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# A structural theorem for codimension-one foliations on $\mathbb{P}^n$, $n \geq 3$, with an application to degree-three foliations

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**Abstract.** Let $\mathcal{F}$ be a codimension-one foliation on $\mathbb{P}^n$: for each point $p \in \mathbb{P}^n$ we define $\mathcal{J}(\mathcal{F}, p)$ as the order of the first non-zero jet $j^k_p(\omega)$ of a holomorphic 1-form $\omega$ defining $\mathcal{F}$ at $p$. The singular set of $\mathcal{F}$ is $\text{sing}(\mathcal{F}) = \{p \in \mathbb{P}^n \mid \mathcal{J}(\mathcal{F}, p) \geq 1\}$. We prove (main Theorem 1.2) that a foliation $\mathcal{F}$ satisfying $\mathcal{J}(\mathcal{F}, p) \leq 1$ for all $p \in \mathbb{P}^n$ has a non-constant rational first integral. Using this fact we are able to prove that any foliation of degree-three on $\mathbb{P}^n$, with $n \geq 3$, is either the pull-back of a foliation on $\mathbb{P}^2$, or has a transverse affine structure with poles. This extends previous results for foliations of degree at most two.

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## Notation

1. $\mathcal{O}_n$: the ring of germs at $0 \in \mathbb{C}^n$ of holomorphic functions. $\mathcal{O}^*_n = \{f \in \mathcal{O}_n \mid f(0) \neq 0\}$. $\mathfrak{m}_n = \{f \in \mathcal{O}_n \mid f(0) = 0\}$.
2. $f \mid g$: $f, g \in \mathfrak{m}_n \setminus \{0\}$ and $f$ divides $g$.
3. $f \nmid g$: $f, g \in \mathfrak{m}_n \setminus \{0\}$ and $f$ does not divide $g$.
4. $[f, g]_0$: the intersection number of $f, g \in \mathfrak{m}_2 \setminus \{0\}$, when $f$ and $g$ have no common factor.
5. $< f, g >$: the ideal generated by $f, g \in \mathcal{O}_p$.
6. $\text{Diff}(\mathbb{C}^n, p)$: the group of germs at $p \in \mathbb{C}^n$ of biholomorphisms $f$ with $f(p) = p$.
7. $i_X(\omega)$: the interior product of the vector field $X$ and the form $\omega$.
8. $L_X$: the Lie derivativative in the direction of the vector field $X$.
9. $j^k_p$: the $k^{\text{th}}$-jet at the point $p$.

## 1. Introduction

In a previous paper [10] we have proved that the space of holomorphic codimension-one foliations and degree-two on $\mathbb{P}^n$, with $n \geq 3$, has six irreducible components. A

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consequence of this classification is that we have two possibilities for a degree-two foliation $\mathcal{F}$ on $\mathbb{P}^n$, with $n \geq 3$: either $\mathcal{F}$ is defined by a meromorphic closed 1-form on $\mathbb{P}^n$, or $\mathcal{F} = g^*(\mathcal{G})$, where $g \colon \mathbb{P}^n \dashrightarrow \mathbb{P}^2$ is a linear map and is $\mathcal{G}$ a degree-two foliation of $\mathbb{P}^2$. A foliation defined by a meromorphic closed 1-form admits a special projective tranverse structure with poles, namely a translation structure. On the other hand, a foliation of the form $g^*(\mathcal{G})$ admits such a structure if, and only if, $\mathcal{G}$ admits one(*cf.* [4]). This is not always the case: a foliation of $\mathbb{P}^2$ which admits a projective or affine transverse structure always has algebraic leaves, whereas for any $d \geq 2$, there are degree-$d$ foliations on $\mathbb{P}^2$ without algebraic invariant curves. The following conjecture is attributed to different authors (Brunella, Lins Neto,...):

**Main Conjecture.** *Any codimension-one holomorphic foliation $\mathcal{F}$ on $\mathbb{P}^n$, with $n \geq 3$, either is a pull-back of a foliation $\mathcal{G}$ on $\mathbb{P}^2$ by a rational map $\Phi \colon \mathbb{P}^n \dashrightarrow \mathbb{P}^2$, or admits a transverse projective structure with poles on some invariant hypersurface.*

In the first case, the leaves of $\mathcal{F}$ are sub-foliated by the levels of $\Phi$ and the dynamic properties of $\mathcal{F}$ are essentialy given by $\mathcal{G}$. In the second, we can associate a triple of meromorphic 1-forms $(\omega_0, \omega_1, \omega_2)$ such that $\omega_0$ defines $\mathcal{F}$ outside its set of poles $|\omega_0|_\infty$ and the triple satisfies the $s\ell(2, \mathbb{C})$ structural relations:

$$\begin{aligned} d\omega_0 &= \omega_0 \wedge \omega_1 \\ d\omega_1 &= \omega_0 \wedge \omega_2 \\ d\omega_2 &= \omega_1 \wedge \omega_2 \end{aligned}$$

inducing the projective structure.

For instance, when $\omega_1 = \omega_2 = 0$, that is $\omega_0$ is closed, the integration of $\omega_0$ on simply connected open sets $U \subset \mathbb{P}^n \setminus |\omega_0|_\infty$ gives $\omega_0 = df_U$, and defines the transverse translation structure: when $U \cap V \neq \emptyset$ we have $f_U = f_V + c_{UV}$, where $c_{UV} \in \mathbb{C}$. On the other hand, if $\omega_2 = 0$ and $\omega_1 \neq 0$ then the transverse structrure is affine.

