P}^n_{1/r}(x^0)$, $\frac{1}{r} = \left(\frac{1}{r_1}, \cdots, \frac{1}{r_n}\right)$. The corresponding scalar product $\langle f, F \rangle$, $f \in \mathcal{H}(P^n_r(x^0))$, $F \in \mathcal{H}'(P^n_r(x^0))$ is given by

$$\text{(1)} \qquad \int_{\Pi^n_{r'}(0)} f(z + x^0) F\left(x^0_1 + \frac{1}{z_1}, \cdots, x^0_n + \frac{1}{z_n}\right) dz_1 \cdots dz_n$$

where $r' = (r'_1, \cdots, r'_n)$, $0 < r'_k < r_k$, the function $F(z)$ being regular on the closed polycylinder $\bar{P}^n_{1/r'}(x^0)$ and

$$\Pi^n_{r'}(0) = [z \in \mathbb{C}^n;\ |z_k| = r'_k]\ .$$

The integral in (1) is independent of the choice of r' satisfying the prescribed condition.

§5. Symbolic integrals.

To introduce symbolic integrals we first consider them in the elementary case of functions $u(x)$ defined on $\mathbb{R}^n$ and then pass to the case of analytic functions $u(x)$ defined in some domain of $\mathbb{R}^n$ or of $\mathbb{C}^n$.

1.° (elementary case). Let $u(x)$ be a locally integrable **function** on $\mathbb{R}^n$. For any point $x^0 \in \mathbb{R}^n$, put

$$M_u(x^0, R) = \frac{1}{\omega_n} \int_{S_1^{n-1}} u(x^0 + R\theta)\, d\sigma(\theta) . \tag{1}$$

For fixed u and x^0, $M_u(x^0, R)$ exists and is finite for almost all R and we write

$$\mathop{\text{\S}\!\!\int}_{x^0} u(x)dx = \int_0^{\infty} \omega_n R^{n-1} M_u(x^0, R)dR \tag{2}$$

when the last integral exists and is finite.

2.° (The case of analytic functions.) Let $u(x)$ be an analytic function in a domain $D \subset \mathbb{R}^n$ (or in $G \subset \mathbb{C}^n$). For any x^0 in its domain, define $M_u(x^0, R)$ by (1) for sufficiently small R. $M_u(x^0, R)$ is an even analytic function of the real variable R. If it has an analytic continuation to the whole real R-axis and if the integral in (2) exists and is finite we define again $\int_{x^0} u(x)dx$ by (2).

Going back to the elementary case we notice that if $u \in L^1(\mathbb{R}^n)$ then

$$\mathop{\text{\S}\!\!\int}_{x^0} u(x)dx = \int_{\mathbb{R}^n} u(x)dx,$$

hence $\mathop{\text{\S}\!\!\int}_{x^0} u(x)dx$ is independent of x^0. This property is essential for what we will call the symbolic integral of u denoted by $\text{\S}\!\!\int u(x)dx$. We will give sufficient conditions for this property to hold and will consider in the sequel the symbolic integral only when these conditions are satisfied.[1]

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[1] This restriction is sufficient for our present purposes, but in future applications it may well be desirable to consider less restrictive conditions for the symbolic integral.

The function u(x) for which we will consider $\S$ will often depend on several other variables besides the variables of integration. These variables will be considered as parameters varying in a domain in some $\mathbb{C}^N$. The uniformity condition will refer also to these parameters. In our applications we will need the symbolic integrals essentially for analytic functions u(x) in some domain $D \subset \mathbb{R}^n$ or $G \subset \mathbb{C}^n$. We will therefore define the uniformity conditions for analytic functions and the reader can easily find the simplification and changes needed in the elementary case.

Uniformity Condition. Let u be an analytic function defined in some domain $D \subset \mathbb{R}^n$, $n \geq 2$, or $G \subset \mathbb{C}^n$ depending also on N parameters y_j varying in a domain $E \subset \mathbb{C}^N$. We will say that u satisfies the uniformity condition if for every compact $K \subset D$ and every compact $K' \subset E$ the integral

$$\int_0^\infty \omega_n R^{n-1} |M_u(x^0, R)|\, dR$$

is uniformly bounded for $x^0 \in K$ and the parameters lying in K'.

For $n = 1$ the uniformity condition will mean: if u(x) is analytic in an interval $D \subset \mathbb{R}^1$ (or $G \subset \mathbb{C}^1$,) we will require that $u(x^0 + \xi)$ has an analytic continuation on the whole real axis ξ and that $\int_{-\infty}^{\infty} |u(x^0 + \xi)|\, d\xi$ be uniformly bounded for x^0 in any compact in the domain of u and any compact of the domain of parameters.

THEOREM I. If u satisfies the uniformity conditions then

a) $\S_{x^0} u(x)dx = \S_{x^1} u(x)dx$ for any x^0 and x^1 in the domain of u.

b) if u is holomorphic in its parameters in their domain, then the $\S\, u(x)dx$ is holomorphic in the parameters.

Remark 1. The necessity of changing the definition of uniformity condition for $n = 1$ can be traced to the fact that S_1^0 is disconnected. With our definition Theorem I is trivial for $n = 1$. A simple example in the elementary case of $u(x) = \operatorname{sgn} x$ for $x \in \mathbb{R}^1$ shows that in this case Theorem I, a) wouldn't be true if we accepted the same uniformity condition for $n = 1$ as for $n > 1$.

Remark 2. It may happen that the uniformity condition is not valid in the whole domain D (or G) and in the whole of E, but is valid in smaller domains D' (or G') and E'. Then we can use the symbolic integral and Theorem I for u restricted to the smaller domain with parameters restricted to the smaller domain.

Example 1. $u(x)$ is harmonic in a domain $D \subset \mathbb{R}^n$, $n \geqq 2$. Here $\int_{x^0} u(x)dx$ exists (and $= 0$) if and only if $u(x^0) = 0$; the uniformity condition is not satisfied.

Example 2. Consider again $u(x)$ harmonic in D and put $v(x) = u(x)e^{-(x-z)^2/4t}$ [1]. Here t and z are parameters, $t \in \mathbb{C}_+^1 = [t \in \mathbb{C}^1, \operatorname{Re} t > 0]$, $z \in \mathbb{C}^n$, The uniformity condition now can be assured in general for $x^0 \in D$, $t \in \mathbb{C}_+^1$ and $z \in \tilde{D}$, where $\tilde{D} \subset \mathbb{C}^n$ is the harmonicity cell of D (see Chapter II, §1.).

