

A note on quasilinear parabolic equations on manifolds

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Abstract. We prove short time existence, uniqueness and continuous dependence on the initial data of smooth solutions of quasilinear locally parabolic equations of arbitrary even order on closed manifolds.

Mathematics Subject Classification (2010): 35K59 (primary); 35K41, 35K52 (secondary).

1. Introduction

Let (M, g) be a compact, smooth Riemannian manifold without boundary of dimension n and let $d\mu$ be the canonical measure associated to the metric tensor g .

We consider the parabolic problem with a smooth initial datum $u_0 : M \rightarrow \mathbb{R}$,

$$\begin{cases} u_t = Q[u] & \text{in } M \times [0, T] \\ u(\cdot, 0) = u_0 & \text{on } M, \end{cases} \quad (1.1)$$

where Q is a smooth, quasilinear, locally elliptic operator of order $2p$, defined in $M \times [0, T)$ for some $T > 0$ which, adopting (as in the rest of the paper) the Einstein convention of summing over repeated indices, can be expressed in local coordinates as

$$\begin{aligned} Q[u](x, t) \\ = A^{i_1 \dots i_{2p}}(x, t, u, \nabla u, \dots, \nabla^{2p-1} u) \nabla_{i_1 \dots i_{2p}}^{2p} u(x, t) + b(x, t, u, \nabla u, \dots, \nabla^{2p-1} u), \end{aligned}$$

where A is a locally elliptic smooth $(2p, 0)$ -tensor of the form

$$A^{i_1 j_1 \dots i_p j_p} = (-1)^{p-1} E_1^{i_1 j_1} \dots E_p^{i_p j_p} \quad (1.2)$$

The authors are partially supported by the Italian project FIRB-IDEAS “Analysis and Beyond”. The second author is partially supported by the Swiss National Fond Grant n. PBEZP2-129520.

Received October 5, 2010; accepted April 30, 2011.

for some $(2, 0)$ -tensors E_1, \dots, E_p and a function b smoothly depending on their arguments.

Local ellipticity here means that for every $L > 0$ there exists a positive constant $\lambda \in \mathbb{R}$ such that each tensor E_ℓ satisfies

$$E_\ell^{ij}(x, t, u, \psi_1, \dots, \psi_{2p-1})\xi_i\xi_j \geq \lambda|\xi|_{g(x)}^2, \quad \text{for every } \xi \in T_x^*M, \quad (1.3)$$

when $x \in M, t \in [0, T]$ with $T < \mathcal{T}, u \in \mathbb{R}$ with $|u| \leq L, \psi_k \in \otimes^k T_x^*M$ with $|\psi_k|_{g(x)} \leq L$. In other words we require that condition (1.3) holds for some positive λ whenever the arguments of E_ℓ^{ij} lie in a compact set K of their natural domain of definition and assume that λ depends only on K . If $\lambda > 0$ can be chosen independent of K (i.e. of L), then we shall say that A is *uniformly elliptic*.

Clearly, this is not the most general notion of quasilinear parabolic problems, due to the special “product” structure of the operator, anyway it covers several important situations. For instance, our definition includes the case of standard locally parabolic equations of order two in non-divergence form. Notice that we make no growth assumptions on the tensor A and the function b .

Interchanging covariant derivatives, integrating by parts and using interpolation inequalities (see [10] for details), the following Gårding’s inequality holds for this class of operators. For every smooth u and $t \in [0, \mathcal{T})$, we have

$$-\int_M \psi A^{i_1 \dots i_{2p}}(u) \nabla_{i_1 \dots i_{2p}}^2 \psi \, d\mu \geq \sigma \|\psi\|_{W^{p,2}(M)}^2 - C \|\psi\|_{L^2(M)}^2 \quad (1.4)$$

$$\forall \psi \in C^\infty(M),$$

where the constants $\sigma > 0$ and $C > 0$ depend continuously only on the C^p -norm of the tensor A and on the C^{3p-1} -norm of the function u at time t (and on the curvature tensor of (M, g) and its covariant derivatives). In particular, if u depends smoothly on time, $\sigma = \sigma(t)$ and $C = C(t)$ are continuous functions of time.

The aim of this note is to prove the following short time existence result.

Theorem 1.1. *For every $u_0 \in C^\infty(M)$ there exists a positive time $T > 0$ such that problem (1.1) has a smooth solution. Moreover, the solution is unique and depends continuously on u_0 in the C^∞ -topology.*

Our interest in having a handy proof of this result is related to geometric evolution problems, like for instance the Ricci flow, the mean curvature flow, the Willmore flow [7], the Q -curvature flow [9], the Yamabe flow [4, 11, 13], etc. In all these problems, the very first step is to have a short time existence theorem showing that for an initial geometric structure (hypersurface, metric) the flow actually starts. Usually, after some manipulations in order to eliminate the degeneracies due to the geometric invariances, one has to face a quasilinear parabolic equation with smooth coefficients and smooth initial data.

If we replace the compact manifold M with a bounded domain $\Omega \subset \mathbb{R}^n$, short time existence for quasilinear systems of order two, with prescribed boundary conditions and initial data, was proven by Giaquinta and Modica [5] in the setting of Hölder spaces.

A different approach to Theorem 1.1 was developed by Polden in his Ph.D. Thesis [10] (see also [6]), by means of an existence result for linear equations in parabolic Sobolev spaces and the inverse function theorem. Unfortunately, as pointed out by Sharples [12], such procedure has a gap in the convergence of the solutions of the “frozen” linear problems to a solution of the quasilinear one.

In the same paper [12] Sharples, pushing further the estimates of Polden and allowing nonsmooth coefficients, was able by means of an iteration scheme to show the existence of a short time solution of the quasilinear problem on a two-dimensional manifold, when the operator is of order two and *in divergence form*.