The main conjecture seems to be reasonable (at least for foliations of small degree) for the following reasons: first of all, if $\mathbb{K}$ is a field of positive characteristic every foliation on a projective manifold over $\mathbb{K}$, in particular on $\mathbb{P}^n_{\mathsf{K}}$, is defined by a closed 1-form (*cf.* [12]). On the other hand, if $\mathcal{F}$ is a foliation on $\mathbb{P}^n$ and $p$ is a prime number then it is possible to define $\mathcal{F}_p$, the reduction modulo $p$ of $\mathcal{F}$. There is a conjecture of Grothendieck-Katz-type which says that if for almost all $p$ the foliation $\mathcal{F}_p$ has a non-constant rational first integral then $\mathcal{F}$ itself has a non-constant rational first integral. Recently F. Touzet has communicated to one of the authors the following result:

**Theorem. (F. Touzet)** *The Grothendieck-Katz conjecture implies that any foliation of degree at most $n-1$ on $\mathbb{P}^n$ either admits a projective transverse structure, or is a pull-back of some foliation on $\mathbb{P}^k$, with $k < n$, by some rational map.*

Concerning the main conjecture, note that the first interesting case which is not covered by the above conditional result is that of foliations of degree-three on

$\mathbb{P}^3$. In fact, one of the goals of this paper is to prove that the conjecture is true for foliations of degree three.

**Theorem 1.1.** *Let $\mathcal{F}$ be a holomorphic codimension-one foliation of degree-three on $\mathbb{P}^n$, with $n \geq 3$. Then:*

- *either $\mathcal{F}$ has a rational first integral,*
- *or $\mathcal{F}$ has an affine transverse structure with poles on an invariant hypersurface,*
- *or $\mathcal{F} = g^*(\mathcal{G})$, where $g\colon \mathbb{P}^n \dashrightarrow \mathbb{P}^2$ is a rational map and $\mathcal{G}$ is a foliation on $\mathbb{P}^2$.*

One of the tools of the proof will be a result of [12] concerning foliations which admit a finite Godbillon-Vey sequence. This result essentially says that such a foliation is either a pull-back of a foliation on a surface or has a transversely projective structure. Let us explain briefly how we can apply the result.

By definition, a degree-$d$ foliation $\mathcal{F}$ on $\mathbb{P}^n$ has $d$ tangencies with a generic straight line of $\mathbb{P}^n$. This implies that $\mathcal{F}$ can be represented in an affine coordinate system $\mathbb{C}^n \simeq E \subset \mathbb{P}^n$ by a polynomial integrable 1-form $\omega_E = \sum_{j=0}^{d+1} \omega_j$, where the coefficients of the 1-form $\omega_j$ are homogeneous polynomials of degree $j$, $0 \leq j \leq d+1$, and $i_R(\omega_{d+1}) = 0$, with $R = \sum_{j=1}^{n} z_j\, \partial_{z_j}$, the radial vector field. The form $\omega_E$ can be considered as a meromorphic 1-form on $\mathbb{P}^n$ with poles of order $d+2$ at the hyperplane of infinity of $E$. Given $p \in E$, we set

$$\mathcal{J}(\mathcal{F}, p) = \min\{k \geq 0 \mid j_p^k(\omega_E) \neq 0\}\ .$$

It can be proved that $\mathcal{J}(\mathcal{F}, p)$ depends only on $p$ and $\mathcal{F}$ and not on $E$ and $\omega_E$. The singular set of $\mathcal{F}$ is defined as

$$\operatorname{sing}(\mathcal{F}) = \{p \in \mathbb{P}^n \mid \mathcal{J}(\mathcal{F}, p) \geq 1\}\,.$$

This set is algebraic and always has irreducible components of codimension two (*cf.* [16]).

Given a degree-three foliation $\mathcal{F}$ of $\mathbb{P}^n$, we will consider two cases:

(1) There exists $p \in \operatorname{sing}(\mathcal{F})$ such that $\mathcal{J}(\mathcal{F}, p) \geq 2$.
(2) For all $p \in \operatorname{sing}(\mathcal{F})$ we have $\mathcal{J}(\mathcal{F}, p) = 1$.

Case (1) will be studied in Section 2. We will see that $\mathcal{F}$ admits a finite Godbillon-Vey sequence in this case and we can apply the result of [12]. In case (2) we will see in Section 3 that $\mathcal{F}$ has a meromorphic first integral.

In Section 3 we will introduce the *Baum-Bott index* of an irreducible component, say $\Gamma$, of codimension-two of $\operatorname{sing}(\mathcal{F})$, which we will denote $\operatorname{BB}(\mathcal{F}, \Gamma)$. As a consequence of the Baum-Bott theorem we will see that $\operatorname{sing}(\mathcal{F})$ always has a codimension-two irreducible component $\Gamma$ with $\operatorname{BB}(\mathcal{F}, \Gamma) \neq 0$.

**Theorem 1.2.** *Let $\mathcal{F}$ be a codimension-one holomorphic foliation on $\mathbb{P}^n$, with $n \geq 3$. Assume that* $\operatorname{sing}(\mathcal{F})$ *has an irreducible component of codimension-two $\Gamma$ such that*

(a) $\mathrm{BB}(\mathcal{F}, \Gamma) \neq 0$.
(b) *The algebraic set* $\{p \in \Gamma \mid \mathcal{J}(\mathcal{F}, p) > 1\}$ *has codimension at least* 4 *in* $\mathbb{P}^n$.