We pass now to multiple symbolic integrals. Let the function u depend on several systems of variables, say x_j, y_k, etc. (possibly with some additional variable parameters). We could consider as a multiple symbolic integral the result of symbolic integration successively

1. This example is connected with the theory of analytic solutions of the heat equation (see Chapter III).

with respect to each system of variables (at each step all the remaining non-integrated variables being considered as parameters). However, to assure that the so-defined multiple S-integrals have certain elementary properties we impose the following conditions:

<u>Uniformity conditions for multiple S-integrals.</u>[1] Let u depend on two systems of variables x and y . Joining y (or x) to the parameters we define $M_u(x^0, R_1)$ (or $M_u(y^0, R_2)$), and assume the corresponding uniformity conditions. In addition we define

$$(3) \qquad M_u(x^0, y^0, R_1, R_2) = \frac{1}{\omega_m \omega_n} \int_{S_1^{m-1}} \int_{S_1^{n-1}} u(x^0 + R_1 \theta_1, y^0 + R_2 \theta_2) d\sigma(\theta_1) d\sigma(\theta_2)$$

$$(4) \qquad \int_{x^0} \int_{y^0} u(x, y) dx dy = \omega_m \omega_n \int_0^\infty \int_0^\infty R_1^{m-1} R_2^{n-1} M_u(x^0, y^0, R_1, R_2) dR_1 dR_2$$

We add then the assumption that for every compact K_1 in the domain of x, every compact K_2 in the domain of y, and every compact K' in the domain of parameters, the integral

$$\omega_m \omega_n \int_0^\infty \int_0^\infty R_1^{m-1} R_2^{n-1} |M_u(x^0, y^0, R_1, R_2)| dR_1 dR_2$$

is uniformly bounded for $x^0 \in K_1$ $y^0 \in K_2$ and the parameters in K'.[2]

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[1] We state them here only for two sets of integration variables. For more such sets, it is clear how the condition should be formulated.

[2] The conditions are stated here for both dimensions m and $n \geqq 2$. If one of them or both are = 1, the uniformity conditions should be changed in a manner similar to that which was done for single S-integrals.

THEOREM II. Let $u(x, y)$ satisfy the uniformity conditions for double S-integrals, then

a) for every x^0 in the domain of x and y^0 in the domain of y we have

$$\mathop{\mathcal{S}}_{x^0}\mathop{\mathcal{S}}_{y^0} u(x,y)dxdy = \mathcal{S}\left(\mathcal{S}\, udx\right)dy = \mathcal{S}\left(\mathcal{S}\, udy\right)dx .$$

This expression will be called the double S-integral of u and denoted by $\mathcal{S}\mathcal{S}\, udxdy$.

b) if u is holomorphic in its parameters then so also is $\mathcal{S}\mathcal{S}\, udxdy$.

For a function $u(x, y)$ we can join the two systems of variables into one $(x;y)$ and consider the corresponding single S-integral by defining

$$M_u((x^0;y^0), R) = \frac{1}{\omega_{m+n}} \int_{S_1^{m+n+1}} u((x^0;y^0)+R\theta)d\sigma(\theta).$$

THEOREM III. Let the uniformity conditions for double S-integrals be satisfied for $u(x, y)$. Then $\mathcal{S}\, u\, d(x;y)$ exists and

$$\mathcal{S}\, u\, d(x;y) = \mathcal{S}\mathcal{S}\, u\, dxdy .$$

We come finally to the notion of convolution for two analytic functions u_1 and u_2 regular in domains G_1 and G_2 respectively of $\mathbb{C}^n$.

We consider the function $u_1(x)u_2(z-x)$ with parameters $z \in \mathbb{C}^n$ in the domain G_z where the function is defined. Clearly $G_z = G_1 \cap (z-G_2)$. The set Ω of all z's such that $G_z \neq \emptyset$ is open. For each $z^0 \in \Omega$ if we consider the ball in $\mathbb{C}^n$, $B_R^{2n}(z^0)$, for sufficiently small R the set $G_R(z^0) = \bigcap_{z \in B_R^{2n}(z^0)} G_z$ is open, non-empty and the function $u_1(x)\, u_2(z-x)$ is defined for all $x \in G_R(z^0)$ and $z \in B_R^{2n}(z^0)$. We assume that for each component of $G_R(z^0)$

the uniformity condition is satisfied with parameter z in $B_R^{2n}(z^0)$. We say then that u_1 and u_2 are convolutable and put

(5) $$(u_1 * u_2)(z) = \oint u_1(x)u_2(z-x)dx .$$

The convolution $u_1 * u_2$ is in general multivalued due to the fact that $G_R(z^0)$ may not be connected. However, if G_1 and G_2 are convex $u_1 * u_2$ forms a single-valued locally holomorphic function defined on the whole of Ω.

THEOREM IV. *If* $u_1 * u_2$ *exists, then* $u_1 * u_2 = u_2 * u_1$. *By a similar proceedure, we can take a finite system of functions* $u_1, \cdots, u_m$ *holomorphic on some domains of* $\mathbb{C}^n$ *and put*

(5') $$(u_1 * u_2 * \cdots * u_m)(z) =$$
$$= \oint u_1(x^1)u_2(x^2)\cdots u_{m-1}(x^{m-1})\, u_m(z-x^1-\cdots-x^{m-1})dx^1dx^2\cdots dx^{m-1} .$$

if the symbolic integral exists.

THEOREM V. *If* $u_1 * u_2 * \cdots * u_m$ *exists then*

$$u_1 * u_2 * \cdots * u_m = u_1 * (u_2 * \cdots * u_m) = (u_1 * u_2) * (u_3 * \cdots * u_m) .$$

CHAPTER II.
POLYHARMONIC FUNCTIONS.

§1. General definitions and properties; Laplacian order and type.

Consider a C^∞ function $u(x)$ in a domain $D \subset \mathbb{R}^n$.

Definition 1. u is of Laplacian order ρ in D $(0 \leq \rho < \infty)$ if for every compact $K \subset D$ and $\varepsilon > 0$ there exists a constant $C_{K,\varepsilon}$ such that $|\Delta^p u(x)| \leq C_{K,\varepsilon}(2p)!^{1-\frac{1}{\rho+\varepsilon}}$ for $x \in K$ and $p = 1, 2, 3, \cdots$.

Definition 2. u is in D of Laplacian order ρ with type $(0 < \rho \leqq \infty)$ if for every compact $K \subset D$ there exists $M_K < \infty$ such that $|\Delta^p u(x)| \leqq M_K^{2p} (2p)!^{1-(1/\rho)}$ for $x \in K$, $p = 1, 2, \cdots$.