Our goal here is instead to simply fill the gap in Polden’s proof. We start with his linear result and we show that his linearization procedure actually works if one linearizes at a suitably chosen function and discusses in details the above mentioned convergence.

As we do not assume any condition on the operator (only its product structure) and on the dimension of the manifold, we have a complete proof of the short time existence of a smooth solution to these quasilinear locally parabolic equations of arbitrary order on compact manifolds and of its uniqueness and smooth dependence on the initial data. We refer the interested reader to the nice and detailed introduction in [12] for the different approaches to the problem.

The paper is organized as follows. In the next section we present the linearization procedure, assuming Polden’s linear result (Proposition 2.2 below) and we prove Theorem 1.1 by means of Lemma 2.5 which is the core of our argument. Roughly speaking, when a candidate solution u stays in some parabolic Sobolev space of order high enough, the functions $u, \nabla u, \dots, \nabla^{2p-1}u$ are continuous (or even more regular), hence the same holds for the tensor A and the function b . This implies that the map $u \mapsto (u_t - Q[u])$ is of class C^1 between some suitable spaces, as it closely resembles a linear map with regular coefficients. This allows the application of the inverse function theorem which, in conjunction with an approximation argument, yields the existence of a solution. The last two sections are devoted to the proof of Lemma 2.5 and to the discussion of the parabolic Sobolev embeddings on which such proof relies.

We mention that the results can be extended to quasilinear parabolic *systems* as the linearization procedure remains the same and Polden’s linear estimates (Proposition 2.2) can be actually easily generalized, assuming a suitable definition of ellipticity. In fact one easily sees that our result applies to all quasilinear systems whose linearization is invertible in the sense of Proposition 2.3 below. For more general definition of elliptic or parabolic operators of higher-order see [2].

In the following the letter C will denote a constant which can change from a line to another and even within the same formula.

ACKNOWLEDGEMENTS. We are grateful to Alessandra Lunardi for useful suggestions. We wish to thank Mariano Giaquinta for several interesting discussions.

2. Proof of the main theorem

We recall Polden’s result for linear parabolic equations. Let us consider the problem

$$\begin{cases} u_t - A^{i_1 \dots i_{2p}} \nabla_{i_1 \dots i_{2p}}^{2p} u - \sum_{k=0}^{2p-1} R_k^{j_1 \dots j_k} \nabla_{j_1 \dots j_k}^k u = b \\ u(\cdot, 0) = u_0, \end{cases} \tag{2.1}$$

where all the tensors A and R_k depend only on $(x, t) \in M \times [0, +\infty)$, are smooth and uniformly bounded with all their derivatives. Moreover, we assume that the tensor A has the product structure (1.2), where each E_ℓ is uniformly elliptic.

The Gårding’s inequality for the linear operator

$$L(u) = A^{i_1 \dots i_{2p}} \nabla_{i_1 \dots i_{2p}}^{2p} u - \sum_{k=0}^{2p-1} R_k^{j_1 \dots j_k} \nabla_{j_1 \dots j_k}^k u$$

reads (see again [10] for details)

$$- \int_M \psi L(\psi) d\mu \geq \frac{\lambda}{2} \|\psi\|_{W^{p,2}(M)}^2 - C \|\psi\|_{L^2(M)}^2 \quad \forall \psi \in C^\infty(M), \tag{2.2}$$

where the constant $C > 0$ depends only on the C^p -norm of the tensors A and R_k . Clearly, by approximation this inequality holds also for every $\psi \in W^{2p,2}(M)$.

Definition 2.1. For any $m \in \mathbb{N}$ and $a \in \mathbb{R}^+$ we define $P_a^m(M)$ to be the completion of $C_c^\infty(M \times [0, +\infty))$ under the *parabolic norm*

$$\|f\|_{P_a^m(M)}^2 = \sum_{j, k \in \mathbb{N} \text{ and } 2pj + k \leq 2pm} \int_{M \times [0, +\infty)} e^{-2at} |\partial_t^j \nabla^k f|^2 d\mu dt$$

and analogously $P^m(M, T)$ as the completion of $C^\infty(M \times [0, T])$ under the norm

$$\|f\|_{P^m(M, T)}^2 = \sum_{j, k \in \mathbb{N} \text{ and } 2pj + k \leq 2pm} \int_{M \times [0, T]} |\partial_t^j \nabla^k f|^2 d\mu dt,$$

for every $T \in \mathbb{R}^+$.

Clearly for every $T \in \mathbb{R}^+$ there is a natural continuous embedding $P_a^m(M) \hookrightarrow P^m(M, T)$.

We have then the following global existence result for problem (2.1), by Polden [10, Theorem 2.3.5].

Proposition 2.2. *For every $m \in \mathbb{N}$ there exists $a \in \mathbb{R}^+$ large enough such that the linear map*

$$\Phi(u) = \left(u_0, u_t - A^{i_1 \dots i_{2p}} \nabla_{i_1 \dots i_{2p}}^{2p} u - \sum_{k=0}^{2p-1} R_k^{j_1 \dots j_k} \nabla_{j_1 \dots j_k}^k u \right) = (u_0, L(u)), \tag{2.3}$$

where $u_0 = u(\cdot, 0)$, is an isomorphism of $P_a^m(M)$ onto $W^{p(2m-1),2}(M) \times P_a^{m-1}(M)$.

In the following it will be easier (though conceptually equivalent) to use the spaces $P^m(M, T)$ instead of the weighted spaces $P_a^m(M)$. For this reason we translate Proposition 2.2 into the setting of $P^m(M, T)$ spaces.

