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Stationary disks and Green functions in almost complex domains

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Abstract. Using generalized Riemann maps, normal forms for almost complex domains (D, J) with singular foliations by stationary disks are defined. Such normal forms are used to construct counterexamples and to determine intrinsic conditions under which the stationary disks are extremal disks for the Kobayashi metric or determine solutions to almost complex Monge-Ampère equation.

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1. Introduction

Let $D \subset M$ be a domain of an almost complex manifold (M, J) with smooth boundary ∂D and $\widetilde{\mathbb{J}}$ the canonical almost complex structure of T^*M determined by J. We recall that a smooth, proper J-holomorphic embedding $f : \Delta \to D$ of the unit disk into D is called *stationary disk* if there exists a map $\widetilde{f} : \overline{\Delta} \to T^*M \setminus \{\text{zero section}\}$, which is $\widetilde{\mathbb{J}}$-holomorphic, projects onto f and is such that, for any $\zeta \in \partial\Delta$, the 1-form $\zeta^{-1} \cdot \widetilde{f}(\zeta) \in T^*_{f(\zeta)}M$ vanishes identically on $T_{f(\zeta)}\partial D$ (see Section 2).

The stationary disks of almost complex domains have been introduced by Coupet, Gaussier and Sukhov in [4,5] (see also [6]). They are useful biholomorphic invariants of almost complex domains and constitute a natural generalization of the stationary disks of strictly linearly convex domains of $\mathbb{C}^n$, considered for the first time in celebrated Lempert's papers on extremal disks and Kobayashi metrics [11,12].

Existence and uniqueness results on stationary disks, with prescribed center and direction, have been established in various contexts, both in the integrable and non-integrable case (see *e.g.* [4,6,11,14,19,20,22]). Moreover, when J is integrable

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and $D \subset (M, J)$ is equivalent to a strictly linearly convex domain in $\mathbb{C}^n$, the family $\mathcal{F}^{(x_o)}$ of stationary disks centered at a fixed $x_o \in D$, determines a smooth foliation of $D \setminus \{x_o\}$ with several important properties [11,12]:

a) it determines a natural diffeomorphism $\Phi : \overline{B}^n \longrightarrow \overline{D}$, which generalizes the usual Riemann map between $\overline{\Delta}$ and any other smoothly bounded domain of $\mathbb{C}$;
b) it consists of disks that are extremal for the Kobayashi metric of D;
c) it can be used to determine a Green pluripotential for D with pole in x_o, *i.e.* a plurisubharmonic function that solves the classical complex Monge-Ampère equation and has a logarithmic pole at x_o.

When J is not integrable, there are several cases, in which the family $\mathcal{F}^{(x_o)}$ of stationary disks, centered at a fixed $x_o \in D$, gives a smooth foliation for $D \setminus \{x_o\}$ (*e.g.* when (D, J) is equivalent with a strictly linearly convex domain of $\mathbb{C}^n$ and J is a small deformation of J_{st}). In these cases, it is still true that $\mathcal{F}^{(x_o)}$ determines a J-biholomorphically invariant, generalized Riemann map $\Phi : \overline{B}^n \longrightarrow \overline{D}$. But in general, *it is no longer true that the disks of $\mathcal{F}^{(x_o)}$ are extremal disks for the Kobayashi metric nor that they can be used to solve the almost complex Monge-Ampère equation*. Here, by "almost-complex Monge-Ampère equation" we mean the differential equation that characterizes the maximal J-plurisubharmonic functions of class $\mathcal{C}^2$ of an almost complex, strongly pseudoconvex domain. By comparison with usual complex Monge-Ampère equation, it can be considered as a very appropriate analogue in almost complex settings.

There are many reasons which justify these phenomena. In case of non-integrable complex structures, the great abundance of J-holomorphic curves, which gives an advantage in many geometrical considerations, turns into a drawback in considering objects as the Kobayashi metric, which reveals to be a weaker and more elusive invariant. In particular, it is natural to expect that the notions of stationary and extremal disks, which involve fine (and different!) properties, become equivalent only when the almost complex structure satisfies appropriate restrictions. In fact, in [9] it is shown that, in general, these notions are different.

As for the construction of Green pluripotentials, additional difficulties emerge. In fact, if one has the integrable setting in mind, the behavior of plurisubharmonic functions in the non-integrable case is quite unexpected. For instance, there are arbitrarily small deformations J of the standard complex structure, with respect to which the logarithm of squared norm $\log|\cdot|^2$ of $\mathbb{C}^n$ is *not* J-plurisubharmonic. Furthermore, the kernel distribution of an almost-complex Monge-Ampère operator, even if appropriate non-degenericity conditions is assumed, is usually *neither integrable, nor J-invariant*. Since all this is in clear contrast with the classical setting, it cannot be expected that for completely arbitrary non-integrable structures one can reproduce the whole pattern of fruitful properties, which relate regular solutions of complex Monge-Ampère equations and Monge-Ampère foliations.

In this paper, with the help of generalized Riemann maps, we determine "normal forms" for almost complex domains (D, J) with singular foliations by stationary disks. We use such normal forms to construct examples and determine intrinsic conditions, under which the disks of the foliations are extremal disks for the

Kobayashi metric or give solutions to the almost complex Monge-Ampère equation. In fact, we are able to determine sufficient conditions on the almost complex structure, which ensure the existence of almost complex Green pluripotential and the equality between the two notions of stationary disks and of extremal disks. It is interesting to note that the class of such structures (called *nice* or *very nice* almost complex structures) is very large in many regards, in fact determined by a finite set of conditions (it is finite-codimensional) in an infinite dimensional space. We hope that such notions will be fruitful also for other questions in almost complex analysis and geometry.

The paper is organized as follows. After a preliminary section, in Section 3 we introduce the notion of almost complex domains of circular type *in normal form*. They are pairs (B^n, J), formed by the unit ball $B^n \subset \mathbb{C}^n$ and an almost complex structure J, which satisfies conditions that guarantee that any radial disk through the origin is stationary for (B^n, J). Since any almost complex domain, admitting a singular foliation by stationary disks, is biholomorphic to a domain in normal form, any problem on such foliations can be reduced to questions on the radial disks of normal forms. In Section 4, we study conditions on J, under which the radial disks of a normal form (B^n, J) are extremal. In Section 5, we define the *almost complex Monge-Ampère equation*, we prove that it characterizes maximal $\mathcal{C}^2$ plurisubharmonic functions and we determine conditions on normal forms (B^n, J), under which the stationary foliation by radial disks determines a Green pluripotential.

Notation. The standard complex structure of $\mathbb{C}^n$ is denoted by J_{st}, the unit ball $\{ |z| < 1 \} \subset \mathbb{C}^n$ is denoted by B^n and, when $n = 1$, by $\Delta = B^1$.

For any $\alpha > 0$ and $\epsilon \in]0, 1[$, a map $f : \overline{\Delta} \longrightarrow M$ into a manifold M is said *of class* $\mathcal{C}^{\alpha,\epsilon}$ if there are coordinates $\xi = (x^1, \dots, x^N) : \mathcal{U} \longrightarrow \mathbb{R}^N$ on a neighborhood of $f(\overline{\Delta})$, such that $\xi \circ f : \overline{\Delta} \longrightarrow \mathbb{R}^n$ is of class $\mathcal{C}^\alpha$ on Δ and Hölder continuous of class $\mathcal{C}^\epsilon$ on $\overline{\Delta}$. If $Y = Y_i^j \frac{\partial}{\partial x^j} \otimes dx^i$ is a tensor field of type $(1, 1)$ on $\mathbb{R}^m$ and $\mathcal{U}$ is a subset of $\mathbb{R}^m$, we denote

$$\|Y\|_{\overline{\mathcal{U}},\mathcal{C}^k} \overset{\text{def}}{=} \sum_{|J| \le k} \sup_{x \in \overline{\mathcal{U}}} \left| \frac{\partial^{|J|} Y_j^i}{\partial x^J}(x) \right|.$$

2. Preliminaries

2.1. Canonical lifts of almost complex structures

Let (M, J) be an n-dimensional complex manifold with integrable complex structure J. In this case, TM and T^*M are naturally endowed with integrable complex structures $\mathbb{J}$ and $\widetilde{\mathbb{J}}$, respectively, corresponding to the atlases of complex charts $\widehat{\xi} : TM|_{\mathcal{U}} \longrightarrow T\mathbb{C}^n \simeq \mathbb{C}^{2n}$, $\widetilde{\xi} : TM^*|_{\mathcal{U}} \longrightarrow T^*\mathbb{C}^n \simeq \mathbb{C}^{2n}$, determined by charts $\xi = (z^i) : \mathcal{U} \subset M \longrightarrow \mathbb{C}^n$ of the atlas of the complex manifold structure of (M, J).

When (M, J) is an almost complex manifold, there is no canonical atlas of complex charts on M and the above construction is meaningless. Nevertheless, *there are natural almost complex structures $\mathbb{J}$ and $\widetilde{\mathbb{J}}$ on TM and T^*M, respectively, also in this more general case* (see [24, Section I.5 and Section VII.7]). Using coordinates they are defined as follows. For a given a system of real coordinates $\xi = (x^1, \dots, x^{2n}) : \mathcal{U} \subset M \longrightarrow \mathbb{R}^{2n}$, we denote by

$$\widehat{\xi} = (x^1, \dots, x^{2n}, q^1, \dots, q^{2n}) : \pi^{-1}(\mathcal{U}) \subset TM \longrightarrow \mathbb{R}^{4n}, \tag{2.1}$$

$$\widetilde{\xi} = (x^1, \dots, x^{2n}, p_1, \dots, p_{2n}) : \widetilde{\pi}^{-1}(\mathcal{U}) \subset T^*M \longrightarrow \mathbb{R}^{4n}, \tag{2.2}$$

the associated coordinates on $TM|_{\mathcal{U}}$ and $T^*M|_{\mathcal{U}}$, determined by the components q^i of vectors $v = q^i \frac{\partial}{\partial x^i}$ and the components p_j of covectors $\alpha = p_j dx^j$. If $J^i_j = J^i_j(x)$ denote the components of $J = J^i_j \frac{\partial}{\partial x^i} \otimes dx^j$, the almost complex structures $\mathbb{J}$ and $\widetilde{\mathbb{J}}$ are defined by the expressions

$$\mathbb{J} = J^a_i \frac{\partial}{\partial x^a} \otimes dx^i + J^a_i \frac{\partial}{\partial q^a} \otimes dq^i + q^b J^a_{i,b} \frac{\partial}{\partial q^a} \otimes dx^i, \tag{2.3}$$

$$\begin{aligned} \widetilde{\mathbb{J}} = {} & J^a_i \frac{\partial}{\partial x^a} \otimes dx^i + J^a_i \frac{\partial}{\partial p_i} \otimes dp_a \\ & + \frac{1}{2} p_a \left(-J^a_{i,j} + J^a_{j,i} + J^a_\ell \left(J^\ell_{i,m} J^m_j - J^\ell_{j,m} J^m_i \right) \right) \frac{\partial}{\partial p_j} \otimes dx^i . \end{aligned} \tag{2.4}$$

These tensor fields can be checked to be independent on the chart (x^i) and:

i) the standard projections $\pi : T^*M \longrightarrow M$, $\widetilde{\pi} : T^*M \longrightarrow M$ are $(\mathbb{J}, J)$-holomorphic and $(\widetilde{\mathbb{J}}, J)$-holomorphic, respectively;
ii) given a (J, J')-biholomorphism $f : (M, J) \longrightarrow (N, J')$ between almost complex manifolds, the tangent and cotangent maps

$$f_* : TM \longrightarrow TN \qquad \text{and} \qquad f^* : T^*N \longrightarrow T^*M$$

are $(\mathbb{J}, \mathbb{J}')$- and $(\widetilde{\mathbb{J}}', \widetilde{\mathbb{J}})$-holomorphic, respectively;
iii) when J is integrable, $\mathbb{J}$ and $\widetilde{\mathbb{J}}$ coincide with above described integrable complex structures of TM and T^*M, respectively (in the integrable case, all derivatives $J^a_{i,j}$ are 0 in holomorphic coordinates).