Definition 3. u is in D of Laplacian order ρ and type M, $0 < \rho \leqq \infty$, $0 \leqq M < \infty$, if for every compact $K \subset D$ and every $\varepsilon > 0$ there exists a constant $C_{K,\varepsilon}$ such that $|\Delta^p u(x)| \leqq C_{K,\varepsilon} (M+\varepsilon)^{2p} (2p)!^{1-(1/\rho)}$, for $x \in K$, $p = 1, 2, \cdots$.

THEOREM I. *u is in D of order ∞ with type if and only if u is analytic in D.*[1]

Functions of infinite Laplacian order and type 0 are called *polyharmonic functions*; such functions form a subclass of analytic functions.

Functions satisfying $\Delta^k u = 0$ are called polyharmonic *of degree* k; they are all polyharmonic of Laplacian order 0.

Definition 4. Let D be a domain in $\mathbb{R}^n$. We denote by $\mathcal{P}^{\rho}(D), \mathcal{P}^{\rho,\infty}(D)$, $\mathcal{P}^{\rho,M}(D)$, $\mathcal{P}^{(k)}(D)$ the class of functions which are in D of Laplacian order ρ, of Laplacian order ρ with type, of Laplacian order ρ and type M, and polyharmonic of degree (k) respectively.

THEOREM II. *Let* $k' < k''$, $0 \leqq M' < M'' < \infty$ *and* $0 < \rho' < \rho'' \leqq \infty$. *We have* $\mathcal{P}^{(k')}(D) \subset \mathcal{P}^{(k'')}(D) \subset \mathcal{P}^{0}(D) \subset \mathcal{P}^{\rho',M'}(D) \subset \mathcal{P}^{\rho',M''}(D) \subset \mathcal{P}^{\rho',\infty}(D)$ $\subset \mathcal{P}^{\rho'}(D) \subset \mathcal{P}^{\rho'',0}(D) \subset \mathcal{P}^{\infty,M'}(D) \subset \mathcal{P}^{\infty,M''}(D) \subset \mathcal{P}^{\infty,\infty}(D)$.

$\mathcal{P}^{\infty,\infty}(D)$ *is the class of all analytic functions in* D *whereas* $\mathcal{P}^{\infty,0}(D)$ *is the class of all polyharmonic functions in* D.

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1. This theorem was alreadv proved in N. Aronszajn, *Acta Math.*, 1935.

Since all the functions in which we are interested are analytic we can extend all our previous definitions _verbatim_, replacing D by a domain $G \subset \mathbb{C}^n$, Δ being the Laplacian in n-complex variables.

Definition 5. The set of all points $x \in \mathbb{C}^n$ satisfying $(x-x^0)^2 = 0$ is called the isotropic cone with vertex x^0, and is denoted by $V(x^0)$.

THEOREM III. _For every domain $D \subset \mathbb{R}^n$ there exists a unique domain $\tilde{D}$ in $\mathbb{C}^n$ such that every polyharmonic function in D has an unrestricted analytic continuation in $\tilde{D}$ (which may be multivalued), $\tilde{D}$ being the largest domain with this property. $\tilde{D}$ is the largest such domain even if we restrict the functions to be harmonic in D (i.e. $\in \wp^{(1)}(D)$). If $\rho < \infty$ and u belongs to $\wp^{\rho}(D)$, $\wp^{\rho,\infty}(D)$, or $\wp^{\rho,0}(D)$, then the extension of u to $\tilde{D}$ belongs to the same classes in $\tilde{D}$. $\tilde{D}$ is obtained as the component containing D of the open set $\mathbb{C}^n \setminus \bigcup_{x^0 \in \mathbb{R}^n \setminus D} V(x^0)$._

$\tilde{D}$ is called the _harmonicity cell_ of D.

In §3, Chapter I we defined directly the harmonicity cell $\tilde{B}^n_R$ of the ball B^n_R.

THEOREM IV. _If $u \in \wp^{\rho,\infty}(D)$ and for some subdomain $D' \subset D$, the restriction $u|_{D'}$ belongs to $\wp^{\rho,0}(D')$, then $u \in \wp^{\rho,0}(D)$; in particular, if u is analytic in D and polyharmonic in D' then it is polyharmonic in D._

Definition 6. A function u defined in a domain $D \subset \mathbb{R}^n$ (or $G \subset \mathbb{C}^n$) is said to be locally of order ρ or order ρ and type $M < \infty$ at a point $x^0 \in D$ (or $\in G$) if for every $\varepsilon > 0$ there exists a neighborhood of x^0, $U_\varepsilon(x^0) \subset D$ (or $\subset G$) such that $u|_{U_\varepsilon(x^0)}$ belongs to $\wp^{\rho+\varepsilon}(U_\varepsilon(x^0))$ or $\wp^{\rho,M+\varepsilon}(U_\varepsilon(x^0))$ respectively.

THEOREM V. u belongs to $\mathfrak{P}^{\rho}(D)$ or $\mathfrak{P}^{\rho,M}(D)$ if and only if at every point $x^0 \in D$, u is locally of order ρ or order ρ and type M respectively. $u \in \mathfrak{P}^{\rho,\infty}(D)$ if and only if at every $x^0 \in D$ u is locally of order ρ and some finite type.

§2. Almansi development of polyharmonic functions.

In Chapter I, §3 we described the Almansi development for general analytic functions. We will give here the special properties of this development in case of polyharmonic functions.

THEOREM I. If u is a polyharmonic function in a domain $G \subset \mathbb{C}^n$ then for every $x^0 \in G$, the Almansi development

$$u(x) = \sum_{k=0}^{\infty} (x-x^0)^{2k} h_k(x^0;x) , \tag{1}$$

is absolutely and uniformly convergent in the largest star-domain with center at x^0 contained in G.

The next theorem allows one to characterize functions of different classes $\mathfrak{P}$ in G for a star-domain G centered at x^0 by the behavior of the harmonic coefficients $h_k(x^0;x)$.