*Then $\mathcal{F}$ has a rational first integral.*

As a consequence, we will get the following:

**Corollary 1.3.** *Let $\mathcal{F}$ be a codimension-one holomorphic foliation on $\mathbb{P}^n$, with $n \geq 3$. If $\mathcal{J}(\mathcal{F}, p) \leq 1$ for all $p \in \mathbb{P}^n$ then $\mathcal{F}$ has a rational first integral.*

**Remark 1.4.** Recall that $p \in \operatorname{sing}(\mathcal{F})$ is of Kupka type if $\mathcal{F}$ is defined in a neighborhood of $p$ by a holomorphic 1-form $\omega$ such that $d\omega(p) \neq 0$. We define $K(\mathcal{F}) = \{p \in \operatorname{sing}(\mathcal{F}) \mid p \text{ is of Kupka type}\}$. If $p \in K(\mathcal{F})$ then $\mathcal{J}(\mathcal{F}, p) = 1$. We would like to observe that if $\operatorname{sing}(\mathcal{F})$ has a smooth irreducible component, say $\Gamma$, with $\Gamma \subset K(\mathcal{F})$, then a theorem due to Calvo Andrade and M. Brunella says that $\mathcal{F}$ has a rational first integral (*cf.* [5,6,11] and [3]). In this sense, Theorem 1.2 is a generalization of Calvo and Brunella's theorem.

**Remark 1.5.** We would like to observe that the conclusion of Corollary 1.3 is not true when we consider codimension-one foliations on more general complex manifolds. For instance, let $M = \mathbb{P}^2 \times \mathbb{P}^k$, with $k \geq 1$, and $\mathcal{F} = \Pi_1^*(\mathcal{G})$, where $\Pi_1 \colon \mathbb{P}^2 \times \mathbb{P}^k \to \mathbb{P}^2$ is the projection on the first factor and $\mathcal{G}$ is a foliation on $\mathbb{P}^2$ of degree at least 2 without non-constant rational first integral and with $\mathcal{J}(\mathcal{G}, p) \leq 1$ for all $p \in \mathbb{P}^2$. Then $\mathcal{F}$ satisfies the hypothesis of Corollary 1.3 but not its conclusion. A natural question which arises is the following:

**Problem 1.6.** For which compact complex manifolds of dimension at least 3 the conclusion of Corollary 1.3 is true?

**Remark 1.7.** We say that a foliation admits a *purely* projective transverse structure (briefly p.p.t.s.) if it has a projective transverse structure with poles, but no affine transverse structure with poles. There are examples of foliations on $\mathbb{P}^3$, for instance the so called Hilbert modular foliations, which admit a p.p.t.s. and are not the pull-back of foliations on $\mathbb{P}^2$ (*cf.* [12]). In fact, these examples have degree at least five.

On the other hand, as a consequence of the proof of Theorem 1.1, any foliation of degree-three on $\mathbb{P}^n$, with $n \geq 3$, that admits a p.p.t.s. is the pull-back of a Riccati foliation on $\mathbb{P}^1 \times \mathbb{P}^1$ (see the third case in the proof of Lemma 2.5). For instance, there are p.p.t.s. Riccati equations on $\mathbb{C}^2$ of the form

$$x(x-1)\,dy - (a_0(x) + a_1(x)\,y + a_2(x)\,y^2)\,dx = 0\ , \tag{1.1}$$

where $a_0, a_1$ and $a_2$ are degree-one polynomials. If $\mathcal{G}$ is a p.p.t.s. foliation defined by (1.1) on $\mathbb{P}^2$ then it has degree-three. In particular, if $\Pi \colon \mathbb{P}^n$ □□□ $\mathbb{P}^2$ is linear then $\Pi^*(\mathcal{G})$ is a p.p.t.s. degree-three foliation on $\mathbb{P}^n$.

Consider a meromorphic integrable 1-form $\Omega$ on $\mathbb{P}^3$ representing $\mathcal{F}$ outside its set of poles. By using the normal type, we will see that there exists a closed meromorphic 1-form $\tilde{\Lambda}$, on some connected neighborhood $U$ of $\Gamma$, such that $d\Omega = \tilde{\Lambda} \wedge \Omega$ on $U$. The extension theorem of [1] and [22] will imply that $\tilde{\Lambda}$ can be extended to a closed meromorphic 1-form $\Lambda$ on $\mathbb{P}^3$ with $d\Omega = \Lambda \wedge \Omega$. Next, working with the pole divisors and residues of $\Lambda$, we will see that $\Lambda = \frac{dF}{F}$, where $F$ is meromorphic on $\mathbb{P}^3$. In particular, we will get $d\left(\frac{\Omega}{F}\right) = 0$, that is, $F$ is an integrating factor of $\Omega$. Finally, by studing $\frac{\Omega}{F}$ around $\Gamma$, we will show that $\mathcal{F}$ has a rational first integral of the form $f_2^m/f_1^n$, where $m.dg(f_2) = n.dg(f_1)$.

**Remark 3.24.** Since $n > m$, the separatrix $\sigma = (x = 0)$ is distinguished. In particular, it extends to a smooth separatrix $\Sigma_1$ of $\mathcal{F}$ along $\Gamma$. When $n > m > 1$ the other separatrix, $\sigma_2 = (y = 0)$, is also distinguished and can be extended to another separatrix, say $\Sigma_2$, of $\mathcal{F}$ along $\Gamma$.

Another fact that we would like to observe is that $f(x, y) := y^m/x^n$ is a mermorphic first integral of $\eta$. On the other hand, $\eta$ has no non-constant holomorphic first integral in a neighborhood $0 \in \mathbb{C}^2$.