Proposition 2.3. *For every $T > 0$ and $m \in \mathbb{N}$ the map Φ given by formula (2.3) is an isomorphism of $P^m(M, T)$ onto $W^{p(2m-1), 2}(M) \times P^{m-1}(M, T)$.*

Proof. The continuity of the second component of Φ is obvious while the continuity of the first component follows as in the Polden’s proof in [10, Proposition 2.2]. Hence, the map Φ is continuous, now we show that it is an isomorphism.

Given any $b \in P^{m-1}(M, T)$ we consider an extension $\tilde{b} \in P_a^{m-1}(M)$ of the function b and we let $\tilde{u} \in P_a^m(M)$ be the solution of problem (2.1) for \tilde{b} . Clearly, $u = \tilde{u}|_{M \times [0, T]}$ belongs to $P^m(M, T)$ and satisfies $\Phi(u) = (u_0, b)$ in $M \times [0, T]$. Suppose that $v \in P^m(M, T)$ is another function such that $\Phi(v) = (u_0, b)$ in $M \times [0, T]$, then setting $w = u - v \in P^m(M, T)$ we have that

$$\begin{cases} w_t - A^{i_1 \dots i_{2p}} \nabla_{i_1 \dots i_{2p}}^{2p} w - \sum_{k=0}^{2p-1} R_k^{j_1 \dots j_k} \nabla_{j_1 \dots j_k}^k w = w_t - L(w) = 0 \\ w(\cdot, 0) = 0. \end{cases}$$

By the very definition of solution in $P^m(M, T)$ (see [10]) and Gårding’s inequality (2.2), we get

$$\begin{aligned} \int_M w^2(x, t) d\mu(x) &= \int_0^t \int_M 2w w_t d\mu ds \\ &= 2 \int_0^t \int_M w L(w) d\mu ds \\ &\leq -\frac{\lambda}{2} \int_0^t \int_M |\nabla^p w|^2 d\mu ds + C \int_0^t \int_M w^2 d\mu ds \\ &\leq C \int_0^t \int_M w^2(x, s) d\mu(x) ds, \end{aligned}$$

as $w(\cdot, t) \in W^{2p, 2}(M)$ for almost every $t \in [0, T]$ and where the constant $C > 0$ depends only on T as the coefficients of the operator L are smooth. Then, by Gronwall’s lemma (in its integral version) it follows that $\int_M w^2(\cdot, t) d\mu$ is zero for every $t \in [0, T]$, as it is zero at time $t = 0$. It follows that w is zero on all $M \times [0, T]$, hence the two functions u and v must coincide.

Since the map $\Phi : P^m(M, T) \rightarrow W^{p(2m-1), 2}(M) \times P^{m-1}(M, T)$ is continuous, one-to-one and onto, it is an isomorphism by the open mapping theorem. \square

Remark 2.4. When u_0 and b are smooth the unique solution u of problem (2.1) belongs to all the spaces $P^m(M, T)$ for every $m \in \mathbb{N}$. As by Sobolev embeddings for any $k \in \mathbb{N}$ we can find a large $m \in \mathbb{N}$ so that $P^m(M, T)$ continuously embeds into $C^k(M \times [0, T])$, we can conclude that u actually belongs to $C^\infty(M \times [0, T])$.

Now we are ready to prove Theorem 1.1. The tensor A and the function b from now on will depend on $x, t, u, \nabla u, \dots, \nabla^{2p-1}u$ as in the introduction. Since M is compact there exists a constant $C > 0$ such that the initial datum satisfies $|u_0| + |\nabla u_0|_g + \dots + |\nabla^{2p-1}u_0|_g \leq C$. Then, since we are interested in existence for short time, possibly modifying the tensor A and the function b outside a compact set with some “cut-off” functions, we can assume that if $|u| + |\nabla u|_g + \dots + |\nabla^{2p-1}u|_g + t \geq 2C$, then

$$E_\ell^{ij}(x, t, u, \nabla u, \dots, \nabla^{2p-1}u) = g^{ij}(x), \quad \text{and} \quad b(x, t, u, \nabla u, \dots, \nabla^{2p-1}u) = 0.$$

In particular we can assume that the tensors E_ℓ are uniformly elliptic.

For a fixed $m \in \mathbb{N}$, we consider the map defined on $P^m(M, T)$ given by

$$\mathcal{F}(u) = (u_0, u_t - Q[u]) = \left(u(\cdot, 0), u_t - A(u) \cdot \nabla^{2p}u - b(u) \right),$$

where in order to simplify we used the notation

$$A(u) \cdot \nabla^{2p}v(x, t) = A^{i_1 \dots i_{2p}}(x, t, u(x, t), \dots, \nabla^{2p-1}u(x, t)) \nabla_{i_1 \dots i_{2p}}^{2p} v(x, t),$$

and

$$b(u)(x, t) = b(x, t, u(x, t), \dots, \nabla^{2p-1}u(x, t))$$

for $u, v \in P^m(M, T)$.

We have seen in Proposition 2.3 that if $A(u)$ and $b(u)$ only depend on $x \in M$ and $t \in [0, T]$ (and not on u and its space derivatives), then \mathcal{F} is a continuous map from $P^m(M)$ onto $W^{p(2m-1), 2}(M) \times P^{m-1}(M)$. This is not the case in general when A and b depend on u and its derivatives, but it is true if $m \in \mathbb{N}$ is large enough and in this case \mathcal{F} is actually C^1 .