We call $\mathbb{J}, \widetilde{\mathbb{J}}$ *canonical lifts of J on TM and T^*M.*

2.2. Blow-ups of almost complex manifolds

Given a point x_o of an almost complex manifold (M, J), we call *blow-up at x_o* the topological manifold $\widetilde{M}$ obtained as follows [19].

Consider a system of complex coordinates $\xi = (z^1, \dots, z^n) : \mathcal{V} \longrightarrow \mathcal{U} \subset \mathbb{C}^n$ on a neighborhood $\mathcal{V}$ of x_o, with $\xi(x_o) = 0$ and which maps $J|_{x_o}$ into the standard complex structure $J_{st}|_0$ of $T_0\mathbb{C}^n \simeq \mathbb{C}^{2n}$. The manifold $\widetilde{M}$ is obtained by gluing $M \setminus \{x_o\}$ with the blow up $\widetilde{\mathcal{U}}$ at 0 of $\mathcal{U} \subset \mathbb{C}^n = \mathbb{R}^{2n}$, identifying $\mathcal{V} \setminus \{0\}$ with $\mathcal{U} \setminus \{0\}$ by means of the map $\xi = (z^i)$. As it was remarked in [19], the smooth manifold structure of $\widetilde{M}$ does not depend on the choice of the coordinates $\xi = (z^i)$. Hence, this manifold $\widetilde{M}$ can be considered as canonically associated with (M, J) and x_o.

Since $M \setminus \{0\} \equiv \widetilde{M} \setminus \pi^{-1}(x_o)$ and $\pi^{-1}(\mathcal{V}) \equiv \pi^{-1}(\widetilde{\mathcal{U}}) \subset \widetilde{B}^n$, we may consider the tensor field J of type $(1, 1)$ on $\widetilde{M}$, with $J_x : T_x\widetilde{M} \to T_x\widetilde{M}$ equal to the almost complex structure of M for any point $x \in \widetilde{M} \setminus \pi^{-1}(x_o)$ and to the standard complex structure of $\widetilde{B}^n$ for any $x \in \pi^{-1}(x_o) \equiv \pi^{-1}(0) \subset \widetilde{B}^n$. Such tensor field is obviously smooth on $M \setminus \pi^{-1}(x_o)$ and, in our discussions, there will be no need to know whether it is smooth also on $\pi^{-1}(x_o)$. It is however possible to check that it is in fact smooth at all points.

3. Normal forms of almost complex domains of circular type

In what follows, $D \subset M$ denotes a domain in a $2n$-dimensional almost complex manifold (M, J) with smooth boundary $\Gamma = \partial D$.

3.1. Almost complex domains of circular type

Let $\mathcal{N}$ be the conormal bundle of $\Gamma = \partial D$, *i.e.* the subset of $T^*M|_\Gamma$

$$\mathcal{N} = \{ \beta \in T_x^*M \ , \ x \in \Gamma \ : \ \ker \beta \subset T_x\Gamma \}.$$

We recall that, given $\alpha \geq 1$ and $\varepsilon > 0$, a *$\mathcal{C}^{\alpha,\varepsilon}$-stationary disk of D* is a map $f : \overline{\Delta} \longrightarrow M$ such that

i) $f|_\Delta$ is a J-holomorphic embedding and $f(\partial\Delta) \subset \partial D$;
ii) there exists a $\mathbb{J}$-holomorphic map $\widetilde{f} : \overline{\Delta} \longrightarrow T^*M$ with $\pi \circ \widetilde{f} = f$, so that

$$\zeta^{-1} \cdot \widetilde{f}(\zeta) \in \mathcal{N} \setminus \{\text{zero section}\} \ \text{ for any } \zeta \in \partial\Delta \tag{3.1}$$

and $\widetilde{\xi} \circ \widetilde{f} \in \mathcal{C}^{\alpha,\varepsilon}(\overline{\Delta}, \mathbb{C}^{2n})$ for some complex coordinates $\widetilde{\xi} = (z^i, w_j)$ around $\widetilde{f}(\overline{\Delta})$.

In (3.1) " $\cdot$ " denotes the usual $\mathbb{C}$-action on T^*M, *i.e.* the action[1]

$$\zeta \cdot \alpha \overset{\text{def}}{=} \operatorname{Re}(\zeta)\alpha - \operatorname{Im}(\zeta) J^*\alpha \quad \text{for any} \quad \alpha \in T^*M, \quad \zeta \in \mathbb{C}. \tag{3.2}$$

If f is stationary, the maps $\widetilde{f}$ satisfying (ii) are called *stationary lifts of f*.

In this paper we are concerned with domains D in an almost complex manifold (M, J), admitting singular foliations by stationary disks with the same properties of the singular foliations by Kobayashi extremal disks of the domains of circular type in $\mathbb{C}^n$. Here is the definition of such domains.

[1] We follow Besse's convention on signs, for which $J^*\alpha(v) = -\alpha(Jv)$ [2]; see also Section 5.1. Due to this, on $\mathbb{C}^n$ we have $J^*_{st}dx^j = dy^j$, $dz^j = dx^j + iJ^*_{st}dx^j$ and $idz^j = -J^*_{st}dz^j$.

Definition 3.1 ([19]). For any point x_o of an almost complex domain $D \subset (M, J)$, we denote by $\mathcal{F}^{(x_o)}$ the family of stationary disks of D with $f(0) = x_o$. We say the $\mathcal{F}^{(x_o)}$ is a *foliation of circular type* if:

i) for any $v \in T_{x_o}D$, there exists a unique disk $f^{(v)} \in \mathcal{F}^{(x_o)}$ with $f_*^{(v)}\left(\frac{\partial}{\partial x}\big|_0\right) = \mu \cdot v$ for some $0 \neq \mu \in \mathbb{R}$;
ii) for a fixed identification $(T_{x_o}D, J_{x_o}) \simeq (\mathbb{C}^n, J_{\mathrm{st}})$, the map from the blow up $\widetilde{B}^n$ at 0 of B^n to the blow up $\widetilde{D}$ at x_o of D

$$\Phi : \widetilde{B}^n \subset \widetilde{\mathbb{C}}^n \longrightarrow \widetilde{D}, \qquad \Phi(v, [v]) \stackrel{\text{def}}{=} \widetilde{f^{(v)}}(|v|) \tag{3.3}$$

is smooth, extends smoothly up to the boundary and determines a diffeomorphism between the boundaries $\Phi|_{\partial B^n} : \partial B^n \longrightarrow \partial D$.

The point x_o is called *center of the foliation* and $\Phi : \widetilde{B}^n \longrightarrow \widetilde{D}$ is called (*generalized*) *Riemann map of* (D, x_o). Any domain $D \subset (M, J)$ admitting a foliation of circular type is called *almost complex domain of circular type*.

There exists a wide class of almost complex domains of circular type. In fact, any bounded, strictly linearly convex domains $\widetilde{D} \subset \mathbb{C}^n$, with smooth boundary and endowed with a small deformation J of the standard complex structure J_{st}, admits singular foliations of circular type made of stationary disks [19, Theorem 4.1]. For such domains, Gaussier and Joo proved in [9] the same existence result for singular foliations, made of the so-called *J-stationary disks* (see later for the definition).

On the other hand, the following fact is well-known (see *e.g.* [8]).

Lemma 3.2. *Let (M, J) be an almost complex manifold and $x_o \in M$. For any integer $k \geq 0$ and $\varepsilon > 0$, there exists a neighborhood $\mathcal{U}$ of x_o, such that $(\mathcal{U}, J)$ is (J, J')-biholomorphic to (B^n, J') for some almost complex structure J' on a neighborhood of $\overline{B}^n \subset \mathbb{C}^n$ with $\|J' - J_{\mathrm{st}}\|_{\overline{B^n},\, \mathcal{C}^k} < \varepsilon$.*

This and previous remarks imply that any point x_o of an almost complex domain (M, J) admits a neighborhood $\mathcal{U}$ containing a domain $D \Subset \mathcal{U}$ of circular type with center x_o.

3.2. Normal forms

By definitions, for any almost complex domain (D, J) of circular type with center x_o, the map $\Phi : \widetilde{B}^n \subset \widetilde{\mathbb{C}}^n \longrightarrow \widetilde{D}$ is a biholomorphism between $(\widetilde{D}, J)$ and $(\widetilde{B}^n, \widetilde{J})$, where $\widetilde{J} \stackrel{\text{def}}{=} \Phi_*^{-1}(J)$.

If we denote by $\pi : \widetilde{B}^n \longrightarrow B^n$, $\pi' : \widetilde{D} \longrightarrow D$ the natural blow down maps and by $J' = \pi_*(\widetilde{J})$ the projected almost complex structure on $B^n \setminus \{0\}$, we have that the map $E = \pi \circ \Phi \circ \pi'^{-1} : D \setminus \{0\}) \longrightarrow B^n \setminus \{0\}$ is a (J, J')-biholomorphism. In general, the tensor field J' does not extend smoothly at $0 \in B^n$. Nonetheless such singularity is "removable" in the following sense.

First of all, we remark that the map E extends uniquely to a *homeomorphism* $E : D \longrightarrow B^n$ by setting $E(x_o) = 0$. So, we may consider the atlas $\mathcal{A}$ on B^n, formed by the charts $\eta = \xi \circ E^{-1}$ determined by charts $\xi : \mathcal{U} \subset D \to \mathbb{R}^{2n}$ of the manifold structure of D. *Such atlas defines a smooth manifold structure on B^n, which coincides with the standard one on $B^n \setminus \{0\}$, but contains charts around* 0 *that in general are non-standard*. By construction, the components of J' in the charts of $\mathcal{A}$ extend smoothly at 0.

We call the pair (B^n, J') *normal form of* (D, J) *determined by* $\mathcal{F}^{(x_o)}$. If we endow B^n with the atlas $\mathcal{A}$, by construction (B^n, J') is an almost complex domain of circular type with center 0 and foliation $\mathcal{F}^{(0)}$ given by the *straight disks*

$$f^{(v)} : \Delta \longrightarrow B^n, \qquad f^{(v)}(\zeta) = \zeta \cdot v, \qquad v \in \mathbb{C}^n. \tag{3.4}$$

Now, let us give an intrinsic characterization of the almost complex structures J on $B^n \setminus \{0\}$ that correspond to normal forms of domains of circular type. For this purpose, we need some new notation and the notion of "almost L-complex structures".

Let $Z \overset{\text{def}}{=} \operatorname{Re}\left(z^i \frac{\partial}{\partial z^i}\right)$ and denote by $\mathcal{Z}$ the J_{st}-invariant distribution on $B^n \setminus \{0\} \subset \mathbb{C}^n$ defined by $\mathcal{Z}_z \overset{\text{def}}{=} < Z_z, J_{\text{st}} Z_z >$ at any $z \neq 0$. We recall that

$$\mathcal{Z}_z = \ker dd^c_{\text{st}} \log \tau_o\big|_x, \quad \text{where } \tau_o(z) \overset{\text{def}}{=} |z|^2, \; d^c_{\text{st}} \overset{\text{def}}{=} J^*_{\text{st}} \circ d \circ J^*_{\text{st}} \tag{3.5}$$

and that $dd^c_{\text{st}}\tau_o(Z, X) = X(\tau_o)$ for any $X \in TB^n \setminus \{0\}$. One can check that $\mathcal{Z}$ is integrable and that its integral leaves are the (images of the) disks (3.4).