Fix an affine chart $(x, y, z) \in \mathbb{C}^3 \subset \mathbb{P}^3$ and a polynomial integrable 1-form $\Omega$ on $\mathbb{C}^3$ which represents $\mathcal{F}|_{\mathbb{C}^3}$. Without lost of generality, we can assume that $\Gamma$ is transverse to the line at infinity $L_\infty = \mathbb{P}^3 \setminus \mathbb{C}^3$.

**Construction of $\tilde{\Lambda}$ in a neighborhood of $\tilde{\Gamma}$.** Let $(U_\alpha)_{\alpha \in A}$ be a covering of $\tilde{\Gamma}$ with the following properties:

(a) $U_\alpha \cap \tilde{\Gamma}$ is connected and non-empty for all $\alpha \in A$.
(b) If $U_{\alpha\beta} \neq \emptyset$ then $U_{\alpha\beta} \cap \tilde{\Gamma}$ is connected and non-empty.
(c) For all $\alpha \in A$ there is a chart $(x_\alpha, y_\alpha, z_\alpha)\colon U_\alpha \to \mathbb{C}^3$ such that $\tilde{\Gamma} \cap U_\alpha = (x_\alpha = y_\alpha = 0)$ and $\mathcal{F}|_{U_\alpha}$ is represented by $\eta_\alpha = m\, x_\alpha\, dy_\alpha - n\, y_\alpha\, dx_\alpha$.

In particular, $\Sigma_1 \cap U_\alpha = (x_\alpha = 0)$, $f_\alpha := y_\alpha^m/x_\alpha^n$ is a meromorphic first integral of $\mathcal{F}|_{U_\alpha}$ and

$$df_\alpha = \frac{y_\alpha^{m-1}}{x_\alpha^{n+1}} . \eta_\alpha \ , \ \forall \alpha \in A \ . \tag{3.16}$$

Fix $U_{\alpha\beta} \neq \emptyset$ and let $\varphi_{\alpha\beta} \in \mathcal{O}^*(U_{\alpha\beta})$ and $g_{\alpha\beta} \in \mathcal{O}^*(U_{\alpha\beta})$ be such that $\eta_\alpha = \varphi_{\alpha\beta} . \eta_\beta$ and $x_\alpha = g_{\alpha\beta} . x_\beta$ on $U_{\alpha\beta}$. From (3.16) we get

$$df_\alpha = a_{\alpha\beta} . df_\beta \ , \ a_{\alpha\beta} = \frac{(y_\alpha/y_\beta)^{m-1}}{g_{\alpha\beta}^{n+1}} \varphi_{\alpha\beta} \ .$$

Note that $a_{\alpha\beta} \in \mathcal{O}^*(U_{\alpha\beta})$. In fact, if $m = 1$ this is clear. On the other hand, if $m > 1$ then by Remark 3.24, there is a separatrix $\Sigma_2$ along $\Gamma$ such that

$$\Sigma_2 \cap U_{\alpha\beta} = (y_\alpha = 0) \cap U_\beta = (y_\beta = 0) \cap U_\alpha \ .$$

As a consequence, there exists $h_{\alpha\beta} \in \mathcal{O}^*(U_{\alpha\beta})$ such that $y_\alpha = h_{\alpha\beta} . y_\beta$. Hence, $a_{\alpha\beta} = h_{\alpha\beta}^{m-1} . \varphi_{\alpha\beta}/g_{\alpha\beta}^{n+1} \in \mathcal{O}^*(U_{\alpha\beta})$.

From $df_\alpha = a_{\alpha\beta}.df_\beta$ we get

$$da_{\alpha\beta} \wedge df_\beta = 0 \implies da_{\alpha\beta} \wedge \eta_\beta = 0$$

and $a_{\alpha\beta}$ is a holomorphic first integral of $\mathcal{F}$ in a neighborhood of $U_{\alpha\beta} \cap \Gamma$. This implies that $a_{\alpha\beta} \in \mathbb{C}^*$, because the normal type has no non-constant holomorphic first integral.

Given $\alpha \in A$, $\Omega|_{U_\alpha}$ and $df_\alpha$ represent $\mathcal{F}$ in the complement of their poles. Hence, there is a meromorphic function $g_\alpha$ on $U_\alpha$ such that $\Omega = g_\alpha . df_\alpha$. Since $df_\alpha = a_{\alpha\beta}.df_\beta$ on $U_{\alpha\beta} \neq \emptyset$, we get

$$\Omega = g_\alpha.df_\alpha = g_\alpha.a_{\alpha\beta}.df_\beta = g_\beta.df_\beta \implies g_\beta = a_{\alpha\beta}.g_\alpha \text{ , on } U_{\alpha\beta} \text{ .}$$

Since $a_{\alpha\beta} \in \mathbb{C}^*$, we get

$$\frac{dg_\alpha}{g_\alpha} = \frac{dg_\beta}{g_\beta} \text{ , on } U_{\alpha\beta}$$

and this implies that there exists a meromorphic 1-form $\tilde{\Lambda}$ on $\tilde{U} := \square_\alpha U_\alpha$ such that $\tilde{\Lambda}|_{U_\alpha} = \frac{dg_\alpha}{g_\alpha}$ for all $\alpha \in A$. Finally, $d\Omega = \tilde{\Lambda} \wedge \Omega$ because

$$d\Omega|_{U_\alpha} = dg_\alpha \wedge df_\alpha = \frac{dg_\alpha}{g_\alpha} \wedge \Omega|_{U_\alpha} = \tilde{\Lambda} \wedge \Omega|_{U_\alpha} \text{ .}$$