THEOREM II. Let $G \subset \mathbb{C}^n$ be a star-domain with center x^0.

a) $u(x) \in \mathfrak{P}^{\rho}(G)$, $0 \leq \rho < \infty$, if and only if for every compact $K \subset G$ and $\varepsilon > 0$ $\exists$constant $C_{K,\varepsilon}$ such that

$$|h_k(x)| \leq C_{K,\varepsilon} (2k)!^{-\frac{1}{\rho+\varepsilon}}, \quad \text{for } x \in K \text{ and } k = 0,1,\cdots.$$

b) $u(x) \in \mathfrak{P}^{\rho,\infty}(G)$, $0 < \rho < \infty$, if and only if for every compact $K \subset G$ there exist constants C_K and M_K such that

$$|h_k(x)| \leq C_K M_K^{2k} (2k)!^{-1/\rho} \text{ for } x \in K,\ k = 0,1,2,\cdots.$$

c) $u(x) \in \mathfrak{P}^{\rho,0}(G)$, $0 < \rho \leq \infty$, if and only if for every compact $K \subset G$ and every $\varepsilon > 0$ there exists a constant $C_{K,\varepsilon}$ such that

$$|h_k(x)| \leqq C_{K,\varepsilon}\, \varepsilon^{2k}(2k)!^{-1/\rho} \quad \underline{\text{for}} \quad x \in K,\ k = 0, 1, 2 \cdots .$$

d) $u(x) \in \wp^{\rho, M}(G)$, $0 < \rho < \infty$, $0 < M < \infty$, <u>if and only if for every compact</u> $K \subset G$ <u>and every</u> $\varepsilon > 0$ <u>there exists a constant</u> $C_{K,\varepsilon}$ <u>such that</u>

$$|(x-x^0)^{2p-2k}\Delta^p((x-x^0)^{2k}h_k(x^0;x))| \leqq C_{K,\varepsilon}(M+\varepsilon)^{2k}\frac{(2k)!^{1-(1/\rho)}}{(2k-2p)!}$$

<u>for every</u> $x \in K$, $k = 0, 1, 2, \cdots$ <u>and</u> $0 \leqq p \leqq k$.

§3. Relations between entire and polyharmonic functions.

There is a distinction in the nature of these relations between the case of one variable $(n = 1)$ and several variables $(n \geqq 2)$.

<u>THEOREM I.</u> <u>Let</u> $u(x)$ <u>be a function defined in a domain</u> $G \subset \mathbb{C}^1$.

a) u <u>is extendable to an entire function in</u> $\mathbb{C}^1$ <u>if and only if it is polyharmonic in</u> G.

b) u <u>is extendable to an entire function of exponential order</u> $\rho < \infty$ <u>if and only if</u> $u \in \wp^{\rho}(G)$.

c) u <u>is extendable to an entire function of exponential order</u> ρ <u>and type</u> $\tau < \infty$ <u>if and only if</u> $u \in \wp^{\rho, M}(G)$, <u>with</u> $M = (\tau\rho)^{1/\rho}$.

In case of several variables, the relations stated in the preceding theorem go only one way.

<u>THEOREM I'.</u> <u>Let</u> u <u>be an entire function in</u> $\mathbb{C}^n$, $n \geqq 2$.

a) u <u>is polyharmonic in</u> $\mathbb{C}^n$.

b) <u>if</u> u <u>is of exponential order</u> $\rho < \infty$, <u>then it is of Laplacian order</u> ρ <u>in</u> $\mathbb{C}^n$.

c) <u>if</u> u <u>is of exponential order</u> $\rho < \infty$ <u>and type</u> $\tau < \infty$, <u>then</u> u <u>is of Laplacian order</u> ρ <u>and type</u> $M = \sqrt{n}(\tau\rho)^{1/\rho}$.

We pass now to the multiplication of polyharmonic functions by entire functions.

In general if $\mathfrak{F}$ is a class of functions defined on a set A, a function u defined on A is called a *multiplier for the class* $\mathfrak{F}$ if for every $f \in \mathfrak{F}$ also $uf \in \mathfrak{F}$.

THEOREM II. For every domain $G \subset \mathbb{C}^n$ u *is a multiplier for the class* $\wp^{\infty, 0}(G)$ *of all polyharmonic functions in* G *if and only if* u *is the restriction to* G *of an entire function in* $\mathbb{C}^n$.

THEOREM III. Let G *be a domain in* $\mathbb{C}^n$. *If* $u \in \wp^{\rho, M}(G)$, $\rho < \infty$, $M < \infty$ *and* v *is an entire function in* $\mathbb{C}^n$ *of exponential order* $\rho_1 < \infty$ *and type* $\tau < \infty$ *then for every founded domain* $G' \subset \overline{G}' \subset G$, $uv \in \wp^{\rho', M'}(G')$ *with* $\rho' = \max[\rho, 2\rho_1]$ *and* $M' = 3^{1/\rho'}[n(\rho_1\tau)^{2/\rho_1} + 2^{2+(2/\rho')}(\rho_1\tau)^{1/\rho_1}R^{-1} + M^2]^{1/2}$, *where* $R = \min[\|x-y\|: x \in \overline{G'}, y \in \partial G]$.

Remark. In the last theorem, the value of ρ' is the best possible in general but the evaluation of M' is possibly not the sharpest.

As immediate corollary of Theorem III we obtain:

Corollary IV. Let G *be a domain in* $\mathbb{C}^n$, $u \in \wp^{\rho}(G)$ *with* $\rho < \infty$ *and let* v *be an entire function of exponential order* $\rho_1 < \infty$. *Put* $\rho' = \max[\rho, 2\rho_1]$. *Then*

a) $uv \in \wp^{\rho'}(G)$;

b) *if* $u \in \wp^{\rho, \infty}(G)$ *and* v *is of exponential order* ρ *and finite type, then* $uv \in \wp^{\rho', \infty}(G)$.

c) *if* $u \in \wp^{\rho, 0}(G)$ *and* v *is of exponential order* ρ_1 *and type* 0 *then* $uv \in \wp^{\rho', 0}(G)$.

CHAPTER III.
ANALYTIC SOLUTIONS OF THE HEAT EQUATION.

§1. General properties and relations with polyharmonic functions of Laplacian order 2.

We will consider functions in a domain $G \subset \mathbb{C}^{n+1} = \mathbb{C}^n \times \mathbb{C}^1$. The first n variables $x_1, \cdots, x_n$ will be called the space variables and the last variable will be called the time variable t.

We will denote the class of analytic solutions of the heat equation

(1) $$\frac{\partial u}{\partial t} = \Delta u ,$$

regular in a domain $G \subset \mathbb{C}^n \times \mathbb{C}^1$ by $\mathfrak{A}\mathfrak{S}\mathcal{H}\mathcal{E}(G)$. If the domain G is of the form $\mathbb{C}^n \times D$ where D is a domain in $\mathbb{C}^1$ we will abbreviate the notation to $\mathfrak{A}\mathcal{H}(D)$.