Consider the blow up $\widetilde{B}^n$ of B^n at 0, the standard identification of $\widetilde{B}^n$ with an open subset of the tautological bundle $\pi : E \longrightarrow \mathbb{C}P^{n-1}$ and the coordinates $\xi : \mathcal{U} \subset E \longrightarrow \mathbb{C}^n, \mathcal{U} \overset{\text{def}}{=} \{ ([v], v) : \; v^n \neq 0 \}$, defined by

$$\begin{aligned} &\xi^{-1}(z^0, \dots, z^{n-1}) \\ &\overset{\text{def}}{=} \left(\left[z^1 : \cdots : z^{n-1} : \sqrt{1 - \sum_{i=1}^{n-1} |z^i|^2} \right]; z^0 \cdot \left(z^1, \dots, z^{n-1}, \sqrt{1 - \sum_{i=1}^{n-1} |z^i|^2} \right) \right). \end{aligned} \tag{3.6}$$

In these coordinates, $\mathcal{Z}^{\mathbb{C}}_z = \operatorname{Span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial z^0}\Big|_z, \frac{\partial}{\partial \overline{z^0}}\Big|_z \right\}$ and $J\left(\frac{\partial}{\partial z^0}\Big|_z\right) = i \frac{\partial}{\partial z^0}\Big|_z$. Now, let us use capital letters $A, B, C, \dots$ for indices that might be indifferently of the form a or $\overline{a}$, so that we may denote the complex coordinates and their conjugates by $(z^A) = (z^a, z^{\overline{a}} \overset{\text{def}}{=} \overline{z^a})$. Let also denote by $(p_A) = (p_a, p_{\overline{a}} \overset{\text{def}}{=} \overline{p_a})$ the complex components of real 1-forms $\omega = p_a dz^a + \overline{p}_a dz^{\overline{a}} \in T^* B^n$. Using these conventions, the canonical lift $\widetilde{\mathbb{J}}$ of an almost complex structure J on $B^n \setminus \{0\}$ is of the form

$$\begin{aligned} \widetilde{\mathbb{J}} = {} & J^B_A \left(\frac{\partial}{\partial z^B} \otimes dz^A + \frac{\partial}{\partial p_A} \otimes dp_B \right) \\ & + \frac{1}{2} p_C \left(-J^C_{A,B} + J^C_{B,A} + J^C_L \left(J^L_{A,M} J^M_B - J^L_{B,M} J^M_A \right) \right) \frac{\partial}{\partial p_B} \otimes dz^A, \end{aligned}$$

where J^A_B are the components of J with respect to the complex vector fields $\left(\frac{\partial}{\partial z^A}\right)$. One way to recover such formula is, for instance, to look at the expression of $\widetilde{\mathbb{J}}$ in terms of Nijenhuis tensor and tautological form of T^*M [21, 24] and write all terms using the complex coordinates (z^A, p_B). Again, we remark that, when J is integrable and $(z^1, \dots, z^n)$ are holomorphic coordinates, $\widetilde{\mathbb{J}} = J^B_A\left(\frac{\partial}{\partial z^B} \otimes dz^A + \frac{\partial}{\partial p_A} \otimes dp_B\right)$.

We can now introduce the "almost L-complex structures": as we will shortly see, a pair (B^n, J) is a domain of circular type in normal form if and only it J is an almost complex structure of this kind (Theorem 3.7). We will also see that these structures are characterized by a finite collection of equations in the space of almost complex structures (Proposition 3.4).

Definition 3.3. We call *almost L-complex structure* any almost complex structure J on $B^n \setminus \{0\}$, smoothly extendible at ∂B^n, such that:

i) $\mathcal{Z}$ is J-stable and $J|_{\mathcal{Z}} = J_{\mathrm{st}}|_{\mathcal{Z}}$;
ii) for any $v \in S^{2n-1}$, the following differential problem on $2n-2$ $\mathbb{C}$-valued maps $g_\alpha, g_{\bar\alpha} : \overline{\Delta} \longrightarrow \mathbb{C}$ of class $\mathcal{C}^{\alpha,\epsilon}$ is solvable (in (3.7), A, B denote indices of the form $\alpha, \bar\alpha, \beta, \bar\beta$, respectively, with $1 \le \alpha, \beta \le n-1$)

$$\begin{cases} \left(\delta^B_A - i(J^B_A|_{\zeta\cdot v})\right) g_{B,\bar\zeta} \\ \quad + \left(-\frac{i}{2}\left(J^B_{A,\bar 0} + i J^B_L J^L_{A,\bar 0}\right)\Big|_{\zeta\cdot v}\right) g_B \\ \quad - \left(\frac{i}{2}\left(J^0_{A,\bar 0} + i J^0_L J^L_{A,\bar 0} + J^{\bar 0}_{A,\bar 0} + i J^{\bar 0}_L J^L_{A,\bar 0}\right)\Big|_{\zeta\cdot v}\right) = 0 \quad \text{when } \zeta \in \Delta, \\ \left((\operatorname{Re}\zeta)\delta^B_A - (\operatorname{Im}\zeta)\, J^B_A|_{\zeta\cdot v}\right) g_B - (\operatorname{Im}\zeta)\left(J^0_A + J^{\bar 0}_A\right)\Big|_{\zeta\cdot v} = 0 \ \text{ when } \zeta \in \partial\Delta, \end{cases} \tag{3.7}$$

where (J^B_A) are the components of J in coordinates of the form (3.6);
iii) there exists a homeomorphism $\xi : \mathcal{U} \longrightarrow \mathcal{V}$ between neighborhoods of $0 \in \mathbb{C}^n$, which is $\mathcal{C}^\infty$ on $\mathcal{U} \setminus \{0\}$ and such that $\xi_*(J)|_{\mathcal{V}\setminus\{\xi(0)\}}$ extends smoothly at 0; in particular, J admits a smooth extension at 0 if B^n is endowed with a (non-standard) atlas containing ξ;
iv) the blow-up $\widetilde{B}^n$ of B^n, determined by J and the non-standard smooth manifold structure described in (iii), is diffeomorphic to the usual blow-up of B^n determined by J_{st}.

Unless explicitly stated, for any almost L-complex structure, we will always assume B^n endowed with the smooth manifold structure described in (iii).

It is useful to remark that condition (ii) of the above definition is satisfied by a very large class of almost complex structures. Moreover, in the integrable case, (ii) is automatically satisfied by the complex structures of the normal forms of domains of circular types (see [18]).

Proposition 3.4. *Let J be an almost complex structure J on $B^n \setminus \{0\}$ such that, for any $v \in S^{2n-1}$ and $\zeta \in \partial\Delta$, the matrices*

$$\mathfrak{C} = \left[\left(\delta^B_A - i\,(J^B_A\big|_{\zeta\cdot v})\right)\right], \quad \mathfrak{D} = \left[\left((\mathrm{Re}\,\zeta)\delta^B_A - (\mathrm{Im}\,\zeta)\; J^B_A\big|_{\zeta\cdot v}\right)\right]$$

are invertible. Let also $\mathbb{F} \in \mathcal{C}^{\alpha-1,\epsilon}(\overline{\Delta}, \mathbb{C}^{2n-2})$ and $\mathbb{G} \in \mathcal{C}^{\epsilon}(\partial\Delta, \mathbb{R}^{4n-4})$ defined by

$$\mathbb{F}(\zeta) = \mathfrak{C}^{-1}(\zeta)\cdot\left(\frac{i}{2}\left(J^0_{A,\bar{0}} + i\,J^0_L J^L_{A,\bar{0}} + J^{\bar{0}}_{A,\bar{0}} + i\,J^{\bar{0}}_L J^L_{A,\bar{0}}\right)\Big|_{\zeta\cdot v}\right),$$
$$\mathbb{G}(\zeta) = (\mathrm{Im}\,\zeta)\left(J^0_A + J^{\bar{0}}_A\right)\Big|_{\zeta\cdot v}.$$

Then, (3.7) *is solvable if and only if* $(\mathbb{F}, \mathbb{G})$ *is in the* (*finite codimensional*) *range of the Fredholm operator described in formula* (3.9) *below.*

Proof. The system (3.7) is equivalent to the "generalized Riemann-Hilbert problem" on maps $g : \overline{\Delta} \longrightarrow \mathbb{C}^{2n-2}$

$$\begin{cases} g,_{\bar{\zeta}} + (\mathfrak{C}^{-1}\mathbb{A})\cdot g = \mathbb{F} & \text{on } \Delta, \\ \mathfrak{D}\cdot g = \mathbb{G} & \text{on } \partial\Delta, \end{cases} \tag{3.8}$$

where $\mathbb{A} : \Delta \to M_{2n-2}(\mathbb{C})$ is $\mathbb{A}(\zeta) \overset{\text{def}}{=} \left[-\frac{i}{2}\left(J^B_{A,\bar{0}} + i\,J^B_L J^L_{A,\bar{0}}\right)\Big|_{\zeta\cdot v}\right]$. The operator

$$\begin{aligned} &R : \mathcal{C}^{\alpha,\epsilon}(\overline{\Delta}, \mathbb{C}^{2n-2}) \longrightarrow \mathcal{C}^{\alpha-1,\epsilon}(\overline{\Delta}, \mathbb{C}^{2n-2}) \times \mathcal{C}^{\epsilon}(\partial\Delta, \mathbb{R}^{4n-4}) \\ &R(h) \overset{\text{def}}{=} \left(\frac{\partial h}{\partial\zeta} + \mathbb{A}\cdot h,\, (\mathrm{Re}\,(\mathfrak{D}\cdot h)\,, \mathrm{Im}\,(\mathfrak{D}\cdot h))\right) \end{aligned} \tag{3.9}$$

is known to be Fredholm if and only if $\mathfrak{D}$ is invertible at all points [23, Section 3.2] and [13, Section VII.3]. The claim follows immediately. □

Remark 3.5. Notice that if the components J^0_A, $J^{\bar{0}}_A$, appearing in (3.7), are identically equal to 0 along the considered disk, the system (3.7) always admits the trivial solutions $g_\alpha \equiv 0 \equiv g_{\bar{\alpha}}$, regardless on the invertibility of $\mathfrak{C}$ and $\mathfrak{D}$. This fact turns out to be quite useful to produce examples.

The interest for almost L-complex structure is motivated by the following.

Lemma 3.6. *If J is an almost L-complex structure on B^n, any straight disk through* 0 *is stationary with respect to J.*

Proof. Using coordinates (3.6), since J satisfies (i) of Definition 3.3, then

$$J^A_0 = i\delta^A_0, \quad J^A_{\bar{0}} = -i\delta^A_{\bar{0}}, \quad J^A_{0,B} = 0, \quad J^A_{\bar{0},B} = 0. \tag{3.10}$$

Given $v = (v^1, \dots, v^{n-1}) \in \mathbb{C}^{n-1}$, consider the straight disk $f : \Delta \longrightarrow B^n$, with tangent direction at 0 given by $\left[v^1 : \cdots : v^{n-1} : \sqrt{1 - \sum_{i=1}^{n-1} |v^i|^2}\right]$, *i.e.*

$$f(\zeta) \stackrel{\text{def}}{=} \xi^{-1}(\zeta, v^1, \dots, v^{n-1}) \qquad \text{for any } \zeta \neq 0.$$

Recall that f is stationary if and only if there is

$$\widetilde{f} : \overline{\Delta} \longrightarrow T^*\widetilde{B}^n, \qquad \widetilde{f}(\zeta) = (\zeta, v^1, \dots, v^{n-1}; g_A(\zeta)_{A=0,\bar{0},1,\bar{1},\dots,n-1,\overline{n-1}})$$

such that: a) $\zeta^{-1} \cdot \widetilde{f}(\zeta)$ is in the conormal bundle $\mathcal{N} \setminus \{\text{zero section}\}$ of ∂B^n for any $\zeta \in \partial\Delta$; b) $\widetilde{f}$ is $\widetilde{\mathbb{J}}$-holomorphic, *i.e.* $\widetilde{f}_*\left(J_{\text{st}}\frac{\partial}{\partial\bar{\zeta}}\right) = \widetilde{\mathbb{J}}\left(\widetilde{f}_*\left(\frac{\partial}{\partial\bar{\zeta}}\right)\right)$.

We claim that a map $\widetilde{f}$ of the above form and satisfying (a) and (b), exists. In fact, by (3.10), condition (b) is equivalent to (here indices A, B might assume any value, $0, \bar{0}$ included)

$$-ig_{A,\bar{\zeta}} - (J_A^B\Big|_f)g_{B,\bar{\zeta}} - \frac{1}{2}g_B\left(\left(J^B_{A,\bar{0}} + iJ^B_L J^L_{A,\bar{0}}\right)\Big|_f\right) = 0. \tag{3.11}$$

By (3.10), in case $A = 0$ equation (3.11) reduces to $2ig_{0,\bar{\zeta}} = 0$, *i.e.* to the requirement of holomorphicity for $g_0 : \Delta \longrightarrow \mathbb{C}$. On the other hand, since the conormal bundle $\mathcal{N}$ is generated at any point by $\omega = z^0 d\bar{z}^0 + \bar{z}^0 dz^0$, (a) is equivalent to the existence of a continuous $\lambda : \partial\Delta \longrightarrow \mathbb{R} \setminus \{0\}$ such that

$$\zeta^{-1}g_0(\zeta) = \bar{\zeta}\lambda(\zeta) \ \text{ for any } \zeta \in \partial\Delta, \tag{3.12}$$

(and hence $g_0 \equiv 1 \equiv g_{\bar{0}}$) and to the requirement that $g_\alpha|_{\partial\Delta}, g_{\bar{\alpha}}|_{\partial\Delta}, \alpha \neq 0$, satisfy the boundary conditions of (3.7). Therefore, inserting $g_0 = g_{\bar{0}} = 1$ into (3.11), by condition (ii) of Definition 3.3, we conclude that there always exists a stationary lift $\widetilde{f}(\zeta) = (\zeta, v^1, \dots, v^{n-1}; g_A(\zeta))$ for f. □

By Lemma 3.6, if J is an almost L-complex structure, (B^n, J) is an almost complex domain of circular type that coincides with its normal form. Conversely, one can directly check that if (B^n, J) is a domain of circular type in normal form, then J satisfies all conditions of Definition 3.3. We have therefore the following intrinsic characterizations of normal forms.