**Extension of $\tilde{\$}$ to a neighborhood of " \ "̃.** Fix $p \in \Gamma \setminus \tilde{\Gamma}$ and a local chart $(V, (u, s, t))$ around $p$ such that $\mathcal{F}|_V$ is represented by

$$\omega = u\,du + (h.\,\zeta_1(h) + u.\,\zeta_2(h))\,dh \text{ , } h = h(s,t)$$

and $\Gamma \cap V = (u = h(s,t) = 0)$. Choose $q \in V \cap \tilde{\Gamma}$ and a chart $(W, (u, v, w))$ around $q$ with $W \subset V$ and $h(s,t) = v$, so that

$$\omega|_W = u\,du + (v\,\zeta_1(v) + u\,\zeta_2(v))\,dv \text{ .}$$

Let $\alpha \in A$ be such that $q \in U_\alpha$. We can assume that $W \subset U_\alpha$. Since $\eta_\alpha|_W$ and $\omega|_W$ represent $\mathcal{F}|_W$ there is $\varphi = \varphi(u, v, w) \in \mathcal{O}^*(W)$ sucht that $\eta_\alpha = \varphi.\omega$ on $W$. This implies $df_\alpha|_W = h.\omega|_W$, where $h(u, v, w) = \varphi.y_\alpha^{m-1}/x_\alpha^{n+1}$ is meromorphic on $W$. In particular, $d(h.\,\omega|_W) = 0$, which implies $d\omega|_W = -\frac{dh}{h} \wedge \omega|_W$ . Since $d\omega|_W$ do not contain terms with $du \wedge dw$ and $dv \wedge dw$, from the last relation we get

$$\frac{\partial h}{\partial w} \equiv 0 \implies h = h(u, v) \text{ .}$$

Therefore, the closed 1-form $\theta := h(u, f(s,t)).\omega$ is meromorphic in some neighborhood $U_p$ of $p$ and extends $df_\alpha$ to this neighborhood. As before, we have $\Omega = g.\theta$, where $g$ is meromorphic on $U_p$ and is an extension of $g_\alpha$ to $U_p$. This implies that $\frac{dg}{g}$ extends $\tilde{\Lambda}$ to $U_p$. In particular, $\tilde{\Lambda}$ can be extended meromorphically to some connected neighborhood $U$ of $\Gamma$. Finally, Theorem 3.23 implies that $\tilde{\Lambda}$ can be extended to a closed meromorphic 1-form $\Lambda$ on $\mathbb{P}^3$ with $d\Omega = \Lambda \wedge \Omega$.

**Poles and residues of \$ .** Let $|\Lambda|_\infty$ be the set of poles of $\Lambda$. Fix $p \in \tilde{\Gamma}$ and $\alpha \in A$ such that $p \in U_\alpha$. Note that $L_\infty = \mathbb{P}^3 \setminus \mathbb{C}^3$ is a pole of $\Omega$ of order $d+2$, where $d = dg(\mathcal{F})$ (*cf.* [2]). Let $(u_\alpha = 0)$ be a reduced equation of $L_\infty \cap U_\alpha$. Since $\Omega|_{U_\alpha}$ and $\eta_\alpha$ represent $\mathcal{F}|_{U_\alpha}$ there is $\phi_\alpha \in \mathcal{O}^*(U_\alpha)$ such that

$$\Omega|_{U_\alpha} = \frac{\phi_\alpha}{u_\alpha^{d+2}} . \eta_\alpha = \frac{\phi_\alpha . x_\alpha^{n+1}}{u_\alpha^{d+2} . y_\alpha^{m-1}} . df_\alpha \implies g_\alpha = \frac{\phi_\alpha . x_\alpha^{n+1}}{u_\alpha^{d+2} . y_\alpha^{m-1}} .$$

From the above expression, we get

$$\Lambda|_{U_\alpha} = \frac{dg_\alpha}{g_\alpha} = (n+1)\frac{dx_\alpha}{x_\alpha} - (m-1)\frac{dy_\alpha}{y_\alpha} - (d+2)\frac{du_\alpha}{u_\alpha} + \frac{d\phi_\alpha}{\phi_\alpha} . \quad (3.17)$$

We have two possibilities:

**1st.** $1 < m < n$. In this case, $|\Lambda|_\infty \cap U_\alpha = (x_\alpha = 0) \cup (y_\alpha = 0) \cup (u_\alpha = 0)$. Since $\Sigma_1 \cap U_\alpha = (x_\alpha = 0)$ and $\Sigma_2 \cap U_\alpha = (y_\alpha = 0)$, they extend to global algebraic irreducible surfaces, which we call again $\Sigma_1$ and $\Sigma_2$, respectively. Moreover, we get $|\Lambda|_\infty \supset \Sigma_1 \cup \Sigma_2 \cup L_\infty$. We assert that $|\Lambda|_\infty = \Sigma_1 \cup \Sigma_2 \cup L_\infty$.

Let $S$ be an irreducible component of $|\Lambda|_\infty$, $S \neq L_\infty$, and let us prove that $S \subset \Sigma_1 \cup \Sigma_2$. We assert that $S$ is $\mathcal{F}$-invariant.