Lemma 2.5. *Assume that*

$$m > \frac{\dim M + 6p - 2}{4p} = \frac{n + 6p - 2}{4p}, \tag{2.4}$$

and $u \in P^m(M, T)$. Then $\mathcal{F}(u) \in W^{p(2m-1), 2}(M) \times P^{m-1}(M, T)$ and the map

$$\mathcal{F} : P^m(M, T) \rightarrow W^{p(2m-1), 2}(M) \times P^{m-1}(M, T)$$

is of class C^1 .

We postpone the proof of this lemma to Section 3.

We fix $m \in \mathbb{N}$ such that the hypothesis of Lemma 2.5 holds and we set

$$\tilde{u}_0(x, t) = \sum_{\ell=0}^{m-1} \frac{a_\ell(x) t^\ell}{\ell!}$$

for some functions $a_0, \dots, a_{m-1} \in C^\infty(M)$ to be determined later. Let $w \in P^m(M, T)$ be the unique solution of the linear problem

$$\begin{cases} w_t = A(\tilde{u}_0) \cdot \nabla^{2p} w + b(\tilde{u}_0) \\ w(\cdot, 0) = u_0. \end{cases}$$

Such solution exists by Proposition 2.3 and it is smooth by Remark 2.4, as u_0 and \tilde{u}_0 are smooth (thus also $A(\tilde{u}_0)$ and $b(\tilde{u}_0)$).

Hence, we have

$$\mathcal{F}(w) = (u_0, w_t - Q[w]) = \left(u_0, (A(\tilde{u}_0) - A(w)) \cdot \nabla^{2p} w + b(\tilde{u}_0) - b(w) \right) =: (u_0, f),$$

where we set $f = (A(\tilde{u}_0) - A(w)) \cdot \nabla^{2p} w + b(\tilde{u}_0) - b(w)$.

If we compute the differential $d\mathcal{F}_w$ of the map \mathcal{F} at the ‘‘point’’ $w \in C^\infty(M \times [0, T])$, acting on $v \in P^m(M, T)$, we obtain

$$\begin{aligned} d\mathcal{F}_w(v) = & \left(v_0, v_t - A^{i_1 \dots i_{2p}}(w) \nabla_{i_1 \dots i_{2p}}^{2p} v - D_w A^{i_1 \dots i_{2p}}(w) v \nabla_{i_1 \dots i_{2p}}^{2p} w \dots \right. \\ & \dots - D_{w_{j_1 \dots j_{2p-1}}} A^{i_1 \dots i_{2p}}(w) \nabla_{j_1 \dots j_{2p-1}}^{2p-1} v \nabla_{i_1 \dots i_{2p}}^{2p} w \\ & \left. - D_w b(w) v \dots - D_{w_{j_1 \dots j_{2p-1}}} b(w) \nabla_{j_1 \dots j_{2p-1}}^{2p-1} v \right), \end{aligned} \tag{2.5}$$

where $v_0 = v(\cdot, 0)$ and we denoted by $D_{w_{j_1 \dots j_k}} A^{i_1 \dots i_{2p}}(w)$, $D_{w_{j_1 \dots j_k}} b(w)$ the derivatives of the tensor A and of the function b with respect to their variables $\nabla_{j_1 \dots j_k}^k w$, respectively.

Then, we can see that $d\mathcal{F}_w(v) = (z, h) \in W^{p(2m-1), 2}(M) \times P^{m-1}(M, T)$ implies that v is a solution of the linear problem

$$\begin{cases} v_t - \tilde{A}^{i_1 \dots i_{2p}} \nabla_{i_1 \dots i_{2p}}^{2p} v - \sum_{k=0}^{2p-1} \tilde{R}_k^{j_1 \dots j_k} \nabla_{j_1 \dots j_k}^k v = h \\ v(\cdot, 0) = z, \end{cases}$$

where

$$\begin{aligned} \tilde{A}^{i_1 \dots i_{2p}} &= A^{i_1 \dots i_{2p}}(w), \\ \tilde{R}_k^{j_1 \dots j_k} &= D_{w_{j_1 \dots j_k}} A^{i_1 \dots i_{2p}}(w) \nabla_{i_1 \dots i_{2p}}^{2p} w + D_{w_{j_1 \dots j_k}} b(w) \end{aligned}$$

are smooth tensors independent of v .

By Proposition 2.3 for every $(z, h) \in W^{p(2m-1),2}(M) \times P^{m-1}(M, T)$ there exists a unique solution v of this problem, hence $d\mathcal{F}_w$ is a Hilbert space isomorphism and the inverse function theorem can be applied, as the map \mathcal{F} is C^1 by Lemma 2.5. Hence, the map \mathcal{F} is a diffeomorphism of a neighborhood $U \subset P^m(M, T)$ of w onto a neighborhood $V \subset W^{p(2m-1),2}(M) \times P^{m-1}(M, T)$ of (u_0, f) .

Getting back to the functions a_ℓ , we claim that we can choose them such that $a_\ell = \partial_t^\ell w|_{t=0} \in C^\infty(M)$ for every $\ell = 0, \dots, m - 1$.

We apply the following recurrence procedure. We set $a_0 = u_0 \in C^\infty(M)$ and, assuming to have defined a_0, \dots, a_ℓ , we consider the derivative

$$\begin{aligned} \partial_t^{\ell+1} w|_{t=0} = \partial_t^\ell \left[A^{i_1 \dots i_{2p}}(x, t, \tilde{u}_0, \nabla \tilde{u}_0, \dots, \nabla^{2p-1} \tilde{u}_0) \nabla_{i_1 \dots i_{2p}}^{2p} w \right. \\ \left. + b(x, t, \tilde{u}_0, \nabla \tilde{u}_0, \dots, \nabla^{2p-1} \tilde{u}_0) \right] \Big|_{t=0} \end{aligned}$$

and we see that the right-hand side contains time-derivatives at time $t = 0$ of $\tilde{u}_0, \dots, \nabla^{2p-1} \tilde{u}_0$ and $\nabla_{i_1 \dots i_{2p}}^{2p} w$ only up to the order ℓ , hence it only depends on the functions a_0, \dots, a_ℓ . Then, we define $a_{\ell+1}$ to be equal to such expression. Iterating up to $m - 1$, the set of functions a_0, \dots, a_{m-1} satisfies the claim.