If $u(x,t) \in \mathfrak{A}\mathfrak{S}\mathcal{H}\mathcal{E}(G)$ then for fixed t^0 the function of x, $u(x, t^0)$, will be called a section (space section) of u at t^0. It is defined in an open set of $\mathbb{C}^n$ which is not necessarily connected. By using Theorem III, §2, Chapter I, one obtains:

THEOREM I. *Let* $u \in \mathfrak{A}\mathfrak{S}\mathcal{H}\mathcal{E}(G)$. *If for a set* $A \subset \mathbb{R}^1$ *of positive Lebesgue measure, the sections of* u *for* $t^0 \in A$ *are all entire functions in* $\mathbb{C}^n$, *then* u *can be extended to a function in* $\mathfrak{A}\mathcal{H}(D)$ *for some domain* $D \subset \mathbb{C}^1$; *however, the extended function will be in general multivalued.*

We can consider the class Ξ of all analytic solutions of the heat equation in all domains $G \subset \mathbb{C}^n \times \mathbb{C}^1$ i.e. $\Xi = \bigcup_G \mathfrak{A}\mathfrak{S}\mathcal{H}\mathcal{E}(G)$. On the class Ξ we define several transformation groups. We give below a list of such transformation groups and describe their action on an element $u(x,t) \in \Xi$. The transformed element will be denoted by $v(x,t)$.

Group $\mathfrak{A}$. This group is formed by all affine orthogonal [1] transformations $a + T$ of $\mathbb{C}^n$ onto itself and

$$v(x,t) = u(a + Tx,\ t).$$

Group $\mathfrak{B}$ is the additive group of $\mathbb{C}^1$ and for $\tau \in \mathbb{C}^1$

$$v(x,t) = u(x, t + \tau).$$

Group $\mathfrak{C}$ is the multiplicative group $\mathbb{C}^1 \setminus (0)$. For $c \in \mathbb{C}^1 \setminus (0)$

$$v(x,t) = u(cx,\ c^2 t).$$

Group $\mathfrak{D}$ is the group of all 2×2 complex matrices $\begin{pmatrix} \alpha, & \beta \\ \gamma, & \delta \end{pmatrix}$ with $\alpha\delta - \beta\gamma = 1$ and for such a matrix

$$v(x,t) = (\gamma t + \delta)^{-n/2}\, e^{\frac{-\gamma x^2}{4(\gamma t + \delta)}}\, u\left(\frac{x}{\gamma t + \delta},\ \frac{\alpha t + \beta}{\gamma t + \delta}\right)$$

Group $\mathfrak{E}$ is the additive group $\mathbb{C}^n$ with the following action of an element $b \in \mathbb{C}^n$

$$v(x,t) = e^{(bx) + b^2 t}\, u(x + 2tb,\ t).$$

The transformation of Ξ corresponding to an element of the above groups will be denoted by $\mathfrak{A}(a + T)$, $\mathfrak{B}(\tau)$, $\mathfrak{C}(c)$, $\mathfrak{D}\begin{pmatrix} \alpha, & \beta \\ \gamma, & \delta \end{pmatrix}$, $\mathfrak{E}(b)$.

The groups $\mathfrak{A}$ and $\mathfrak{D}$ determine all the other transformations; in fact $\mathfrak{B}(\tau) = \mathfrak{D}\begin{pmatrix} 1, & \tau \\ 0, & 1 \end{pmatrix}$, $\mathfrak{C}(c) = c^{-n/2}\mathfrak{D}\begin{pmatrix} c, & 0 \\ 0, & 1/c \end{pmatrix}$, $\mathfrak{E}(b) = \left[\mathfrak{D}\begin{pmatrix} 0, & 1/2 \\ -2, & 0 \end{pmatrix}\right]^{-1} \mathfrak{A}(b+I)\mathfrak{D}\begin{pmatrix} 0, & 1/2 \\ -2, & 0 \end{pmatrix}$

The transformations of each of the above groups transform functions defined in a domain G into functions defined on a well determined domain G' and they present a linear isomorphism between the vector spaces $\mathfrak{A}\mathfrak{S}\mathcal{N}\mathfrak{E}(G)$

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1. An orthogonal transformation T is a linear transformation of $\mathbb{C}^n$ onto $\mathbb{C}^n$ such that for every $x \in \mathbb{C}^n$, $(Tx)^2 = x^2$.

and $\mathfrak{ASHE}(G')$.

This last statement, however, is not literally true in case of the group $\mathfrak{D}$. The transformations of this group present two difficulties. The first is due to the fact that the transformation $\mathfrak{D}\begin{pmatrix}\alpha, \beta \\ \gamma, \delta\end{pmatrix}$ transforms the variable t by a fractional transformation which may send a finite value of t figuring in the domain G into infinity. To avoid this difficulty a transformation of group $\mathfrak{D}$ will be applied to functions defined in a domain G only if no point of G is transformed into infinity.

The second difficulty occurs only for odd dimension n when the transformed functions have the factor $(\gamma t + \delta)^{-n/2}$ which is a two-valued function. To avoid this difficulty we will restrict ourselves to domains G which are simply-connected and will consider the transformation in question as representing two distinct transformations of $\mathfrak{ASHE}(G)$ onto $\mathfrak{ASHE}(G')$ differing by the factor -1 .

THEOREM II. Under the restrictions described above and if we provide the linear spaces $\mathfrak{ASHE}(G)$ with the topology of uniform convergence on compacts,[1.] then the transformations of all groups described above are topological linear isomorphisms onto.

Remark 1. The restrictions imposed above on the nature of the domain G and the transformation $\mathfrak{D}\begin{pmatrix}\alpha, \beta \\ \gamma, \delta\end{pmatrix}$ avoid the difficulties but do not solve them in the general case. To solve them one would have to consider domains $G \subset \mathbb{C}^n \times \overline{\mathbb{C}^1}$,[2.] introduce the notion of regularity of a solution of the heat equation for $t = \infty$ (which can be done) and for

1. which makes $\mathfrak{ASHE}(G)$ into a Montel space.

2. $\overline{\mathbb{C}^1}$ is the Riemannian sphere $\mathbb{C}^1 \cup (\infty)$.

odd dimension n consider multivalued solutions.

We pass now to the characterization of the sections of the analytic solution of the heat equation.

THEOREM III. *A function* $v(x)$ *defined in a domain* $U \subset \mathbb{C}^n$ *is the section of a solution* $u \in \mathcal{A}\mathcal{S}\mathcal{H}\mathcal{E}(U \times B^2_R(t^0))$[1] *at the time* t^0 *if and only if* v *is polyharmonic in* U *of order* 2 *and type* $\frac{1}{\sqrt{2R}}$. *If this condition is satisfied then*

$$(2) \qquad u(x,t) = \sum_{p=0}^{\infty} \Delta^p v(x) \frac{(t-t^0)^p}{p!} .$$

Consider $u(x,t) \in \mathcal{A}\mathcal{S}\mathcal{H}\mathcal{E}(G)$. For any $(x^0;t^0) \in G$ we will call a *B-circle of* u *in the direction* θ *at* $(x^0;t^0)$ the largest circle $B^2_R(t^0 + Re^{i\theta})$ with the property that for any smaller circle $B^2_{R'}(t^0 + R'e^{i\theta})$ there exists a neighborhood U of x^0 in $\mathbb{C}^n$ such that u has a regular extension to $U \times B^2_{R'}(t^0 + R'e^{i\theta})$. We will call a *B-domain of* u *at* $(x^0;t^0)$ the union of t^0 and all the B-circles at $(x^0;t^0)$ in all directions θ.