In fact, fix a smooth point $p \in S \setminus (L_\infty \cup \text{sing}(\mathcal{F}))$. Consider a local chart $\psi = (x_1, x_2, x_3) \colon W \to \mathbb{C}^3$ around $p$ such that $\psi(p) = 0$, $W \cap (L_\infty \cup \text{sing}(\mathcal{F})) = \emptyset$ and $S \cap W = |\Lambda|_\infty \cap W = (x_3 = 0)$. We can write

$$\Lambda|_W = \frac{\theta}{x_3^k} , \ \theta = A_1\, dx_1 + A_2\, dx_2 + A_3\, dx_3 ,$$

where $A_i \in \mathcal{O}(W)$, $i = 1, 2, 3$, $x_3 \nmid A_i$ for some $i = 1, 2, 3$, and $k \geq 1$. From $d\Lambda = 0$, we get

$$x_3^{-k}\, d\theta - k\, x_3^{-(k+1)}\, dx_3 \wedge \theta = 0 \implies d\theta = k\,\frac{dx_3}{x_3} \wedge \theta ,$$

which implies that $x_3 \mid A_1, A_2$ and $x_3 \nmid A_3$.Therefore, we can write $\theta = x_3\,\alpha + A_3\, dx_3$, where $\alpha$ is holomorphic on $W$. Since $d\Omega = \Lambda \wedge \Omega$, we get

$$x_3^k\, d\Omega|_W = \theta \wedge \Omega|_W \implies A_3\, dx_3 \wedge \Omega|_W = x_3(x_3^{k-1} d\Omega|_W - \alpha \wedge \Omega|_W) .$$

From the last relation above, we obtain that $\frac{dx_3}{x_3} \wedge \Omega|_W := \beta$ is holomorphic. Hence, $S$ is $\mathcal{F}$-invariant, because $dx_3 \wedge \Omega|_W = x_3\, \beta$, where $\beta$ is holomorphic.

Since $S$ is $\mathcal{F}$-invariant and $\Gamma \cap S \neq \emptyset$, $S$ must contain some separatrix of $\mathcal{F}$ along $\Gamma$. In particular, $S \cap U_\alpha \neq \emptyset$, which implies that $S \cap U_\alpha \subset (x_\alpha = 0) \cup (y_\alpha = 0)$. Therefore, either $S = \Sigma_1$, or $S = \Sigma_2$.

Let $f_1$, $f_2$, $f_3$ be irreducible homogeneous polynomials on $\mathbb{C}^4$, $f_3$ of degree-one, such that $f_i = 0$ is an equation of $\Sigma_i$, $i = 1, 2$, and $f_3 = 0$ is an equation of $L_\infty$ (in homogeneous coordinates). By (3.17) the residues of $\Lambda$ are $n + 1$ (on $\Sigma_1$), $-(m - 1)$ (on $\Sigma_2$) and $-(d + 2)$ (on $L_\infty$). Therefore, $\Lambda$ can be written in homogeneous coordinates as $dF/F$, where

$$F = \frac{f_1^{n+1}}{f_2^{m-1} . f_3^{d+2}} .$$

**2nd.** $n > m = 1$. In this case, $|\Lambda|_\infty \cap U_\alpha = (x_\alpha = 0) \cup (u_\alpha = 0)$. With the same argument of the 1st case, we get $|\Lambda|_\infty = \Sigma_1 \cup L_\infty$. Let $f_1$, $f_3$ be irreducible homogeneous polynomials on $\mathbb{C}^4$, $f_3$ of degree-one, such that $f_1 = 0$ is an equation of $\Sigma_1$ and $f_3 = 0$ is an equation of $L_\infty$ (in homogeneous coordinates). By (3.17) the residues of $\Lambda$ are $n + 1$ (on $\Sigma_1$) and $-(d + 2)$ (on $L_\infty$). Therefore, $\Lambda$ can be written in homogeneous coordinates as $dF/F$, where

$$F = \frac{f_1^{n+1}}{f_3^{d+2}} .$$

**The first integral.** Let $\Pi \colon \mathbb{C}^4 \setminus \{0\} \to \mathbb{P}^3$ be the canonical projection and $(x_0, x_1, x_2, x_3)$ be homogeneous coordinates such that $L_\infty = (f_3 = x_0 = 0)$ and the previous affine chart $\mathbb{C}^3 \subset \mathbb{P}^3$ is $(x_0 = 1)$. In this chart,

$$\Pi(x_0, x_1, x_2, x_3) = \left( \frac{x_1}{x_0}, \frac{x_2}{x_0}, \frac{x_3}{x_0} \right) .$$

Since $dg(\mathcal{F}) = d$ we can write $\Pi^*(\Omega) = \frac{1}{x_0^{d+2}}\, \omega$, where the coefficients of $\omega$ are homogenous of degree $d + 1$ and $i_R(\omega) = 0$, $R = \sum_i x_i \partial_{x_i}$. If $m = 1$ we set $f_2^{m-1} := 1$. With this convention, we can write $F = \frac{f_1^{n+1}}{f_2^{m-1} . x_0^{d+2}}$. On the other hand, the relation $d\Omega = \frac{dF}{F} \wedge \Omega$ is equivalent to $d(F^{-1}\,\Omega) = 0$, and so the form

$$\mu := \frac{\Omega}{F} = \frac{f_2^{m-1}\,\omega}{f_1^{n+1}}$$

is closed. Since it is closed and its pole divisor is $f_1^{n+1}$, it can written as

$$\mu = \lambda\, \frac{df_1}{f_1} + d \left( \frac{h}{f_1^n} \right)$$

where $\lambda \in \mathbb{C}$, $h$ is a homogeneous polynomial and $dg(h) = n\, dg(f_1)$.