Then, $a_\ell = \partial_t^\ell \tilde{u}_0|_{t=0} = \partial_t^\ell w|_{t=0}$ and it easily follows by the “structure” of the function $f \in C^\infty(M \times [0, T])$, that we have $\partial_t^\ell f|_{t=0} = 0$ and $\nabla^j \partial_t^\ell f|_{t=0} = 0$ for any $0 \leq \ell \leq m - 1$ and $j \in \mathbb{N}$.

We consider now for any $k \in \mathbb{N}$ the “translated” functions $f_k : M \times [0, T] \rightarrow \mathbb{R}$ given by

$$f_k(x, t) = \begin{cases} 0 & \text{if } t < 1/k \\ f(x, t - 1/k) & \text{if } 1/k \leq t \leq T. \end{cases}$$

Since $f \in C^\infty(M \times [0, T])$ and $\nabla^j \partial_t^\ell f|_{t=0} = 0$ for every $0 \leq \ell \leq m - 1$ and every $j \in \mathbb{N}$, all the functions $\nabla^j \partial_t^\ell f_k \in C^0(M \times [0, T])$ for every $0 \leq \ell \leq m - 1$ and $j \geq 0$, it follows easily that

$$\nabla^j \partial_t^\ell f_k \rightarrow \nabla^j \partial_t^\ell f \quad \text{in } L^2(M \times [0, T]) \text{ for } 0 \leq \ell \leq m - 1, j \geq 0,$$

hence $f_k \rightarrow f$ in $P^m(M, T)$.

Hence, there exists a function $\tilde{f} \in P^{m-1}(M, T)$ such that (u_0, \tilde{f}) belongs to the neighborhood V of $\mathcal{F}(w)$ and $\tilde{f} = 0$ in $M \times [0, T']$ for some $T' \in (0, T]$. Since $\mathcal{F}|_U$ is a diffeomorphism between U and V , we can find a function $u \in U$ such that $\mathcal{F}(u) = (u_0, \tilde{f})$. Clearly such $u \in P^m(M, T')$ is a solution of problem (1.1) in $M \times [0, T']$. Since $u \in P^m(M, T')$ implies that $\nabla^{2p-1} u \in C^0(M \times [0, T'])$, parabolic regularity implies that actually $u \in C^\infty(M \times [0, T'])$.

We now prove uniqueness by a standard energy estimate, which we include for completeness. In the sequel for simplicity we relabel T the time T' found above.

Suppose that we have two smooth solutions $u, v : M \times [0, T] \rightarrow \mathbb{R}$ of Problem (1.1). Setting $w := u - v$, we compute in an orthonormal frame

$$\begin{aligned} \frac{d}{dt} \int_M |\nabla^p w|^2 d\mu &= \int_M 2\nabla_{i_1 \dots i_p}^p w \nabla_{i_1 \dots i_p}^p (A(u) \cdot \nabla^{2p} u - A(v) \cdot \nabla^{2p} v) d\mu \\ &\quad + \int_M 2\nabla_{i_1 \dots i_p}^p w \nabla_{i_1 \dots i_p}^p (b(u) - b(v)) d\mu \\ &= 2 \int_M \nabla_{i_1 \dots i_p}^p w \nabla_{i_1 \dots i_p}^p (A(u) \cdot \nabla^{2p} w) d\mu \\ &\quad + 2 \int_M \nabla_{i_1 \dots i_p}^p w \nabla_{i_1 \dots i_p}^p ((A(u) - A(v)) \cdot \nabla^{2p} v) d\mu \\ &\quad + 2(-1)^p \int_M \nabla_{i_1 \dots i_p}^p \nabla_{i_1 \dots i_p}^p w (b(u) - b(v)) d\mu \\ &\leq 2 \int_M \nabla_{i_1 \dots i_p}^p w \nabla_{i_1 \dots i_p}^p (A(u) \cdot \nabla^{2p} w) d\mu \\ &\quad + 2 \int_M |\nabla^{2p} w| (|A(u) - A(v)| |\nabla^{2p} v| + |b(u) - b(v)|) d\mu, \end{aligned}$$

where the integrals over M are intended at time $t \in [0, T]$.

Now we consider the integral $\int_M \nabla_{i_1 \dots i_p}^p w \nabla_{i_1 \dots i_p}^p (A^{j_1 \dots j_{2p}}(u) \nabla_{j_1 \dots j_{2p}}^{2p} w) d\mu$. Expanding the derivative $\nabla_{i_1 \dots i_p}^p (A^{j_1 \dots j_{2p}}(u) \nabla_{j_1 \dots j_{2p}}^{2p} w)$ we will get one special term $A^{j_1 \dots j_{2p}}(u) \nabla_{i_1 \dots i_p j_1 \dots j_{2p}}^{3p} w$ and several other terms of the form $B(x, t, u, \dots, \nabla^{3p-1} u) \# \nabla^q w$ with $2p \leq q < 3p$, for some tensor B smoothly depending on its arguments, where the symbol $\#$ means metric contraction on some indices. For each of these terms, integrating repeatedly by parts, we can write

$$\begin{aligned} &\int_M \nabla^p w \# B(x, t, u, \dots, \nabla^{3p-1} u) \# \nabla^q w d\mu \\ &= \sum_{\ell=p}^{2p} \int_M \nabla^\ell w \# D_\ell(x, t, u, \dots, \nabla^{4p-1} u) \# \nabla^{q-p} w d\mu \end{aligned}$$

where the tensors D_ℓ are smoothly depending on their arguments.