We introduce the fundamental solution $E(x,t)$ of the heat equation

$$(3) \qquad E(x,t) = \frac{1}{(4\pi t)^{n/2}} e^{\frac{-x^2}{4t}} .$$

THEOREM III. *Let* $B^2_R(t^0 + R^{i\theta})$ *be the B-circle of* u *in the direction* θ *at* $(x^0;t^0)$. *Let* $0 < R' < R$ *and* U *be a circled domain in* $\mathbb{C}^n$ *centered at* x^0 *such that* u *is regular on* $U \times B^2_{R'}(x^0 + R'e^{i\theta})$. *Then if* $v(x)$ *is the restriction of the section of* u *at* t^0 *to* U *we have*

$$(4) \qquad u(z,t') = \mathop{\int\!\!\!\!\!\!S} v\left(e^{\frac{i\theta}{2}} x + x^0\right) E\left(e^{\frac{-i\theta}{2}} (z-x^0) - x; e^{-i\theta}(t'-t^0)\right) dx ,$$

the symbolic integral existing for $z \in U$ *and* $t' \in B^2_{R'}(t^0 + R'e^{i\theta})$.

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1. Here we identify $\mathbb{C}^1$ with $\mathbb{R}^2$.

In the proof of (4) we consider x^0, t^0 and θ as fixed, only z and t' as variable parameters and $x \in U-x^0$. Then we prove that (putting $x' = e^{(-i\theta)/2}(z-x^0)$)

$$u(z,t') = \oint_{x'} v\left(e^{\frac{i\theta}{2}} x + x^0\right) E(x-x',\ e^{-i\theta}(t'-t^0))dx .$$

This needs the Almansi development of $v\left(e^{\frac{i\theta}{2}} x + x^0\right)$ at x', formula (4), §3, Chapter I, Hadamard's theorem on multiplication of singularities and Borel transformation (that is where the name B-circle comes from). For $\oint_{x''}$ with $x'' \in U-x^0$ we use the transformation $\mathcal{E}\left(\frac{x''-x'}{2e^{-i\theta}(t'-t^0)}\right)$ to reduce it to the case $\oint_{x'}$. Finally we use the analytic capacity (already mentioned in §1, Chapter I) to obtain the uniformity condition relative to x'' and parameters z and t'.

Applying (4) to varying directions θ we obtain an expression of the solution u at all points $(x^0;t)$ with t in the B-domain of u at $(x^0;t^0)$ in terms of the section v at t^0.

§2. Developments in Hermite Polynomials.

We recall that Hermite polynomials in one variable x are given by

(1) $$H_k(x) = (-1)^k e^{x^2} \frac{\partial^k}{\partial x^k} e^{-x^2}, \quad k = 0,1,2,\dots .$$

$H_k(x)$ is a polynomial of order k with leading coefficient 2^k and all terms of the same parity.

We will consider Hermite polynomials in n variables defined for systems $k = (k_1, \cdots k_n)$ of nonnegative integers by

(2) $$H_k(x) = H_{k_1}(x_1)\cdots H_{k_n}(x_n) = (-1)^{|k|} e^{x^2} D^k e^{-x^2},$$

where $D^k = \frac{\partial^{k_1}}{\partial x_1^{k_1}} \cdots \frac{\partial^{k_n}}{\partial x_n^{k_n}}$.

It is well-known that the functions

(3) $$\psi_k(x) = \frac{1}{2^{|k|/2}(k)!^{1/2}\pi^{n/4}} e^{\frac{-x^2}{2}} H_k(x) ,$$

form an orthonormal complete system in $L^2(\mathbb{R}^n)$. This fact implies immediately the following proposition.

Proposition I. Let c be a strictly positive number. Then the functions

(4) $$\frac{1}{2^{|k|/2}(k)!^{1/2}\pi^{n/4}} H_k(cx)$$

form an orthonormal and complete system in $L^2(\mathbb{R}^n, c^n e^{-(cx)^2} dx)$.

The next proposition is proved immediately by using the differential equations characterizing the H_k's .

Proposition II. The function given by

(5) $$2^{|k|}(-t)^{|k|/2} H_k\left(\frac{x}{2(-t)^{1/2}}\right)$$

is a polynomial in x as well as t which is a solution of the heat equation with section at $t = 0$ equal to $2^{|k|} x|^k$.

THEOREM III. Let $u(x,t)$ be a solution of the heat equation regular at a point $(x^0;t^0)$, then

a) if t' belongs to the B-domain of u at $(x^0;t^0)$, the section of u at t' possesses, in a circled domain $U \subset \mathbb{C}^n$ with center x^0, a development in Hermite polynomials

(6) $$u(x,t') = \sum_{p=0}^{\infty} \sum_{|k|=p} A_k (t^0-t')^{|k|/2} H_k\left(\frac{x-x^0}{2\sqrt{t^0-t'}}\right) , \quad A_k = \frac{1}{k!} D_x^k u(x^0,t^0),$$

the series in p being uniformly convergent on compacts in U .

b) <u>if, in addition,</u> t^0 <u>belongs to the B-circle of</u> u <u>at</u> $(x^0;t')$ <u>in the direction</u> θ, <u>then the coefficients</u> A_k <u>can be expressed in terms of</u> $u(x,t')$ <u>by the formula</u>

$$(7) \qquad A_k = \oint u\left(e^{\frac{i\theta}{2}} x + x^0, t'\right) E(x, e^{-i\theta}(t^0-t')) \frac{1}{(2\sqrt{t^0-t'})^{|k|}} H_k\left(\frac{x}{2e^{-i\theta/2}\sqrt{t^0-t'}}\right) dx \, .$$

c) <u>if, furthermore, we assume that for some circled domain</u> $U_0 \subset C^n$ <u>with center</u> x^0 <u>and for certain</u> $R > \left|\frac{t'-t^0}{2}\right|$, <u>the solution u is regular in the domain</u> $U_0 \times B_R^2\left(\frac{t^0+t'}{2}\right)$ <u>then the infinite series in (6) is uniformly convergent on every compact in</u> U_0 <u>and formula (7) can be further simplified by choosing</u> $\theta = \mathrm{Arg}(t^0-t')$:

$$(7') \qquad A_k = \oint u\left(e^{\frac{i\theta}{2}} x + x^0, t'\right) E(x, |t^0-t'|) \frac{1}{(2\sqrt{t^0-t'})^{|k|}} H_k\left(\frac{x}{2|t^0-t'|^{1/2}}\right) dx \, .$$