Since $0 = i_R(\mu) = \lambda\, dg(f_1)$, we get $\lambda = 0$. It follows that $h/f_1^n$ is a rational first integral of $\mathcal{F}$. If $m > 1$ then $\Sigma_2 = (f_2 = 0)$ is $\mathcal{F}$-invariant. Hence, there exists $b \in \mathbb{C}$ such that $(f_2 = 0) \subset (h + b\, f_1^n = 0)$. In particular, there exist $k \in \mathbb{N}$ and a homogeneous polynomial $g$ such that $g.f_2^k = h + b\, f_1^n$, where $f_1, f_2 \nmid g$ and $dg(g) + k\, dg(f_2) = n\, dg(f_1)$. This implies

$$\frac{f_2^{m-1}}{f_1^{n+1}}\,\omega = d\left(\frac{h}{f_1^n}\right) = d\left(\frac{g.f_2^k}{f_1^n}\right)$$
$$\implies f_2^{m-1}\,\omega = f_2^{k-1}\,(f_1\, f_2\, dg + k\, f_1\, g\, df_2 - n\, g\, f_2\, df_1) \implies m = k$$

and $g$ is a constant, because otherwise in a point $q \in (g = f_1 = f_2 = 0) \cap \Gamma$ we would have $j_q^1(\omega) > 1$. This implies that $\mathcal{F}$ has a first integral of the form $f_2^m/f_1^n$. When $m = 1$, we have that $h$ is irreducible and we take $f_2 = h$. This finishes the proof of Theorem 1.2 in dimension three.

### 3.3. Proof of Theorem 1.2 in dimension $n \geq 4$

The idea is to use the case of dimension three and the following known result (*cf.* [8]):

**Theorem 3.25.** *Let $\mathcal{G}$ be a codimension-one holomorphic foliation on $\mathbb{P}^n$, $n \geq 3$. Assume that there is a $k$-plane $E \simeq \mathbb{P}^k$, $2 \leq k < n$ such that $E$ is in general position with $\mathcal{G}$ and $\mathcal{G}|_E$ is represented by a closed meromorphic 1-form $\omega$ on $E$ outside its poles. Then $\omega$ can be extended to a closed meromorphic 1-form $\Omega$ on $\mathbb{P}^n$ representing $\mathcal{G}$ outside its poles. In particular, if $\mathcal{G}|_E$ has a rational first integral then it can be extended to rational first integral of $\mathcal{G}$.*

Recall that $E$ is in general position with $\mathcal{G}$ if:

(a) $E$ is not $\mathcal{G}$-invariant.
(b) The divisor of tangencies between $\mathcal{G}$ and $E$ has codimension at least 2 in $E$.

Moreover, the set of $k$-planes in general position with $\mathcal{G}$ is a Zariski open and dense subset of the respective grassmanian (*cf.* [8]).

Let $\mathcal{F}$ be a codimension-one foliation on $\mathbb{P}^n$, $n \geq 4$, such that $\text{sing}_2(\mathcal{F})$ has an irreducible component $\Gamma$ with $\text{BB}(\mathcal{F}, \Gamma) \neq 0$ and the set $X := \{p \in \Gamma \mid \mathcal{J}(\mathcal{F}, p) > 1\}$ has codimension at least 4 in $\mathbb{P}^n$. Set $\mathcal{N}_\Gamma = \{p \in \Gamma \mid p$ is a nilpotent singularity of $\mathcal{F}\}$ and $K_\Gamma = \{p \in \Gamma \mid p$ is a singularity of Kupka type of $\mathcal{F}\}$. Since $\text{cod}_{\mathbb{P}^n}(X) \geq 4$ and $\text{cod}_{\mathbb{P}^n}(\Gamma) = 2$, we have $\Gamma = \mathcal{N}_\Gamma \cup K_\Gamma \cup X$ and

- Either $\Gamma = \mathcal{N}_\Gamma \cup X$, or $K_\Gamma$ is a Zariski open and dense subset of $\Gamma$.

When $\mathcal{N}_\Gamma \cup X = \emptyset$ then $\Gamma \subset K(\mathcal{F})$ and so Theorem 1.2 is true by [6,11] and [3]. Therefore, from now on we will assume that $\mathcal{N}_\Gamma \cup X \neq \emptyset$. In view of Theorem 3.25, the next result will reduce the problem to the case $n = 3$.

**Lemma 3.26.** *In the above situation, there is a* $(n-1)$*-plane* $\mathbb{P}^{n-1} \simeq E \subset \mathbb{P}^n$ *in general position with* $\mathcal{F}$ *and such that:*

(a) $\Gamma \cap E \subset \mathrm{sing}_2(\mathcal{F}|_E)$.
(b) *The set* $X_E := \{p \in \Gamma \cap E \mid \mathcal{J}(\mathcal{F}|_E, p) > 1\}$ *has codimension at least* 4 *in* $E$.
(c) *If* $\Gamma'$ *is an irreducible component of* $\Gamma \cap E$ *then* $\mathrm{BB}(\mathcal{F}|_E, \Gamma') \neq 0$.