Theorem 3.7. *A pair (B^n, J) is a domain of circular type in normal form if and only if J is an almost L-complex structure.*

3.3. Deformation tensors of normal forms

Consider now the J_{st}-invariant distribution $\mathcal{H}$ on $B^n \setminus \{0\}$ defined by

$$\mathcal{H}_z \overset{\text{def}}{=} \{ X \in T_x M \ : \ dd^c_{\mathrm{st}}\tau_o(Z, X) = dd^c_{\mathrm{st}}\tau_o(J_{\mathrm{st}}Z, X) = 0 \}. \tag{3.13}$$

One can directly check that $T_z M = \mathcal{Z}_z \oplus \mathcal{H}_z$ and that $\mathcal{H}_z$ coincides with the holomorphic tangent space to the sphere $S_{|z|} = \{ w \ : |w| = |z| \}$. In particular, for any $0 < c < 1$, *the pair* $(\mathcal{H}|_{S_c}, J_{\mathrm{st}})$ *is the CR structure of the sphere* $S_c = \{ \tau_o = c \}$. It is known that the distributions $\mathcal{Z}$ and $\mathcal{H}$ extend smoothly on the blow up $\widetilde{B}$ and that also such extensions are J_{st}-invariant (see *e.g.* [15,16,18]).

Recall that any complex structure J_z on a tangent space $T_z B^n$ is uniquely determined by its $-i$-eigenspaces $(T_z B^n)^{01}_J$ in $T^{\mathbb{C}}_z B^n$. If $\mathcal{Z}^{\mathbb{C}}_z = \mathcal{Z}^{10}_z + \mathcal{Z}^{01}_z$ and $\mathcal{H}^{\mathbb{C}}_z = \mathcal{H}^{10}_z + \mathcal{H}^{01}_z$ are decompositions into J_{st}-eigenspaces, a generic complex structure $J_z : T_z B^n \longrightarrow T_z B^n$ is completely determined by the tensors

$$\phi^{\mathcal{Z}}_z \in \mathrm{Hom}(\mathcal{Z}^{01}_z, \mathcal{Z}^{10}_z), \quad \phi^{\mathcal{H}}_z \in \mathrm{Hom}(\mathcal{H}^{01}_z, \mathcal{H}^{10}_z), \quad \phi^{\mathcal{Z},\mathcal{H}}_z \in \mathrm{Hom}(\mathcal{Z}^{01}_z, \mathcal{H}^{10}_z),$$
$$\phi^{\mathcal{H},\mathcal{Z}}_z \in \mathrm{Hom}(\mathcal{H}^{01}_z, \mathcal{Z}^{10}_z),$$

which determine the $-i$-eigenspace $(T_z B^n)^{01}_J$ as the complex subspace

$$\begin{aligned}(T_z B^n)^{01}_J = &\Big(\mathcal{Z}^{01} + \phi^{\mathcal{Z}}_z(\mathcal{Z}^{01}) + \phi^{\mathcal{Z},\mathcal{H}}_z(\mathcal{Z}^{01})\Big)\\ &+ \Big(\mathcal{H}^{01} + \phi^{\mathcal{H}}_z(\mathcal{H}^{01}) + \phi^{\mathcal{H},\mathcal{Z}}_z(\mathcal{H}^{01})\Big).\end{aligned} \tag{3.14}$$

We call *deformation tensor of J with respect to to* J_{st} the tensor field $\phi \in (T^{01*} \otimes T^{10})(B^n \setminus \{0\})$, defined at any point by $\phi_z \overset{\text{def}}{=} \phi^{\mathcal{Z}}_z + \phi^{\mathcal{Z},\mathcal{H}}_z + \phi^{\mathcal{H}}_z + \phi^{\mathcal{H},\mathcal{Z}}_z$. From Definition 3.3 (i), it follows immediately that

Proposition 3.8. *A generic almost L-complex structure J is uniquely determined by a deformation tensor of the form*

$$\phi_z = \phi^{\mathcal{H}}_z + \phi^{\mathcal{H},\mathcal{Z}}_z \qquad \textit{for any } z \in B^n \setminus \{0\}. \tag{3.15}$$

and, conversely, any deformation tensor as in (3.15) *gives an almost L-complex structure, provided that the corresponding J extends to ∂B^n and* 0 *as required in* (ii)-(iv) *of of Definitions* 3.3.

Remark 3.9. Conditions (iii)-(iv) of Definition 3.3 are requirements that might be hard to check. However, to construct examples, it is often sufficient to observe that, given a deformation tensor ϕ_o that satisfy those conditions (*e.g.* $\phi_o \equiv 0$), also the deformation tensors of the form $\phi = \phi_o + \delta\phi$, in which $\delta\phi$ vanishes identically on some neighborhood $\mathcal{U}$ of 0, satisfy them.

Notice also that if J is an almost complex structure, determined by a deformation tensor of the form $\phi = \phi^{\mathcal{H}}$, for any given disk $f(\zeta) = \zeta \cdot v$, we may choose coordinates of the form (3.6), in which $v = (0, \dots, 0, 1)$ and hence the components J^0_A, $J^{\bar{0}}_A$, appearing in (3.7), are identically equal to 0 along the disk. In this case, (3.7) always admits the trivial solutions $g_\alpha \equiv 0 \equiv g_{\bar{\alpha}}$. On the other hand, condition (ii) corresponds to the solvability of the system that determines stationary lifts and hence it is independent on the choice of the coordinate system. All this implies that when $\phi = \phi^{\mathcal{H}}$, condition (ii) of Definition 3.3 is always automatically satisfied.

4. Extremal disks and critical foliations

4.1. Critical and extremal disks

In this section, we recall the notion of "critical disks", recently introduced by Gaussier and Joo in [9] in their studies on the extremality with respect to the Kobayashi metric of J-holomorphic disks. We have to point out that "critical disks" is not the name used in [9]. In that paper, such disks are called "disks vanishing the first order variations".

For this, we first need to remind of a few concepts related with the geometry of the tangent bundle of a manifold M. We recall that the *vertical distribution in* $T(TM)$ is the subbundle of $T(TM)$ defined by

$$T^V(TM) = \bigcup_{(x,v)\in TM} T^V_{(x,v)}M, \qquad T^V_{(x,v)}M = \ker \pi_*|_{(x,v)}.$$

For any $x \in M$, let us denote by $(\cdot)^V : T_xM \longrightarrow T^V_{(x,v)}M$ the map $\left(w^i \frac{\partial}{\partial x^i}\Big|_x \right)^V \overset{\text{def}}{=} w^i \frac{\partial}{\partial q^i}\Big|_{(x,v)}$. It is possible to check that this map does not depend on the choice of coordinates and that it determines a natural map from TM to $T(TM)$ (see [24]). For any $w \in TM$, the corresponding vector $w^V \in T(TM)$ is called *vertical lift of* w.

Definition 4.1 ([10]). Let $f : \overline{\Delta} \longrightarrow M$ be a $\mathcal{C}^{\alpha,\epsilon}$, J-holomorphic embedding with $f(\partial\Delta) \subset \partial D$. We call *infinitesimal variation of* f any $\mathbb{J}$-holomorphic map $W : \overline{\Delta} \longrightarrow TM$ of class $\mathcal{C}^{\alpha-1,\epsilon}$ with $\pi \circ W = f$ (here, $\pi : TM \longrightarrow M$ is the natural projection). An infinitesimal variation W is called *attached to* ∂D *and with fixed center* if

a) $\alpha(W_\zeta) = 0$ for any $\alpha \in \mathcal{N}_{f(\zeta)}$, $\zeta \in \partial\Delta$,
b) $W|_0 = 0$.

It is called *with fixed central direction* if in addition it satisfies

c) $W_*\left(\frac{\partial}{\partial \operatorname{Re}\zeta}\Big|_0 \right) \in T^V_{W_0}(TM)$ and it is equal to $\lambda \left(f_*\left(\frac{\partial}{\partial \operatorname{Re}\zeta}\Big|_0 \right) \right)^V$ for some $\lambda \in \mathbb{R}$.

The disk f is called *critical* if for any infinitesimal variation W, attached to ∂D and with fixed central direction, one has $W_*\left(\frac{\partial}{\partial \operatorname{Re}\zeta}\Big|_0\right) = 0$.

Remark 4.2. The previous definition is motivated by the following facts. When $f^{(t)} : \overline{\Delta} \longrightarrow M$, $t \in]-a, a[$, is a smooth 1-parameter family of J-holomorphic disks of class $\mathcal{C}^{\alpha,\epsilon}$ with $f^{(0)} = f$, it is simple to check that $W \overset{\text{def}}{=} \frac{df^{(t)}}{dt}\Big|_{t=0}$ is a variational field on f. Moreover, if $f^{(t)}$ is such that, for all $t \in]a, a[$

$$f^{(t)}(\partial\Delta) \subset \partial D, \quad f^{(t)}(0) = f(0), \quad f_*^{(t)}\left(\frac{\partial}{\partial \operatorname{Re}\zeta}\Big|_0\right) \in \mathbb{R} f_*\left(\frac{\partial}{\partial \operatorname{Re}\zeta}\Big|_0\right), \tag{4.1}$$

then W satisfies (a)-(c). On the other hand, a disk f is a *locally extremal disk* if for any J-holomorphic disk $g : \overline{\Delta} \longrightarrow M$ of class $\mathcal{C}^{\alpha,\epsilon}$, with image contained in some neighborhood of $f(\overline{\Delta})$ and such that, for some $\lambda \in \mathbb{R}$,

$$g(\partial\Delta) \subset \partial D, \quad g(0) = f(0) = x_o, \quad g_*\left(\frac{\partial}{\partial \operatorname{Re}\zeta}\Big|_0\right) = \lambda f_*\left(\frac{\partial}{\partial \operatorname{Re}\zeta}\Big|_0\right),$$

then $\lambda \leq 1$. One can directly check that *any locally extremal disk f, with $f(\partial\Delta) \subset \partial D$, is critical* (see [10, proof of Theorem 4.3]). Conversely, by Section 5 and [9, Theorem 6.4], *in case $D \subset \mathbb{C}^n$ is strictly convex on a neighborhood of $f(\overline{\Delta})$ (in suitable cartesian coordinates) and J is sufficiently close to J_{st}, any critical disk f is locally extremal.*

Notice that, when J is close to J_{st}, the critical disks are characterized by properties that closely resemble those that define the stationary disks. Disks with such properties are called *J-stationary disks* [9].

It is well-known that, when J is integrable, the disks that are stationary coincide with the disks that are critical (see *e.g.* [11, 14]). This equality is no longer valid for generic non-integrable complex structures. In [9], counterexamples are given.

Next theorem gives conditions that imply the equality between stationary and critical disks and will be used in the sequel. The claim and the proof are refinements of a result and arguments given in [9]. In the statement, $f : \overline{\Delta} \longrightarrow M$ is a J-holomorphic embedding, of class $\mathcal{C}^{\alpha,\epsilon}$ with $f(\partial\Delta) \subset \partial D$, and $\mathfrak{Var}_o(f)$ denotes the class of infinitesimal variations of f attached to ∂D and with fixed center.