<u>Remark1.</u> If the section $u(z,t')$ is regular on the whole hyperplane $e^{\frac{i\theta}{2}} \mathbb{R}^n + x^0$ and if, restricted to this hyperplane, it is in $L^2\left(\mathbb{R}^n, \left(\frac{1}{2|t^0-t'|^{1/2}}\right)^n e^{-x^2/(4|t-t'|)} dx\right)$, then formula (6) with its A_k's determined by (7') gives the standard development of the function $u\left(e^{\frac{i\theta}{2}} x + x^0, t'\right)$ in the orthonormal and complete system

$$\frac{1}{2^{|k|/2}(k!)^{1/2}\pi^{n/4}} H_k\left(\frac{x}{2|t^0-t'|^{1/2}}\right) .$$

<u>§3. The spaces $\mathcal{GX}$ and $\mathcal{GX}_R$ and their duals.</u>

We will be interested in a very special kind of spaces of the type $\mathcal{GX}(D)$ (see §1) namely, the spaces $\mathcal{GX}_R = \mathcal{GX}(B_R^2(R))$, $0 < R < \infty$ and the space $\mathcal{GX} \equiv \mathcal{GX}_\infty = \mathcal{GX}(\mathbb{C}_+^1)$.[1]

The proper functional spaces $\mathcal{GX}_R$ with their natural topology

1. $\mathbb{C}_+^1 = [z \in \mathbb{C}^1 : \mathrm{Re}\, z > 0]$.

(uniform convergence on compacts) are Montel spaces. For $R' < R \leqq \infty$ $\mathcal{G}\mathcal{X}_R$ is identified (by restriction) with a subspace of $\mathcal{G}\mathcal{X}_{R'}$ which is dense but not closed in $\mathcal{G}\mathcal{X}_{R'}$.

The transformations of the groups $\mathcal{G}$ and $\mathcal{E}$ are automorphisms for each $\mathcal{G}\mathcal{X}_R$. Transformations $\mathcal{C}(c)$ for real $c \neq 0$ form an isomorphism of $\mathcal{G}\mathcal{X}_R$ onto $\mathcal{G}\mathcal{X}_{R/c^2}$. Among the transformations of the group $\mathcal{D}$ we will single out $\mathcal{D}\begin{pmatrix} 1 & 0 \\ \frac{1}{2R} - \frac{1}{2R'} & 1 \end{pmatrix}$ which is an isomorphism of $\mathcal{G}\mathcal{X}_R$ onto $\mathcal{G}\mathcal{X}_{R'}$ and $\mathcal{D}\begin{pmatrix} 1 & 0 \\ \frac{-1}{\beta} & 1 \end{pmatrix}$ with purely imaginary $\beta \neq 0$. This last transformation is an automorphism of $\mathcal{G}\mathcal{X}_\infty$ onto itself which transforms $t = 0$ into $t = \infty$ and vice versa. The theorems of the preceding two sections take an especially simple form for functions in $\mathcal{G}\mathcal{X}_R$. For reference's sake we will state here these theorems and some of their corollaries.

<u>THEOREM I.</u> <u>A function</u> $v(x)$ <u>is the section at</u> t^0 <u>of a function in</u> $\mathcal{G}\mathcal{X}(B_R^2(t^0))$ <u>if and only if</u> v <u>is an entire function of</u> $x \in \mathbb{C}^n$ <u>polyharmonic of Laplacian order</u> 2 <u>and type</u> $\frac{1}{\sqrt{2R}}$.

If t^0 is in the t-domain of $\mathcal{G}\mathcal{X}_R$ we can choose a transformation of the group $\mathcal{D}$ which will transform this domain onto a circle and t^0 into its center. We obtain thus the corollary:

<u>Corollary II.</u> <u>A function</u> $v(x)$ <u>is the section of a function</u> $u \in \mathcal{G}\mathcal{X}_R$, $R < \infty$ <u>at a point</u> $t^0 \in B_R^2(R)$ <u>if and only if the function</u>

$$\exp\left(\frac{(R-\overline{t^0})x^2}{4(R^2-|R-t^0|^2)}\right)$$

<u>is an entire function of Laplacian order</u> 2 <u>and type</u> $\left(\frac{R}{2(R^2-|R-t^0|^2)}\right)^{1/2}$.

Corollary II'. $v(x)$ is the section at $t^0 \in \mathbb{C}^1_+$ of a function in $\mathcal{GX} \equiv \mathcal{GX}_\infty$ if and only if $e^{(x^2)/(8\,\mathrm{Re}\,t^0)}\, v(x)$ is an entire function of Laplacian order 2 and type $\frac{1}{2\sqrt{\mathrm{Re}\,t^0}}$.

For the class $\mathcal{GX}_R$ consider t^0 in its t-domain and the largest circle $B(R, t^0, \theta)$ contained in the t-domain of $\mathcal{GX}_R$ with diameter in the direction θ, and starting at t^0.[1] Every B-circle of a function $u \in \mathcal{GX}_R$ at t^0, x^0 in the direction θ contains $B(R, t^0, \theta)$. We will call this circle the minimal B-circle for the class $\mathcal{GX}_R$ at t^0, θ. The corresponding minimal B-domain $= \bigcup_\theta B(R, t^0, \theta) \cup (t^0)$ is always equal to the t-domain of $\mathcal{GX}_R$.

THEOREM III. For t^0 in the t-domain of $\mathcal{GX}_R$ we have

(1) $$u(z,t) = \oint u\left(e^{\frac{i\theta}{2}} x, t^0\right) E\left(e^{\frac{-i\theta}{2}} z - x,\ e^{-i\theta}(t-t^0)\right) dx\,,$$

the symbolic integral existing for all $z \in \mathbb{C}^n$ and $t \in B(R, t^0, \theta)$.

Thus formula (1) allows one to express the function u in the whole domain of $\mathcal{GX}_R$ in terms of the section at t^0 by choosing θ suitably (the choice of θ is not unique).

Theorem III of §2 would be needed especially in the case when $t^0 > 0$.

THEOREM IV. Let $u \in \mathcal{GX}_R$ and $t' \in B^2_R(R)$, $0 < t^0 < 2R$. Then the section $u(x, t')$ can be developed in a series of Hermite polynomials.