*Proof.* Fix an affine chart $(z_1, ..., z_n) \in \mathbb{C}^n \subset \mathbb{P}^n$ and a polynomial 1-form $\Omega$ representing $\mathcal{F}$ in this chart. Given $p \in \mathbb{C}^n \cap \mathcal{N}_\Gamma$ there is $\ell_p \in \mathbb{C}[z_1, ..., z_n]$, of degree-one, such that $\ell_p(p) = 0$ and

$$j^1_p(\Omega) = \ell_p \, d\ell_p \ .$$

Note that the hyperplane $H_p = \overline{(\ell_p = 0)} \in \check{\mathbb{P}}^n$ does not depend on the affine chart containing $p$. As a consequence, the correspondence $p \mapsto H_p$ defines an analytic map $H \colon \mathcal{N}_\Gamma \to \check{\mathbb{P}}^n$. Since $\dim(\mathcal{N}_\Gamma) \leq n-2$, we get $\dim(H(\mathcal{N}_\Gamma)) \leq n-2$. In particular, the set

$$A := \check{\mathbb{P}}^n \setminus \overline{H(\mathcal{N}_\Gamma)}$$

is a Zariski open and dense subset of $\check{\mathbb{P}}^n$. Let $B = \{E \in A \mid E$ is in general position with $\mathcal{F}\}$.

Note that $B$ is a Zariski open and dense subset of $\check{\mathbb{P}}^n$. Moreover, if $E \in B$ then all points of $\mathcal{N}_\Gamma \cap E$ are nilpotent singularities of $\mathcal{F}|_E$. In fact, fix $p \in \mathcal{N}_\Gamma \cap E$, an affine coordinate system $z = (z_1, ..., z_n) \in \mathbb{C}^n \subset \mathbb{P}^n$ and a polynomial 1-form $\Omega$ representing $\mathcal{F}$ in this chart, such that $z(p) = 0$ and $E \cap \mathbb{C}^n = (z_n = 0)$. Let $\ell_p$ be a degree-one polynomial with $\ell_p(p) = 0$, $H_p \cap \mathbb{C}^n = (\ell_p = 0)$ and $j^1_p(\Omega) = \ell_p \, d\ell_p$. Since $\ell_p(0) = 0$ and $E \neq H_p$, we can set $\ell_p(z) = \sum_{j=1}^{n} a_j z_j$, where $a_j \neq 0$ for some $j \in \{1, ..., n-1\}$. The polynomial $\tilde{\ell}_p := \ell_p|_{E \cap \mathbb{C}^n}$ is non-constant. In particular,

$$j^1_0(\Omega|_E) = \tilde{\ell}_p \, d\tilde{\ell}_p \neq 0 \ .$$

Therefore, $p$ is a nilpotent singularity of $\mathcal{F}|_E$.

Now, consider an algebraic stratification $\mathrm{sing}(\mathcal{F}) := S_0 \supset S_1 \supset ... \supset S_r = \emptyset$, where $\dim(S_0) = n-2$, $\dim(S_{j+1}) < \dim(S_j)$ and $S_j \setminus S_{j+1}$ is a smooth manifold, for all $0 \leq j < r$. By transversality theory, there exists $E \in B$ transverse to all manifolds $S_j \setminus S_{j+1}$, $0 \leq j < r$. We assert that $E$ satisfies properties (a), (b) and (c).

In fact, since $\Gamma \subset \mathrm{sing}_2(\mathcal{F})$ we must have $\Gamma \setminus S_1 \neq \emptyset$, and so $\mathrm{cod}(\Gamma \cap E) = 2$, which implies (a), because $\Gamma \cap E \subset \mathrm{sing}(\mathcal{F}|_E)$. On the other hand, since $K_\Gamma$ is smooth of codimension-two, we get $K_\Gamma \subset S_0 \setminus S_1$. In particular, $E$ is transverse to $K_\Gamma$ and this implies that $K_\Gamma \cap E \subset K(\mathcal{F}|_E)$. Therefore, $\mathcal{J}(\mathcal{F}|_E, p) \leq 1$ for all $p \in (\Gamma \setminus X) \cap E$. This implies also that $X_E = X \cap E$. Since $X \subset S_1$, by transversality we get $\mathrm{cod}_E(X_E) \geq 4$.

Finally, if $\Gamma'$ is an irreducible component of $\Gamma \cap E$ then $\mathrm{BB}(\mathcal{F}|_E, \Gamma')$ can be computed in any dimension two transverse section, say $\Lambda$, through any point in the smooth part of $\Gamma \cap E$. If we take such a point in the smooth part of $\Gamma$ then we see that $\Lambda$ is also transverse to $\Gamma$ at this point, which implies

$$\mathrm{BB}(\mathcal{F}|_E, \Gamma') = \mathrm{BB}(\mathcal{F}, \Gamma) \neq 0 \ . \qquad \square$$

By using Lemma 3.26 inductively $n-3$ times we get:

**Corollary 3.27.** *In the situation of Lemma* 3.26 *there is a 3-plane* $\mathbb{P}^3 \simeq E \subset \mathbb{P}^n$*, in general position with* $\mathcal{F}$*, with* $\mathcal{J}(\mathcal{F}|_E, p) \leq 1$*, for all* $p \in \Gamma \cap E$*, and* $\mathrm{BB}(\mathcal{F}|_E, \Gamma') \neq 0$*, for all irreducible components of* $\Gamma'$ *of* $\Gamma \cap E$.

In particular, $\mathcal{F}|_E$ has a rational first integral of the form $f_1^m/f_2^n$, where $gcd(m,n) = 1$, $1 \leq m < n$, $m\,dg(f_1) = n\,dg(f_2)$ and $f_1$, $f_2$ are irreducible. By Theorem 3.25 this first integral can be extended to a rational first integral of $\mathcal{F}$. This finshes the proof of Theorem 1.2. $\square$

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