Theorem 4.3. *Assume that $D \subset M$ is of the form $D = \{ \rho < 0 \}$ for some J-plurisubharmonic ρ (see Section 5.1, for definition) and that $\mathfrak{Var}_o(f)$ contains a $(2n-2)$-dimensional J-invariant vector space, generated by infinitesimal variations e_i, Je_i, $1 \leq i \leq n-1$, such that the maps $\zeta^{-1}\cdot e_i(\zeta)$, $\zeta^{-1}\cdot Je_i(\zeta) : \overline{\Delta} \longrightarrow TM$ are of class $\mathcal{C}^{\alpha,\epsilon}$ on $\overline{\Delta}$. Assume also that, for any $\zeta \in \overline{\Delta}$, the set $\{e_i(\zeta), Je_i(\zeta)\} \subset T_{f(\zeta)}M$ span a subspace, which is complementary to $T_{f(\zeta)}f(\Delta) \subset T_{f(\zeta)}M$. Then f is critical if and only if it is stationary.*

Proof. Let $e_0 : \overline{\Delta} \longrightarrow TM$ be the map defined by $e_0(\zeta) = f_*\left(\frac{\partial}{\partial \operatorname{Re}\zeta}\Big|_\zeta\right)$. By hypotheses, the collection

$$\Big(e_0(\zeta), Je_0(\zeta), \zeta^{-1}\cdot e_1(\zeta), \dots, \zeta^{-1}\cdot e_{n-1}(\zeta), \zeta^{-1}\cdot Je_{n-1}(\zeta)\Big) \tag{4.2}$$

is a basis for $T_{f(\zeta)}M$ for all $\zeta \in \Delta$ and we may consider a system of coordinates $\xi : (x^0, x^1, \dots, x^{2n-2}, x^{2n-1}) = (z^0, \dots, z^{n-1}) : \mathcal{U} \longrightarrow \mathbb{R}^{2n} = \mathbb{C}^n$ on a neighborhood $\mathcal{U}$ of $f(\overline{\Delta})$ such that

$$\begin{aligned}&\frac{\partial}{\partial x^0}\bigg|_{f(\zeta)} = e_0(\zeta), \quad \frac{\partial}{\partial x^1}\bigg|_{f(\zeta)} = J(e_0(\zeta)),\\ &\frac{\partial}{\partial x^{2i}}\bigg|_{f(\zeta)} = \zeta^{-1}\cdot e_i(\zeta), \quad \frac{\partial}{\partial x^{2i+1}}\bigg|_{f(\zeta)} = J(\zeta^{-1}\cdot e_i(\zeta)) \ \text{ for all } \ \zeta \in \Delta\setminus\{0\}.\end{aligned} \tag{4.3}$$

If we identify $\mathcal{U}$ with $\xi(\mathcal{U}) \subset \mathbb{C}^n$, we have that $J|_y = J_{\rm st}|_y$ for all $y \in f(\Delta)$ and the maps f, e_i and Je_i are of the form (here, any vector valued map is denoted by a pair, formed by the base point and the vector components):

$$\begin{aligned} f(\zeta) &= (\operatorname{Re}\zeta, \operatorname{Im}\zeta, 0, \dots, 0),\\ e_i(\zeta) &= ((\operatorname{Re}\zeta, \operatorname{Im}\zeta, 0, \dots, 0); (0, \dots, \underbrace{\operatorname{Re}(\zeta)}_{2i\text{-th place}}, \dots, 0)),\\ Je_i(\zeta) &= ((\operatorname{Re}\zeta, \operatorname{Im}\zeta, 0, \dots, 0); (0, \dots, \underbrace{\operatorname{Im}(\zeta)}_{(2i+1)\text{-th place}}, \dots, 0)).\end{aligned} \tag{4.4}$$

A map $W = (f, v^j \frac{\partial}{\partial x^j}) : \Delta \longrightarrow T\mathbb{R}^{2n} \simeq TM$ is an infinitesimal variation (*i.e.* $\mathbb{J}$-holomorphic) if and only if it is solution of the p.d.e. system

$$\frac{\partial v^i}{\partial \operatorname{Re}\zeta} + J^i_j\Big|_f \frac{\partial v^j}{\partial \operatorname{Im}\zeta} + \frac{\partial J^i_j}{\partial x^k}\bigg|_f v^k \frac{\partial f^j}{\partial \operatorname{Im}\zeta} = \frac{\partial v^i}{\partial \operatorname{Re}\zeta} + {J_{\rm st}}^i_j \frac{\partial v^j}{\partial \operatorname{Im}\zeta} + \frac{\partial J^i_1}{\partial x^k}\bigg|_f v^k = 0. \tag{4.5}$$

By hypotheses, the fields in (4.4) are solutions of (4.5). From this and the fact that the components $J^i_j|_{f(\Delta)} = {J_{\rm st}}^i_j|_{f(\Delta)}$ are constant on $f(\Delta)$ we get that

$$\frac{\partial J^i_1}{\partial x^k} = \frac{\partial J^i_j}{\partial x^0} = \frac{\partial J^i_j}{\partial x^1} = 0 \qquad \text{at all points of } f(\Delta). \tag{4.6}$$

From this and the explicit formulae in coordinates for $\mathbb{J}$ and $\widetilde{\mathbb{J}}$, one can directly check that a map $W = (f, w^j \frac{\partial}{\partial z^j} + \overline{w}^j \frac{\partial}{\partial \overline{z}^j}) : \Delta \longrightarrow T\mathbb{C}^n \simeq TM$ (respectively $\widetilde{f} = (f, g_i dz^i + \overline{g}_i d\overline{z}^i) : \Delta \longrightarrow T^*\mathbb{C}^n \simeq T^*M$) is $\mathbb{J}$- (respectively $\widetilde{\mathbb{J}}$-) holomorphic if and only if the functions w^j (respectively g_i) are holomorphic in the classical sense. Assume that f is stationary and that $\widetilde{f} = ((\zeta, 0, \dots, 0); g_i dz^i + \overline{g}_i d\overline{z}^i)$ is a stationary lift of f and $W : \overline{\Delta} \longrightarrow TM$ is an infinitesimal variation, attached to ∂D and with fixed central direction. Then, for any $\zeta \neq 0$, $W(\zeta)$ is the form $W(\zeta) = \zeta \cdot \mu^j(\zeta) \cdot e_i(\zeta) + \overline{\zeta \cdot \mu^j(\zeta) \cdot e_i(\zeta)}$, for some holomorphic $\mu^j : \Delta \longrightarrow \mathbb{C}$, and $W_*\left(\frac{\partial}{\partial \operatorname{Re} \zeta}\Big|_0\right) = \mu^j(0) \cdot e_j(0)$. By Definition 4.1 (c), we get that $\mu^0(0) \in \mathbb{R}$ and $\mu^i(0) = 0$ for $i \neq 0$. On the other hand, by Definition 4.1(b), the function

$$\varphi : \overline{\Delta} \longrightarrow \mathbb{R}, \qquad \varphi(\zeta) = \widetilde{f}(\zeta)\left(\zeta^{-1} \cdot W(\zeta)\right) = \operatorname{Re}(\mu^j g_j)(\zeta),$$

is so that $\varphi|_{\partial\Delta} \equiv 0$. Since φ is harmonic, $\varphi \equiv 0$ and in particular $\mu^0(0)\operatorname{Re}(g_0(0)) = 0$. By [9, Corollary 2.5], $\widetilde{f}\left(\frac{\partial}{\partial \operatorname{Re}\zeta}\Big|_0\right) = g_0(0) \neq 0$ and $\mu^0(0) = 0$, *i.e.* f is critical. Conversely, assume f critical and let $\widetilde{f} : \overline{\Delta} \longrightarrow T^*M$ be defined by $\widetilde{f}(\zeta) = (\zeta, 0 \dots, 0; dz^0 + d\overline{z}^0)$. By previous observations, $\widetilde{f}$ is $\widetilde{\mathbb{J}}$-holomorphic and it is an embedding. From the fact that $Je_0|_\zeta$, $e_i|_\zeta$, $Je_i|_\zeta$ span the tangent spaces $T_{f(\zeta)}\partial D$, $\zeta \in \partial\Delta$, we get that $\widetilde{f}$ is a stationary lift of f and that f is stationary. □

4.2. Critical foliations of normal forms

Let (B^n, J) be an almost complex circular domain in normal form. We recall that, for any $v \in S^{2n-1} \subset T_0 B^n$, the straight disk $f(\zeta) = \zeta \cdot v$ is stationary for (B^n, J).

For any $w \in T_v S^{2n-1}$ and any smooth curve $\gamma_t^{(w)}$ in S^{2n-1} with $\gamma_0^{(w)} = v$ and $\dot{\gamma}_0^{(w)} = w$, we may consider the infinitesimal variation, with fixed center and attached to ∂B^n, defined by

$$W^{(w)} : \overline{\Delta} \to T\mathbb{C}^{2n}, \qquad W^{(w)}(\zeta) = \left.\frac{d\left(\zeta \cdot \gamma_t^{(w)}\right)}{dt}\right|_{t=0} = \zeta \cdot w \in T_{f(\zeta)}\mathbb{C}^n. \quad (4.7)$$

These maps form a $(2n-2)$-subspace $\widetilde{\mathfrak{Var}}_o(f)$ of the vector space of infinitesimal variations in $\mathfrak{Var}_o(f)$ such that:

a) $\left\{W(\zeta),\ W \in \widetilde{\mathfrak{Var}}_o(f)\right\} = \mathcal{H}_{f(\zeta)}$ for any $\zeta \neq 0$;

b) for any $W \in \widetilde{\mathfrak{Var}}_o(f)$, the map $\alpha(\zeta) \stackrel{\text{def}}{=} |\zeta|^{-1}|W(\zeta)|$ is constant on $\overline{\Delta} \setminus \{0\}$.

Moreover,

Lemma 4.4. *The space $J\widetilde{\mathfrak{Var}}_o(f)$ is included in $\mathfrak{Var}_o(f)$ if and only if $\mathcal{L}_{Z^{01}} J = 0$, with $Z^{01} = \overline{z}^i \frac{\partial}{\partial \overline{z}^i}$.*

Proof. Consider an open subset $\mathcal{V} \subset S^{2n-1} \subset T_0 B^n$ and a field of real frames $(e_1^o(v), J_0 e_1^o(v) \dots, J_0 e_{n-1}^o(v)), v \in \mathcal{V}$, for the holomorphic subspaces $H_v \subset T_v S^{2n-1}$ of S^{2n-1}. We denote by $e_i(v, \cdot)$, $(J_0 e_i)(v, \cdot)$ the corresponding infinitesimal variations along $f^{(v)}(\zeta) = \zeta \cdot v$, *i.e.*

$$e_1(v; \zeta) \stackrel{\text{def}}{=} \zeta \cdot e_1^o(v), \qquad \dots, \qquad (J_0 e_{n-1})(v; \zeta) \stackrel{\text{def}}{=} \zeta \cdot (J_0 e_{n-1}^o(v)).$$

Notice that the points $f^{(v)}(\zeta)$, with $(v, \zeta) \in \mathcal{V} \times \{\Delta \setminus \{0\}\}$, fill an open subset $\mathcal{U} \subset B^n \setminus \{0\}$, that the ordered set of vector fields

$$\left(e_0(v;\zeta) \overset{\text{def}}{=} f_*^{(v)}\left(\frac{\partial}{\partial x}\bigg|_\zeta\right), Je_0(v;\zeta) \overset{\text{def}}{=} f_*^{(v)}\left(\frac{\partial}{\partial y}\bigg|_\zeta\right), e_1(v;\zeta), \dots, (J_0 e_{n-1})(v;\zeta)\right)$$

is a frame field on $\mathcal{U} \simeq \mathcal{V} \times \{\Delta \setminus \{0\}\}$ and that the field $\frac{1}{2}(e_0(v;\zeta) + iJe_0(v;\zeta))$ $= f_*^{(v)}\left(\frac{\partial}{\partial \bar\zeta}\Big|_\zeta\right)$ is a generator for $\mathcal{Z}^{01}$. One can also check that

$$\mathcal{L}_{Z^{01}} e_i = \mathcal{L}_{Z^{01}}(J_0 e_i) = 0 \qquad \text{for } 1 \le i \le n-1. \tag{4.8}$$