Since $u \in C^\infty(M \times [0, T])$, all the tensors D_ℓ are bounded, hence we can estimate

$$\begin{aligned} \int_M \nabla_{i_1 \dots i_p}^p w \nabla_{i_1 \dots i_p}^p (A(u) \cdot \nabla^{2p} w) d\mu &\leq \int_M \nabla_{i_1 \dots i_p}^p w A^{j_1 \dots j_{2p}}(u) \nabla_{i_1 \dots i_p j_1 \dots j_{2p}}^{3p} w d\mu \\ &\quad + C \sum_{r=p}^{2p-1} \sum_{\ell=p}^{2p} \int_M |\nabla^\ell w| |\nabla^r w| d\mu. \end{aligned}$$

where C is a constant independent of time (actually C depends only on the structure of A). Interchanging the covariant derivatives we have

$$\nabla_{i_1 \dots i_p j_1 \dots j_{2p}}^{3p} w = \nabla_{j_1 \dots j_{2p} i_1 \dots i_p}^{3p} w + \sum_{q=0}^{3p-1} R_q \# \nabla^q w$$

where the tensors R_q are functions of the Riemann tensor and its covariant derivatives, hence they are smooth and bounded. We can clearly deal with this sum of terms as above, by means of integrations by parts, obtaining the same result. Then we conclude, also using Gårding’s inequality (1.4)

$$\begin{aligned} & \int_M \nabla_{i_1 \dots i_p}^p w \nabla_{i_1 \dots i_p}^p (A(u) \cdot \nabla^{2p} w) \, d\mu \leq \int_M \nabla_{i_1 \dots i_p}^p w A^{j_1 \dots j_{2p}}(u) \nabla_{j_1 \dots j_{2p}}^{2p} \nabla_{i_1 \dots i_p}^p w \, d\mu \\ & + C \sum_{r=p}^{2p-1} \sum_{\ell=p}^{2p} \int_M |\nabla^\ell w| |\nabla^r w| \, d\mu \\ & \leq -\alpha \int_M |\nabla^{2p} w|^2 \, d\mu + C \sum_{r=p}^{2p-1} \sum_{\ell=p}^{2p} \int_M |\nabla^\ell w| |\nabla^r w| \, d\mu, \end{aligned}$$

for some positive constant α . Getting back to the initial computation and using Peter-Paul inequality we get

$$\begin{aligned} \frac{d}{dt} \int_M |\nabla^p w|^2 \, d\mu & \leq -2\alpha \int_M |\nabla^{2p} w|^2 \, d\mu + C \sum_{r=p}^{2p-1} \sum_{\ell=p}^{2p} \int_M |\nabla^\ell w| |\nabla^r w| \, d\mu \\ & + C \int_M |\nabla^{2p} w| (|A(u) - A(v)| |\nabla^{2p} v| + |b(u) - b(v)|) \, d\mu \\ & \leq -2\alpha \int_M |\nabla^{2p} w|^2 \, d\mu + C \sum_{r=p}^{2p-1} \sum_{\ell=p}^{2p-1} \int_M |\nabla^\ell w| |\nabla^r w| \, d\mu \\ & + \sum_{r=0}^{2p-1} \left(\varepsilon_r \int_M |\nabla^{2p} w|^2 \, d\mu + C_{\varepsilon_r} \int_M |\nabla^r w|^2 \, d\mu \right) \\ & + \delta \int_M |\nabla^{2p} w|^2 \, d\mu + C_\delta \int_M (|A(u) - A(v)|^2 + |b(u) - b(v)|^2) \, d\mu \\ & \leq -\alpha \int_M |\nabla^{2p} w|^2 \, d\mu + C \sum_{r=0}^{2p-1} \int_M |\nabla^r w|^2 \, d\mu \\ & + C_\delta \int_M (|A(u) - A(v)|^2 + |b(u) - b(v)|^2) \, d\mu, \end{aligned}$$

where we chose $\delta + \sum_{r=0}^{2p-1} \varepsilon_r = \alpha$ and we used the fact that $|\nabla^{2p} v|$ is bounded.

As the tensor A and the function b are smooth, we can easily bound

$$|A(u) - A(v)|^2 + |b(u) - b(v)|^2 \leq C \sum_{r=0}^{2p-1} |\nabla^r u - \nabla^r v|^2 = C \sum_{r=0}^{2p-1} |\nabla^r w|^2,$$

so finally

$$\frac{d}{dt} \int_M |\nabla^p w|^2 d\mu \leq -\alpha \int_M |\nabla^{2p} w|^2 d\mu + C \sum_{r=0}^{2p-1} \int_M |\nabla^r w|^2 d\mu.$$

Now we have, using again Gårding's and Peter-Paul inequalities,

$$\begin{aligned} \frac{d}{dt} \int_M w^2 d\mu &= 2 \int_M w (A(u) \cdot \nabla^{2p} u - A(v) \cdot \nabla^{2p} v) d\mu + 2 \int_M w (b(u) - b(v)) d\mu \\ &= 2 \int_M w A(u) \cdot \nabla^{2p} w d\mu + 2 \int_M w ((A(u) - A(v)) \cdot \nabla^{2p} v + b(u) - b(v)) d\mu \\ &\leq -\beta \int_M |\nabla^p w|^2 d\mu + C \int_M w^2 d\mu + C \int_M w (A(u) - A(v) + b(u) - b(v)) d\mu \\ &\leq -\beta \int_M |\nabla^p w|^2 d\mu + C \int_M w^2 d\mu + C \int_M (|A(u) - A(v)|^2 + |b(u) - b(v)|^2) d\mu. \end{aligned}$$