(2) $$u(x,t') = \sum_{p=0}^{\infty} \sum_{|k|=p} A_k (t^0 - t')^{|k|/2} H_k\left(\frac{x}{2\sqrt{t^0 - t'}}\right),$$

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1. For $R = \infty$ and $\theta = 0$ (i.e. the direction of the positive axis) this circle will actually be the half-plane $\mathrm{Re}\,(t - t^0) > 0$.

the series in p being uniformly convergent for (x,t') in any compact of $\mathbb{C}^n \times B_R^2(R)$. The coefficients A_k can be expressed by :

(3) $$A_k = \frac{1}{k!} D_u^k u(0,t^0) \text{ for all } t' \in B_R^2(R),$$

(3') $$A_k = \oint u(x,t')E(x,t^0-t') \frac{1}{\left(2\sqrt{t^0-t'}\right)^{|k|}} H_k\left(\frac{x}{2\sqrt{t^0-t'}}\right) dx \text{ for } 0 < t' < t^0 .$$

Before we consider the topological dual of $\mathcal{G}\mathcal{X}_R$ we will first represent $\mathcal{G}\mathcal{X}_R$ itself as a dual of relatively simple function space. We consider the class of functions $\mathcal{V}_R$ which is obtained by taking all functions of x,t of the form $E(x-\zeta,\ t+\tau)$, their derivatives relative to the variable parameters ζ and τ for $\zeta \in \mathbb{C}^n$ and $\tau \in B_R^2(R)$, and all finite linear combinations of these functions. We obtain thus a linear function space. As topology in this space we take the strongest locally convex topology for which the difference quotients of $D_{\zeta,t}^k E(x-\zeta,\ t+\tau)$ relative to ζ_j or τ converge to the corresponding derivative. It is easy to see that such topology exists and is Hausdorffian.

THEOREM V. The topological dual $\mathcal{V}'_R$ is exactly the space $\mathcal{G}\mathcal{X}_R$ if we identify an element $f \in \mathcal{V}'_R$ with the function

$$f(z,t') = \langle f,\ E(x-z,\ t+t') \rangle \in \mathcal{G}\mathcal{X}_R .$$

The scalar product between a function $u \in \mathcal{G}\mathcal{X}_R$ and a function $v \in \mathcal{V}_R$ can be written in the form

$$\langle u, v \rangle = \oint u(x,t')v(x,-t')dx$$

where the symbolic integral exists and is independent of t' for sufficiently small positive t'.[1.]

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1. In fact, if in the expression of v figure the values τ_j of the parameter τ the integral will exist for t' such that $\tau_j - t' \in B_R^2(R)$ for all j's .

The Mackey topology on $\mathcal{A}\mathcal{N}_R$ corresponding to the pairing $<\mathcal{A}\mathcal{N}_R, V_R>$ is strictly weaker than the canonical topology of $\mathcal{A}\mathcal{N}_R$. Hence the dual $\mathcal{A}\mathcal{N}'_R \supsetneq V_R$. To describe the dual $\mathcal{A}\mathcal{N}'_R$ we notice that $\mathcal{A}\mathcal{N}_R$ is a closed subspace of $\mathcal{A}(\mathbb{C}^n \times B^2_R(R))$. Hence every continuous, linear functional of $\mathcal{A}\mathcal{N}'_R$ can be obtained (in infinitely many ways) by taking an analytic functional $F \in \mathcal{A}'(\mathbb{C}^n \times B^2_R(R))$ and restricting it to $\mathcal{A}\mathcal{N}_R$. By formula 1, §4, Chapter I, F can be identified with a function $F(z,t)$ regular on $\overline{P}^n_{1/r}(0) \times \overline{P}^1_{1/R}(R)$ and the scalar product is given by

$$< u(z,t),\ F > = \int_{\Pi^n_{r'}(0) \times \Pi^1_{R'}(0)} u(z, t+R) F\left(\frac{1}{z_1}, \cdots, \frac{1}{z_n}, \frac{1}{t} + R\right) dz_1, \cdots dz_n\, dt,$$

the function F being regular on $\overline{P}^n_{1/r'}(0) \times \overline{P}^1_{1/R'}(R)$ and $0 < R' < R$.

Similar procedure can be applied to the case $R = \infty$ where $B^2_R(R)$ is replaced by $\mathbb{C}^1_+$. The representation leads to the following theorem:

THEOREM VI. Each functional $v \in \mathcal{A}\mathcal{N}'_R$, $R < \infty$ can be identified with the function $v(x',t') \in \mathcal{A}\mathcal{N}_R$

$$v(x',t') = < E(x'-x,\ t'+t),\ v > .$$

Each such function is regular for $x' \in \mathbb{C}^n$, $t' \in \mathbb{C}^1 \setminus B^2_{R'}(-R)$ for certain $R' < R$. For $0 < t' < R-R'$ the function $v(x,-t')$ is convolutable with every function $u(x,t) \in \mathcal{A}\mathcal{N}_R$ and the scalar product is given by

$$< u, v > = (u(x,t') * v(x,-t'))(0)$$

the last value being independent of t'.

THEOREM VI'. For $R = \infty$ the only difference from the statement in Theorem VI is that $v(x',t')$ is regular for $x \in \mathbb{C}^n$, $t' \in \mathbb{C}^1 \setminus B^2_{R'}(-R_1)$

<u>for some</u> R' <u>and</u> R_1 <u>satisfying</u> $R' < R_1$ <u>and in the formula for scalar product</u> $< u, v >$, t' <u>should satisfy</u> $0 < t' < R_1 - R'$.

The last theorems do not characterize completely the function $v(x', t)$ in the dual $\mathfrak{G}\mathcal{H}'_R$. The complete characterization was obtained by M. S. Baouendi in case $R = \infty$.

<u>THEOREM VII.</u> <u>(Baouendi).</u> $v(x, t) \in \mathfrak{G}\mathcal{H}'$ <u>if and only if</u> $v \in \mathfrak{G}\mathcal{H}$, <u>is regular for</u> $t = 0$ <u>and there exist constants</u> $C > 0$, $M \geqq 0$, $A > B \geqq 0$ <u>such that for</u> $x \in \mathbb{C}^n$

$$|v(x, 0)| \leqq Ce^{M|x| + B|x^2| - A \operatorname{Re} x^2} .$$

One can obtain a corresponding characterization for $v \in \mathfrak{G}\mathcal{H}'_R$ by applying to $\mathfrak{G}\mathcal{H}$ the isomorphism $\mathfrak{D}\begin{pmatrix} 1 & , & 0 \\ \frac{1}{-2R} & , & 1 \end{pmatrix}$ which transforms $\mathfrak{G}\mathcal{H}$ onto $\mathfrak{G}\mathcal{H}_R$ and $\mathbb{C}^1_+$ onto $B^2_R(R)$.