We claim that $\mathcal{L}_{Z^{01}} Je_i = \mathcal{L}_{Z^{01}} J(J_0 e_i) = 0$ (or, equivalently, that $\mathcal{L}_{Z^{01}} J|_{\mathcal{U}} = 0$) if and only if the fields $Je_i(v;\zeta)$ and $J(J_0 e_i)(v;\zeta)$ are in $\mathfrak{Var}_o(f^{(v)})$; by arbitrariness of $\mathcal{V} \subset S^{2n-1}$ this conclude the proof. To check the claim, let us fix a straight disk $f^{(v)}(\zeta) = \zeta \cdot v$. By construction, the fields $\zeta^{-1} \cdot e_i(v;\zeta)$ and $\zeta^{-1} \cdot (J_0 e_i)(v;\zeta)$ are of class $\mathcal{C}^\infty$ at any $\zeta \in \overline{\Delta}$. We may therefore consider a system of coordinates $\xi = (z^0 = x^0 + ix^1, \dots, z^{n-1} = x^{2n-2} + ix^{2n-1})$ around $f^{(v)}(\overline{\Delta})$ satisfying (4.3). As in the proof of Theorem 4.3, we have that a vector field $W : \Delta \longrightarrow TB^n$ along $f^{(v)}(\Delta)$ is $\mathbb{J}$-holomorphic if and only if the complex functions $w^j : \Delta \longrightarrow \mathbb{C}$ such that

$$W_\zeta = \mathrm{Re}(w^j(\zeta))\, e_j(f^{(v)}(\zeta)) + \mathrm{Im}(w^j(\zeta))(J_0 e_j)(f^{(v)}(\zeta))$$

are holomorphic. By (4.8), such holomorphicity condition is equivalent to $\mathcal{L}_{Z^{01}} W = 0$. Since, by construction, the fields Je_i and $J(J_0 e_i)$ satisfy (a) and (b) of Definition 4.1, they are in $\mathfrak{Var}_o(f^{(v)})$ if and only if they are $\mathbb{J}$-holomorphic, *i.e.* if and only if $\mathcal{L}_{Z^{01}} Je_i = \mathcal{L}_{Z^{01}} J(J_0 e_i) = 0$. □

Let us introduce the following definition.

Definition 4.5. Let (B^n, J) be an almost complex domain of circular type in normal form. We call it *nice* if the distribution $\mathcal{H}$ defined in (3.13) is J-invariant. We call it *very nice* if for any straight disk $f(\zeta) = \zeta \cdot v$, the associated vector space $\widetilde{\mathfrak{Var}}_o(f)$ is J-invariant.

An almost complex domain (D, J) of circular type with center x_o is called *nice* (respectively *very nice*) if it has a nice (respectively very nice) normal form.

Motivation for considering such notions comes from the following

Proposition 4.6. *The stationary disks of the circular type foliation $\mathcal{F}^{(x_o)}$ of a very nice almost complex domain (D, J) of the form $D = \{\ \rho < 0\ \}$ for some J-plurisubharmonic ρ, are critical.*

Proof. With no loss of generality, assume that $(D, J) = (B^n, J)$ is in normal form and that its foliation of circular type is given by the straight disks $f(\zeta) = \zeta \cdot v$, $v \in S^{2n-1} \subset T_0 B^n \simeq \mathbb{C}^n$. Fix $v \in S^{2n-1}$ and let $H_v \subset T_v S^{2n-1}$ be the holomorphic tangent space at v and $(e^o_1, \dots, e^o_{n-1})$ a basis over $\mathbb{C}$ for H_v. Consider the infinitesimal variations in $\widetilde{\mathfrak{Var}}_o(f)$ defined in (4.8)

$$e_1(\zeta) \overset{\text{def}}{=} \zeta \cdot e^o_1, \qquad \dots, \qquad e_{n-1}(\zeta) \overset{\text{def}}{=} \zeta \cdot e^o_{n-1}, \qquad \zeta \in \overline{\Delta}.$$

By construction, the fields $\zeta^{-1} \cdot e_i$ and $\zeta^{-1} \cdot Je_i$, defined at the points of $\overline{\Delta}$, are of class $\mathcal{C}^\infty$. Being (B^n, J) very nice, they span $\mathcal{H}_z \subset T_{f(\zeta)}\mathbb{C}^n$, which is complementary to $T_{\varphi(\zeta)} f(\overline{\Delta})$. By Theorem 4.3, the conclusion follows. □

By definitions any very nice domain is nice. The converse is not true, as next proposition and example show.

Proposition 4.7. *Let (B^n, J) be an almost complex domain of circular type in normal form, with J given by a deformation tensor $\phi = \phi^{\mathcal{H}} + \phi^{\mathcal{H},\mathcal{Z}}$. It is nice if and only if $\phi^{\mathcal{H},\mathcal{Z}} \equiv 0$, while it is very nice if and only if*

$$\phi^{\mathcal{H},\mathcal{Z}} \equiv 0 \qquad \textit{and} \qquad \mathcal{L}_{Z^{01}}\phi^{\mathcal{H}} = 0. \tag{4.9}$$

Proof. It follows from definitions and Lemma 4.4. □

Example 4.8. Let $\phi = \phi^{\mathcal{H}}$ be a deformation tensor in $\mathrm{Hom}(\mathcal{H}^{01}_z, \mathcal{H}^{10}_z)$ at any $z \in B^n$, which is non zero only on a relatively compact subset, whose closure does not contain the origin. By Remark 3.9 and Proposition 4.7, $\phi = \phi^{\mathcal{H}}$ determines an almost complex structure J such that (B^n, J) is nice. On the other hand, by assumptions, there are straight disks f with $\phi^{\mathcal{H}}|_{f(\Delta)} \not\equiv 0$ and with $\phi^{\mathcal{H}}|_{\mathcal{V}} \equiv 0$ on some open subset of $\mathcal{V} \subset f(\Delta)$. Due to this, the equality $\mathcal{L}_{Z^{01}}\phi^{\mathcal{H}} = 0$ is not satisfied and (B^n, J) is not very nice.

5. Almost complex Monge-Ampère operators

5.1. Plurisubharmonic functions and pseudoconvex manifolds

Let (M, J) be an almost complex manifold and $\Omega^k(M)$, $k \geq 0$, the space of k-forms of M. We denote by $d^c : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$ the classical d^c-operator

$$d^c\alpha = (-1)^k (J^* \circ d \circ J^*)(\alpha),$$

where J^* denotes the usual action of J on k-forms, *i.e.* $J^*\beta(v_1, \dots, v_k) \overset{\text{def}}{=} (-1)^k \beta(Jv_1, \dots, Jv_k)$ (see *e.g.* [2]). If J is integrable, it is well known that

$$d^c = i(\overline{\partial} - \partial), \quad \partial\overline{\partial} = \frac{1}{2i} dd^c, \quad dd^c = -d^c d$$

and that $dd^c u$ is a J-Hermitian 2-form for any $\mathcal{C}^2$-function u. Unfortunately, *when J is not integrable, $d^c d \neq -dd^c$ and the 2-forms $dd^c u$, with $u \in \mathcal{C}^2(M)$, are usually not J-Hermitian*. In fact, one has that

$$dd^c u(JX_1, X_2) + dd^c u(X_1, JX_2) = 4N_{X_1 X_2}(u), \tag{5.1}$$

where $N_{X_1X_2}$ is the Nijenhuis tensor evaluated on X_1, X_2 and is in general non zero. This fact suggests the following definition.

Definition 5.1. Let $u : \mathcal{U} \subset M \longrightarrow \mathbb{R}$ be of class $\mathcal{C}^2$. We call *J-Hessian of u at x* the symmetric form $\mathcal{H}ess(u)_x \in S^2T_xM$, whose associated quadratic form is $\mathcal{L}(u)_x(v) = dd^cu(v, Jv)_x$. By polarization formula and (5.1), one has that, for any $v, w \in T_xM$,

$$\begin{aligned}\mathcal{H}ess(u)_x(v,w) &= \frac{1}{2}\left(dd^cu(v,Jw) + dd^cu(w,Jv)\right)\Big|_x \\ &= dd^cu(v,Jw)_x - 2N_{vw}(u).\end{aligned} \tag{5.2}$$

We remark that $\mathcal{H}ess(u)_x$ is not only symmetric, but also J-Hermitian, *i.e.* $\mathcal{H}ess(u)_x(Jv, Jw) = \mathcal{H}ess(u)_x(v,w)$ for any v, w. It is therefore associated with the Hermitian antisymmetric tensor

$$\mathcal{H}ess(u)(J\cdot,\cdot) = \frac{1}{2}\left(dd^cu(\cdot,\cdot) + dd^cu(J\cdot,J\cdot)\right) = \frac{1}{2}\left(dd^cu + J^*dd^cu\right). \tag{5.3}$$

The quadratic form $\mathcal{L}(u)_x(v) = dd^cu(v, Jv)|_x$ is the so-called *Levi form of u at x* (see *e.g.* [6]) and it is tightly related with the notion of J-plurisubharmonicity. On this regard, we recall that an upper semicontinuous function $u : \mathcal{U} \subset M \longrightarrow \mathbb{R}$ is called *J-plurisubharmonic* if, for any J-holomorphic disk $f : \Delta \longrightarrow \mathcal{U} \subset M$, the composition $u \circ f : \Delta \longrightarrow \mathbb{R}$ is subharmonic. By simple arguments (similar to those used for complex manifolds), whenever u is in $\mathcal{C}^2(\mathcal{U})$ one has that *u is J-plurisubharmonic if and only if* $\mathcal{L}(u)_x(v) = \mathcal{H}ess(u)_x(v,v) \geq 0$ *for any* $x \in \mathcal{U}$ *and* $v \in T_xM$.

This motivates the following generalizations of classical notions (see *e.g.* [7]). In the following, for any $\mathcal{U} \subset M$, the symbol $\mathrm{Psh}(\mathcal{U})$ denotes the class of J-plurisubharmonic functions on $\mathcal{U}$.

Definition 5.2. Let (M, J) be an almost complex manifold and $\mathcal{U} \subset M$ an open subset. We say that $u \in \mathrm{Psh}(\mathcal{U})$ is *strictly J-plurisubharmonic* if:

a) $u \in L^1_{\mathrm{loc}}(\mathcal{U})$;
b) for any $x_o \in \mathcal{U}$ there exists a neighborhood $\mathcal{V}$ of x_o and $v \in \mathcal{C}^2(\mathcal{V}) \cap \mathrm{Psh}(\mathcal{V})$ for which $\mathcal{H}ess(v)_x$ is positive definite at all points and $u - v$ is in $\mathrm{Psh}(\mathcal{V})$.

In particular, $u \in \mathrm{Psh}(\mathcal{U}) \cap \mathcal{C}^2(\mathcal{U})$ is strictly plurisubharmonic if and only if $\mathcal{H}ess(u)_x$ is positive definite at any $x \in \mathcal{U}$.

The almost complex manifold (M, J) is called *weakly* (respectively *strongly*) *pseudoconvex* if it admits a $\mathcal{C}^2$ exhaustion $\tau : M \longrightarrow]-\infty, \infty[$, which is plurisubharmonic (respectively strictly plurisubharmonic)[2].

[2] Strongly pseudoconvex manifolds are called *almost complex Stein manifolds* in [8].

5.2. Maximal plurisubharmonic functions

J-plurisubharmonic functions share most of the basic properties of classical plurisubharmonic functions. For instance, for any open domain $\mathcal{U} \subset M$, the class $\mathrm{Psh}(\mathcal{U})$ is a convex cone and a lattice, as for domains in complex manifolds. In fact, given $u_i \in \mathrm{Psh}(\mathcal{U})$ and $\lambda_i \in \mathbb{R}$, also the functions $u = \sum_{i=1}^n \lambda_i u_i$ and $u' = \max\{ u_1, \dots, u_n \}$ are in $\mathrm{Psh}(\mathcal{U})$.

It is therefore natural to consider the following notion of "maximal" J-plurisubharmonic functions. This and next theorem indicate which operator should be considered as natural generalization of the classical complex Monge-Ampère operator.

Definition 5.3. Let D be a domain in a strongly pseudoconvex almost complex manifold (M, J). A function $u \in \mathrm{Psh}(D)$ is called *maximal* if for any open $\mathcal{U} \Subset D$ and $h \in \mathrm{Psh}(\mathcal{U})$ satisfying the condition

$$\limsup_{z \to x} h(z) \leq u(x) \qquad \text{for all } x \in \partial\mathcal{U}, \tag{5.4}$$

one has that $h \leq u|_{\mathcal{U}}$.