Estimating the last integral as before and putting the two computation together we obtain

$$\frac{d}{dt} \int_M (|\nabla^p w|^2 + w^2) d\mu \leq -\frac{\alpha}{2} \int_M |\nabla^{2p} w|^2 d\mu + C \sum_{r=0}^{2p-1} \int_M |\nabla^r w|^2 d\mu.$$

In order to deal with the last term, we apply the following Gagliardo-Nirenberg interpolation inequalities (see [3, Proposition 2.11] and [1, Theorem 4.14]): for every $0 \leq r < 2p$ and $\varepsilon > 0$ there exists a constant C_ε such that

$$\|\nabla^r f\|_{L^2(M)}^2 \leq \varepsilon \|\nabla^{2p} f\|_{L^2(M)}^2 + C_\varepsilon \|f\|_{L^2(M)}^2$$

for every function $f \in W^{2p,2}(M)$.

Hence, for some $\varepsilon > 0$ small enough we get,

$$\begin{aligned} &\frac{d}{dt} \int_M (|\nabla^p w|^2 + w^2) d\mu \\ &\leq -\frac{\alpha}{4} \int_M |\nabla^{2p} w|^2 d\mu + C \sum_{r=0}^{2p-1} \varepsilon \int_M |\nabla^{2p} w|^2 d\mu + C \sum_{r=0}^{2p-1} C_\varepsilon \int_M w^2 d\mu \\ &\leq C \int_M w^2 d\mu. \end{aligned}$$

From this ordinary differential inequality and Gronwall’s lemma, it follows that if the quantity $\int_M (|\nabla^p w|^2 + w^2) d\mu$ is zero at some time t_0 , then it must be zero for every time $t \in [t_0, T]$. Since at $t = 0$ we have $w(\cdot, 0) = u_0 - v_0 = 0$, we are done.

We now prove the continuous dependence of a solution $u \in C^\infty(M \times [0, T])$ on its initial datum $u_0 = u(\cdot, 0) \in C^\infty(M)$. Fix any $m \in \mathbb{N}$ satisfying condition (2.4), so that by the Sobolev embeddings $u \in P^m(M, T)$ implies $\nabla^{2p-1}u \in C^0(M \times [0, T])$. By the above argument, $u = (\mathcal{F}|_U)^{-1}(u_0, 0) \in P^m(M, T)$ where $\mathcal{F}|_U$ is a diffeomorphism of an open set $U \subset P^m(M, T)$ onto $V \subset W^{p(2m-1), 2}(M) \times P^{m-1}(M, T)$, with $(u_0, 0) \in V$. Then, assuming that $u_{k,0} \rightarrow u_0$ in $C^\infty(M)$ as $k \rightarrow \infty$, we also have $u_{k,0} \rightarrow u_0$ in $W^{p(2m-1), 2}(M)$, hence for k large enough $(u_{k,0}, 0) \in V$ and there exists $u_k \in U$ such that $\mathcal{F}(u_k) = (u_{k,0}, 0)$. This is the unique solution in $P^m(M, T)$ (hence in $C^\infty(M \times [0, T])$) by parabolic bootstrap with initial datum $u_{k,0}$. Moreover, since $\mathcal{F}|_U$ is a diffeomorphism, we have $u_k \rightarrow u$ in $P^m(M, T)$.

By uniqueness, we can repeat the same procedure for any $m \in \mathbb{N}$ satisfying condition (2.4) concluding that $u_k \rightarrow u$ in $P^m(M, T)$ for every such $m \in \mathbb{N}$, hence in $C^\infty(M \times [0, T])$.

3. Proof of Lemma 2.5

We shall write $P^m = P^m(M, T)$, $L^q = L^q(M \times [0, T])$, $C^0 = C^0(M \times [0, T])$ etc..., so that for instance $C^0(P^m; C^1)$ will denote the space of continuous maps from $P^m(M, T)$ to $C^1(M \times [0, T])$. The first component of \mathcal{F} , i.e. the map $u \mapsto u(\cdot, 0)$ is linear and bounded from P^m to $W^{p(2m-1), 2}(M)$, by Proposition 2.3, therefore it is C^1 . Obviously the map $u \mapsto \partial_t u$ is linear and bounded from P^m to P^{m-1} , hence also C^1 . Thus, it remains to show that the two maps

$$\mathcal{F}_A(u) := A(u) \cdot \nabla^{2p}u, \quad \mathcal{F}_b(u) := b(u)$$

belong to $C^1(P^m; P^{m-1})$.

We first prove that $\mathcal{F}_A, \mathcal{F}_b \in C^0(P^m; P^{m-1})$. By an induction argument, it is easy to see that for every $k \in \mathbb{N}$

$$\nabla^k(A(u) \cdot \nabla^{2p}u) = \sum_{j=0}^k \sum_{\substack{i_1, i_2, \dots, i_{j+1} \geq 1 \\ i_1 + \dots + i_{j+1} \leq k + 2p + (2p-1)j}} \partial^j A(u) \# \nabla^{i_1}u \# \dots \# \nabla^{i_{j+1}}u, \quad (3.1)$$

where $\partial^j A(u)$ denotes the j -th derivative of A with respect to any of its arguments and $D \# E$ denotes an arbitrary contraction with the metric of two tensors D and E .