Theorem 5.4. *Let $D \subset M$ be a domain of a strongly pseudoconvex almost complex manifold (M, J) of dimension $2n$. A function $u \in \mathrm{Psh}(D) \cap \mathcal{C}^2(D)$ is maximal if and only if it satisfies*

$$\left(dd^c u + J^*(dd^c u)\right)^n = 0. \tag{5.5}$$

Proof. Let $\tau : M \longrightarrow]-\infty, +\infty[$ be a $\mathcal{C}^2$ strictly plurisubharmonic exhaustion for M and assume that u satisfies (5.5). We need to show that for any $h \in \mathrm{Psh}(\mathcal{U})$ on an $\mathcal{U} \Subset D$ that satisfies (5.4), one has that $h \leq u|_{\mathcal{U}}$. Suppose not and pick $\mathcal{U} \Subset D$ and $h \in \mathrm{Psh}(\mathcal{U})$, so that (5.4) is true but there exists $x_o \in \mathcal{U}$ with $u(x_o) < h(x_o)$. Let $\lambda > 0$ so small that

$$h(x_o) + \lambda\,(\tau(x_o) - M) > u(x_o), \qquad \text{where } M = \max_{y \in \overline{\mathcal{U}}} \tau(y),$$

and denote by $\widehat{h}$ the function

$$\widehat{h} \overset{\text{def}}{=} h + \lambda(\tau - M)|_{\mathcal{U}}\,. \tag{5.6}$$

By construction, $\widehat{h} \in \mathrm{Psh}(\mathcal{U})$, satisfies (5.4) and $(\widehat{h} - u)(x_o) > 0$. In particular, $\widehat{h} - u$ achieves its maximum at some inner point $y_o \in \mathcal{U}$. Now, we remark that (5.5) is equivalent to say that, for any $x \in D$, there exists $0 \neq v \in T_x M$ so that

$$\left(dd^c u + J^*(dd^c u)\right)_x (v, Jv) = \mathcal{H}ess_x(u)(v, v) = 0. \tag{5.7}$$

Let $0 \neq v_o \in T_{y_o} M$ be a vector for which (5.7) is true and let $f : \Delta \longrightarrow M$ be a J-holomorphic disk so that $f(0) = y_o$ and with

$$f_*\left(\left.\frac{\partial}{\partial x}\right|_0\right) = v_o, \qquad f_*\left(\left.\frac{\partial}{\partial y}\right|_0\right) = f_*\left(J_{\mathrm{st}}\left.\frac{\partial}{\partial x}\right|_0\right) = J v_o.$$

Then, consider the function $G : \Delta \longrightarrow \mathbb{R}$ defined by

$$G \stackrel{\text{def}}{=} \widehat{h} \circ f - u \circ f = h \circ f + (\lambda\tau - \lambda M - u) \circ f. \tag{5.8}$$

We claim that there exists a disk $\Delta_r = \{|\zeta| < r\}$ such that $G|_{\Delta_r}$ is subharmonic. In fact, since τ is $\mathcal{C}^2$ and strictly plurisubharmonic and $\mathcal{H}ess(u)_{y_o}(v_o, v_o) = 0$, we have that

$$0 < \mathcal{H}ess((\lambda\tau - \lambda M - u))_{y_o}(v_o, v_o) = 2i\ \partial\overline{\partial}((\lambda\tau - \lambda M - u) \circ f)\big|_0 .$$

Hence, by continuity, there exists $r > 0$ so that

$$0 < 2i\ \partial\overline{\partial}((\lambda\tau - \lambda M - u) \circ f)\big|_\zeta \qquad \text{for any} \quad \zeta \in \overline{\Delta_r}.$$

It follows that $(\lambda\tau - \lambda M - u) \circ f|_{\Delta_r}$ is strictly subharmonic and that $G|_{\Delta_r}$ is subharmonic, being sum of subharmonic functions. At this point, it suffices to observe that, since y_o is a point of maximum for $\widehat{h} - u$ on $f(\Delta) \subset \mathcal{U}$, then $0 = f^{-1}(y_o) \in \Delta_r$ is an inner point of maximum for $G|_{\Delta_r}$. In fact, from this and the maximum principle, we get that $G|_{\Delta_r}$ is constant and hence that $h \circ f|_{\Delta_r}$ is $\mathcal{C}^2$ with $2i\ \partial\overline{\partial}(h \circ f)\big|_{\Delta_r} < 0$, contradicting the hypothesis on subharmonicity of $h \circ f$.

Conversely, assume that $u \in \mathcal{C}^2(D) \cap \text{Psh}(D)$ is maximal, but that (5.5) is not satisfied, *i.e.* that there exists $y_o \in D$ for which $\mathcal{H}ess_{y_o}(u)(v, v) > 0$ for any $0 \neq v \in T_{y_o}M$. By Lemma 3.2, there exist a relatively compact neighborhood $\mathcal{U}$ of y_o and a (J, J')-biholomorphism between $(\mathcal{U}, J)$ and (B^n, J'), with J' arbitrarily close in $\mathcal{C}^2$ norm to the standard complex structure. Due to this, we may assume that $\tau = \tau_o \circ \varphi$, with $\tau_o(z) = |z|^2$, is a $\mathcal{C}^2$ strictly J-plurisubharmonic exhaustion on $\mathcal{U}$, tending to 1 at the points of $\partial\mathcal{U}$. Hence, there is a constant $c > 0$ such that

$$\mathcal{H}ess_x(u + c(1 - \tau))(v, v) = \mathcal{H}ess_x(u)(v, v) - c\mathcal{H}ess_x(\tau)(v, v) \geq 0,$$

for all $x \in \mathcal{U}$ and $v \in T_x M \simeq \mathbb{R}^{2n}$ with $|v| = 1$. This means that

$$\widehat{h} \stackrel{\text{def}}{=} (u + c(1 - \tau))|_{B_{y_o}(r)}$$

is in $\mathcal{C}^2(\mathcal{U}) \cap \text{Psh}(\mathcal{U})$, satisfies (5.4) and, by maximality of u, satisfies $\widehat{h} \leq u$ at all points of $\mathcal{U}$. But there is also an $\epsilon > 0$ such that $\emptyset \neq \tau^{-1}([0, 1 - \epsilon[) \subsetneq \mathcal{U}$ and hence such that, on this subset, $\widehat{h} \geq u + c\epsilon > u$, contradicting the maximality of u. □

5.3. Green functions of nice circular domains

The results of previous section show that (5.5) is a natural analogue of classical complex Monge-Ampère equation for domains in $\mathbb{C}^n$ and that the solutions of (5.5) are interesting biholomorphic invariants of strongly pseudoconvex domains. This motivates the following generalized notion of Green functions (see *e.g.* [1]).

Definition 5.5. Let D be a relatively compact domain in a strongly pseudoconvex, almost complex manifold (M, J). We call *almost pluricomplex Green function with pole at* $x_o \in D$ an exhaustion $u : \overline{D} \longrightarrow [-\infty, 0]$ such that

i) $u|_{\partial D} = 0$ and $u(x) \simeq \log \|x - x_o\|$ when $x \to x_o$, for some Euclidean metric $\|\cdot\|$ on a neighborhood of x_o;
ii) it is J-plurisubharmonic;
iii) it is a solution of the generalized Monge-Ampere equation $(dd^c u + J^*(dd^c u))^n = 0$ on $D \setminus \{x_o\}$.

Notice that, if a Green function with pole x_o exists, by a direct consequence of property of maximality (Theorem 5.4) it is unique.

Consider now an almost complex domain D of circular type in (M, J) with center x_o. Denoting by $\Phi : \widetilde{B}^n \longrightarrow \widetilde{D}$ the corresponding Riemann map, we call *standard exhaustion of* D the map

$$\tau_{(x_o)} : D \longrightarrow [0, 1[, \qquad \tau_{(x_o)}(x) = \begin{cases} |\Phi^{-1}(x)|^2 & \text{if } x \neq 0, \\ 0 & \text{if } x = x_o. \end{cases}.$$

When D is in normal form, *i.e.* $D = (B^n, J)$ with J almost L-complex structure, its standard exhaustion is just $\tau_o(z) = |z|^2$.

Proposition 5.6. *Let D be a domain of circular type in (M, J) with center x_o and standard exhaustion $\tau_{(x_o)}$. If $u = \log \tau_{(x_o)}$ is J-plurisubharmonic, then u is an almost pluricomplex Green function with pole at x_o.*

Proof. With no loss of generality, we may assume that the domain is in normal form, *i.e.* $D = (B^n, J)$ and $\tau_{(x_o)}(z) = \tau_o(x) = |x|^2$. Since τ_o is smooth on $B^n \setminus \{0\}$ and $u = \log \tau_o$ is J-plurisubharmonic, we have that $\mathcal{H}ess(u)_x \geq 0$ for any $x \neq 0$. On the other hand, for any straight disk $f : \Delta \longrightarrow B^n$ of the form $f(\zeta) = v \cdot \zeta$, we have that $u \circ f$ is harmonic and $\mathcal{H}ess(u)_{f(\zeta)}(v, v) = 0$ for any $\zeta \neq 0$. This means that $\mathcal{H}ess(u)_x \geq 0$ has at least one vanishing eigenvalue at any point of $B^n \setminus \{0\}$ and means that (5.5) is satisfied. Other conditions of Definition 5.5 can be checked directly. □

When J is integrable, *the standard exhaustion $u = \log \tau_{(x_o)}$ of the normal form of a domain of circular type is automatically plurisubharmonic* [18]. In the almost complex case, this is no longer true, as the following example shows.

Example 5.7. Consider a quadruple of vector fields $(Z, J_{\text{st}}Z, E, J_{\text{st}}E)$ on $\widetilde{B}^2$, determined as follows. The field Z has been defined in Section 3.2 and, in coordinates (3.6), is of the form $Z_z = \text{Re}\left(z^0 \left.\frac{\partial}{\partial z^0}\right|_z\right)$ at any $z \in \widetilde{B}^2 \setminus \pi^{-1}(0)$. The field E is any vector field in the distribution $\mathcal{H}$ that satisfies the conditions

$$[Z, E] = [J_{\text{st}}Z, E] = 0, \qquad [E, J_{\text{st}}E] = -J_{\text{st}}Z. \tag{5.9}$$

It is uniquely determined, up to a smooth family of unitary transformations of the subspaces $\mathcal{H}_z \subset T_z\widetilde{B}^2$. Notice that the standard holomorphic bundle $T^{10}\widetilde{B}^2$ is generated at all points by the complex vector fields $Z^{10} = Z - iJ_{\mathrm{st}}Z$, $E^{10} = E - iJ_{\mathrm{st}}E$. In the following, we denote by $(E^{10*}, E^{01*}, Z^{10*}, Z^{01*})$ the field of complex coframes, which is dual to $(E^{10}, E^{01} = \overline{E^{10}}, Z^{10}, Z^{01} = \overline{Z^{10}})$ at all points.

Consider a deformation tensor $\phi \in \mathrm{Hom}(\mathcal{H}^{01}, \mathcal{Z}^{10} + \mathcal{H}^{10})$ of the form $\phi_z = h(z)Z^{10}_z \otimes E^{01*}_z$ for some smooth real valued function $h : \widetilde{B}^n \longrightarrow \mathbb{R}$, which is constant on all spheres $S_c = \{ \tau_o(z) = c \}$ (*i.e.* $h = h(|z^0|)$) and is equal to 0 on an open neighborhood of $\pi^{-1}(0) = \mathbb{C}P^1$.

By definitions and Remark 3.9, the deformation tensor ϕ determines an almost complex structure J, with J-holomorphic spaces $T^{10}_{J_z}\widetilde{B}^n = \mathbb{C}Z^{10}_z \oplus \mathbb{C}\widetilde{E}^{10}_z$, $\widetilde{E}^{10}_z \overset{\text{def}}{=} E^{10}_z + h(z)Z^{01}_z$, that satisfies (i), (iii) and (iv) of Definition 3.3. We claim that the system (3.7) is always solvable, so that J is an almost L-complex structure and $(\widetilde{B}^n, J)$ is an almost complex domain of circular type in normal form. This claim can be checked observing that the components of J in coordinates (3.6) along the disk $f(\zeta) = \zeta \cdot v$ with $v = (0, 1)$, are such that

$$J^1_1 = i = -J^{\bar{1}}_{\bar{1}}, \quad J^1_{\bar{1}} = J^{\bar{1}}_1 = 0$$

and $J^0_A = J^0_A(|z^0|)$, $J^{\bar{0}}_A = J^{\bar{0}}_A(|z^0|)$ for any $A = 1, \bar{1}$. So, (3.7) reduces to

$$\begin{cases} g_{1,\bar{\zeta}} = \mathbb{F}_1 & \text{on } \Delta, \\ g_1 = k(\zeta^2 - 1) & \text{on } \partial\Delta, \end{cases} \tag{5.10}$$

with $\mathbb{F}_1 : \overline{\Delta} \longrightarrow \mathbb{C}$ depending only on $\rho = |\zeta|$ and k constant. Consider the map

$$\widetilde{\mathbb{F}}_1 : \overline{\Delta} \longrightarrow \mathbb{C}, \qquad \widetilde{\mathbb{F}}_1 \overset{\text{def}}{=} -\frac{1}{\pi}\int_{\overline{\Delta}} \frac{\mathbb{F}_1(|w|)}{w - \zeta} dw \wedge d\bar{w}.$$

It is such that

$$\widetilde{\mathbb{F}}_{1,\bar{\zeta}} = \mathbb{F}_1 \quad \text{and} \quad \int_{\partial\Delta} \widetilde{\mathbb{F}}_1 \zeta^n d\zeta = 2i \int_{\overline{\Delta}} \mathbb{F}_1(\rho)\rho^{n+1}e^{in\vartheta} d\rho \wedge d\vartheta = 0$$

for any $n \geq 0$. Hence, if $h_1 : \overline{\Delta} \longrightarrow \mathbb{C}$ is a holomorphic map such that $h_1|_{\partial\Delta} = \widetilde{\mathbb{F}}_1|_{\partial\Delta}$, the map $g_1 = \widetilde{\mathbb{F}}_1 - h_1 + k(\zeta^2 - 1)$ is a solution to (5.10). Since the solvability of (3.7) is independent on the choice of coordinates, this concludes the proof of the claim.

Now, we want to show that *if $h \not\equiv 0$, the function $u = \log \tau_o$ is not J-plurisubharmonic*. For this, we first observe that, for any pair of real vector fields X, Y, if we set $X^{10} = X - iJX$, $Y^{01} = Y + iJY$

$$\begin{aligned} \mathcal{H}ess(u)(X, Y) &= \frac{1}{2}\,\mathrm{Im}\, dd^c u(X^{10}, Y^{01}) \\ &= \frac{1}{2}\,\mathrm{Im}\Big(iX^{10}(Y^{01}(u)) + iY^{01}(X^{10}(u)) + J[X^{10}, Y^{01}](u)\Big). \end{aligned} \tag{5.11}$$

We also recall that (here, $(\cdot)_Z \overset{\text{def}}{=} Z(\cdot)$ denotes derivation along Z)

$$Z^{10}(u) = 1 = Z^{01}(u), \quad E^{10}(u) = 0, \quad \widetilde{E}^{10}(u) = h = \widetilde{E}^{01}(u), \tag{5.12}$$

$$\widetilde{E}^{10}(\widetilde{E}^{01}(u)) = hh_Z = \widetilde{E}^{01}(\widetilde{E}^{10}(u)), \tag{5.13}$$

$$[E^{10}, E^{01}] = -2i[E, JE] = -i2JZ, \tag{5.14}$$

$$J[\widetilde{E}^{10}, \widetilde{E}^{01}](u) = J[E^{10}, E^{01}](u) + ihh_Z(Z^{10} + Z^{01})(u) = i2(1 + hh_Z), \tag{5.15}$$

$$J[\widetilde{E}^{10}, Z^{01}](u) = ih_Z Z^{01}(u) = ih_Z. \tag{5.16}$$

Hence, if we set $\widetilde{E} = \mathrm{Re}(\widetilde{E}^{10})$ and $J\widetilde{E} = \mathrm{Re}(i\widetilde{E}^{10})$, using (5.11) - (5.16) we may conclude that

$$\mathcal{H}ess(\widetilde{E}, \widetilde{E}) = 1 + 2hh_Z, \qquad \mathcal{H}ess(\widetilde{E}, J\widetilde{E}) = 0,$$
$$\mathcal{H}ess(\widetilde{E}, Z) = h_Z, \quad \mathcal{H}ess(\widetilde{E}, JZ) = \mathcal{H}ess(Z, Z) = \mathcal{H}ess(Z, JZ) = 0,$$

so that the matrix H, with entries given by the components of $\mathcal{H}ess(u)_z$ in the basis $\mathcal{B} = (e_1 = \widetilde{E}_z, e_2 = J\widetilde{E}_z, e_3 = Z_z, e_4 = JZ_z)$, is

$$H = \begin{pmatrix} 1 + 2hh_Z & 0 & h_Z & 0 \\ 0 & 1 + 2hh_Z & 0 & h_Z \\ h_Z & 0 & 0 & 0 \\ 0 & h_Z & 0 & 0 \end{pmatrix}.$$

Since the eigenvalues of H are $\lambda_\pm = \frac{(1+2hh_Z) \pm \sqrt{(1+2hh_Z)^2 + 4h_Z^2}}{2}$, we conclude that u is J-plurisubharmonic if and only if $h_Z \equiv 0$, *i.e.* if and only if $h \equiv 0$.

From previous example, we see that the standard exhaustion $\tau_{(x_o)}$ of an almost complex domain (D, J) of circular type is *in general* not a Green function, independently on how J is close to an integrable complex structure. However, the property remains valid if one restricts to the class of nice domains and to small deformations of integrable structures, as it is shown in next theorem. This property nicely relates to [9, Theorem 6.4] (see also Remark 4.2) on the existence of extremal disks for domains with small deformations of an integrable complex structure.

Theorem 5.8. *Let D be a nice circular domain with standard exhaustion $\tau_{(x_o)}$ and normal form (B^n, J). If J is a sufficiently small $\mathcal{C}^1$-deformation of J_{st}, then $u = \log \tau_{(x_o)}$ is the Green function with pole at x_o.*

Proof. By Proposition 5.6, it suffices to show that, when J is sufficiently close to J_{st}, then $u = \log |z|^2$ is J-pseudoconvex on $B^n \setminus \{0\}$. For this, we first claim that if (B^n, J) is nice, then $\mathcal{H}ess(u)_z(\mathcal{Z}, \mathcal{H}) = 0$ at any $z \neq 0$. Since $\mathcal{Z}$ and $\mathcal{H}$ are both J-invariant, this is equivalent to claim that, that for any $z \neq 0$,

$$\left(dd^c u + J^*(dd^c u)\right)_z (Z, X) = 0, \qquad Z \overset{\text{def}}{=} \mathrm{Re}\left(z^i \frac{\partial}{\partial z^i}\right), \quad X \in \mathcal{H}. \tag{5.17}$$

But this follows from the fact that X and JX are tangent to the level sets of $u = \log \tau_o$, that $[\mathcal{Z}, \mathcal{H}] \subset \mathcal{H}$ and hence that

$$dd^c u(Z, X) = -Z(JX(u)) + X(JZ(u)) + (J[Z, X])(u) = X(J_{\mathrm{st}} Z(u)) = 0$$
$$J^* dd^c u(Z, X) = JZ(X(u)) - JX(Z(u)) + (J[JZ, JX])(u) = -JX(Z(u)) = 0.$$

Since $\mathcal{H}ess(u)|_{\mathcal{Z}\times\mathcal{Z}} = 0$, it remains to check that $\mathcal{H}ess(u)|_{\mathcal{H}_z\times\mathcal{H}_z} \geq 0$. Since any sphere $S^{2n-1}_c = \{ z \in B^n \ , \ \tau_o(z) = c \}, c \in]0, 1]$, is strongly J_{st}-pseudoconvex and its (real) holomorphic tangent distribution distribution is $\mathcal{H}|_{S^{2n-1}_c}$, if J is sufficiently close to J_{st}, the sphere S^{2n-1}_c is strongly pseudoconvex also with respect to J by a continuity argument. This is the same of saying that $\mathcal{H}ess(u)|_{\mathcal{H}_z\times\mathcal{H}_z} \geq 0$ for any $z \neq 0$, as needed. □

5.4. Concluding remarks

It is well known that on (integrable!) complex manifolds there is a tight connection between the existence of regular plurisubharmonic solutions u of maximal rank for homogeneous complex Monge-Ampère equations and existence of foliations by Riemann surfaces. In fact, such foliation is given by the complex curves along which the solution u is harmonic.

This idea was exploited by Lempert in his work on strictly convex domains in $\mathbb{C}^n$ [11]. In this case, the foliation is made of the extremal disks for the Kobayashi metric through a fixed point x_o, which coincide with stationary disks through x_o. The function u is the pull-back on each disk of the standard Green function with pole at 0 of $\Delta \subset \mathbb{C}$, so that any strictly convex domain is a domain of circular type. Conversely, the singular foliation by holomorphic disks, determined by the exhaustion u of a domain of circular type, is a foliation by Kobayashi extremal disks [17].

Also for almost complex domains, the plurisubharmonic $\mathcal{C}^2$ functions, which are harmonic along the leaves of a foliation of circular type $\mathcal{F}^{(x_o)}$, are solutions of an almost complex Monge-Ampère equation:

Proposition 5.9. *Let $D \subset M$ be a domain in a strongly pseudoconvex manifold (M, J) with a foliation of circular type $\mathcal{F}^{(x_o)}$ of (D, x_o). Let also $u : D \longrightarrow]-\infty, +\infty[$ be a function which is in* $\mathrm{Psh}(D) \cap \mathcal{C}^2(D \setminus \{x_o\})$ *and so that $u \circ f : \Delta \setminus \{0\} \longrightarrow \mathbb{R}$ is harmonic for any $f \in \mathcal{F}^{(x_o)}$. Then u is a solution of the generalized Monge-Ampère equation* (5.5) *on $D \setminus \{x_o\}$.*

Proof. First of all, we claim that u satisfies the generalized Monge-Ampere equation (5.5) at all points of $D \setminus \{x_o\}$. In fact, by (ii) of Definition 3.1, for any $y \in D$ we know that there exists a disk $f : \Delta \longrightarrow D$ in $\mathcal{F}^{(x_o)}$ with $y = f(\zeta)$ for some $\zeta \in \Delta$. Since $u \circ f$ is harmonic, if we denote by $v = f_*\left(\frac{\partial}{\partial(\mathrm{Re}\,\zeta)}\right)$, we have that

$$H(u)_y(v, v) = (dd^c u + J^*(dd^c u))_y(v, Jv) = \Delta(u \circ f)_\zeta = 0, \qquad (5.18)$$

from which it follows immediately that $(dd^c u + J^*(dd^c u))^n_y = 0$. The conclusion follows immediately from Theorem 5.4. □

Notice that, by Example 5.7, the previous remark is not so useful to determine almost pluricomplex Green functions, since in general the function $u = \log \tau_{(x_o)}$, which is determined by the geometric construction, is not plurisubharmonic. Furthermore, we know that, in general, stationary disks of an almost complex domain D are not extremal disks for the Kobayashi metric of D [9]. Nevertheless, the above properties of strictly convex domains of $\mathbb{C}^n$ remain valid in a large class of almost complex domains, as it is illustrated in the following theorem, which is direct consequence of our results.

Theorem 5.10. *Let D be an almost complex domain of circular type with center x_o in (M, J) strongly pseudoconvex. If the normal form (B^n, J') of (D, J) is very nice with J' sufficiently close to J_{st}, then*

a) *the stationary foliation $\mathcal{F}^{(x_o)}$ consists of extremal disks with respect to Kobayashi metric;*
b) *the function $u = \log \tau_{(x_o)}$ is the almost pluricomplex Green function of D with pole x_o;*
c) *the distribution $\mathcal{Z}_z = \ker \mathcal{H}ess(u)_z$ is integrable and the closures of its integral leaves are the disks in $\mathcal{F}^{(x_o)}$.*

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