Taking into account formula (3.1) with $k \leq 2p(m - 1)$, in order to prove that the map $u \mapsto \nabla^{2p(m-1)}(A(u) \cdot \nabla^{2p}u)$ belongs to $C^0(P^m; L^2)$ we have to show that any map of the form

$$u \mapsto \partial^j A(u) \# \nabla^{i_1}u \# \dots \# \nabla^{i_{j+1}}u \quad (3.2)$$

belongs to $C^0(P^m; L^2)$ whenever

$$i_1 + \dots + i_{j+1} \leq 2pm + (2p - 1)j \quad \text{and} \quad i_1, \dots, i_{j+1} \geq 1. \tag{3.3}$$

The case $r = 0$ and $\ell = 2p - 1$ of the Sobolev embeddings (4.3) below and condition (2.4) imply that if $u \in P^m$ then $\nabla^{2p-1}u \in C^0$ (and the immersion is bounded), hence all the maps $u \mapsto \partial^j A(u)$ belong to $C^0(P^m; C^0)$.

We can assume from now on that $j \geq 1$, since in the case $j = 0$, we get the term $A(u)\#\nabla^{2p+k}u$ which is continuous from P^m to L^2 as a function of u for $k \leq 2p(m - 1)$.

As for the factors $\nabla^{i_\ell}u$ appearing in formula (3.2), first we assume that each i_ℓ is such that we are in case (4.1) of Sobolev embeddings, *i.e.*

$$\frac{1}{q_\ell} := \frac{1}{2} - \frac{2pm - i_\ell}{n + 2p} > 0, \tag{3.4}$$

so that the map $u \mapsto \nabla^{i_\ell}u$ lies in $C^0(P^m; L^{q_\ell})$. By Hölder’s inequality, the condition

$$\frac{1}{q} := \sum_{\ell=1}^{j+1} \frac{1}{q_\ell} = \sum_{\ell=1}^{j+1} \left(\frac{1}{2} - \frac{2pm - i_\ell}{n + 2p} \right) \leq \frac{1}{2}, \tag{3.5}$$

implies that the map $u \mapsto \nabla^{i_1}u\#\dots\#\nabla^{i_{j+1}}u$ belongs to $C^0(P^m; L^q)$, hence also to $C^0(P^m; L^2)$, as L^q embeds continuously into L^2 for $q \geq 2$. Then, if we show inequality (3.5), the map defined by formula (3.2) belongs to $C^0(P^m; L^2)$. From inequalities (2.4), (3.3) and $j \geq 1$ it follows,

$$\begin{aligned} \sum_{\ell=1}^{j+1} \frac{1}{q_\ell} &\leq \frac{j + 1}{2} - \frac{2pm(j + 1) - 2pm - (2p - 1)j}{n + 2p} \\ &= \frac{1}{2} + \frac{j}{2} - \frac{(2pm - 2p + 1)j}{n + 2p} < \frac{1}{2}. \end{aligned} \tag{3.6}$$

Now, if for some i_ℓ , say i_1, \dots, i_s , we have $\frac{2pm - i_\ell}{n + 2p} > \frac{1}{2}$, then we are in case (4.3) of Sobolev embeddings and the corresponding maps $u \mapsto \nabla^{i_\ell}u$ belong to $C^0(P^m; C^0)$, hence we can avoid to estimate such factors, as for $A(u)$. Then, since (3.4) holds for $\ell \in \{s + 1, \dots, j + 1\}$, arguing again by induction, in this case we have to deal with functions $u \mapsto \nabla^{i_{s+1}}u\#\dots\#\nabla^{i_{j+1}}u$ under the conditions

$$i_{s+1} + \dots + i_{j+1} \leq 2pm + (2p - 1)(j - s) \quad \text{and} \quad i_{s+1}, \dots, i_{j+1} \geq 1.$$

Then, computing as in inequality (3.6) one shows

$$\begin{aligned} \sum_{\ell=s+1}^{j+1} \frac{1}{q_\ell} &\leq \frac{j + 1 - s}{2} - \frac{2pm(j + 1 - s) - 2pm - (2p - 1)(j - s)}{n + 2p} \\ &= \frac{1}{2} + \frac{j - s}{2} - \frac{(2pm - 2p + 1)(j - s)}{n + 2p} \\ &\leq \frac{1}{2}, \end{aligned} \tag{3.7}$$

where we intend that if $s = j + 1$ there is nothing to sum. Notice that the last inequality is strict if $s \neq j$, and in the case $s = j$ the map $u \mapsto \nabla^{i_{j+1}}u$ is continuous from P^m to L^2 as $i_{j+1} \leq 2pm$.

If in addition for some i_ℓ , say i_{s+1}, \dots, i_r , we have $\frac{2pm-i_\ell}{n+2p} = \frac{1}{2}$ (i.e. we are in the critical case (4.2) of the Sobolev embeddings), we know that for such indices the maps $u \mapsto \nabla^{i_\ell}u$ belong to $C^0(P^m; L^q)$ for every $1 \leq q < \infty$. Then inequality (3.7) still holds true if we choose q_{s+1}, \dots, q_r large enough, since, unless $s = r = j$, the last inequality in (3.7) is strict.

Hence, we conclude as before that the map $u \mapsto \nabla^{2p(m-1)}(A(u) \cdot \nabla^{2p}u)$ lies in $C^0(P^m; L^2)$.

The time or mixed space-time derivatives $\partial_t^r \nabla^k(A(u) \cdot \nabla^{2p}u)$ with $2pr + k \leq 2p(m - 1)$ can be treated in a similar way, observing that the functions $\partial_t^r \nabla^\ell u$ have the same integrability of $\nabla^{2pr+\ell}u$ from the point of view of the embeddings (4.1)-(4.3).

Starting from formula (3.1) and differentiating in time, again by an induction argument, one gets

$$\begin{aligned} & \partial_t^r \nabla^k(A(u) \cdot \nabla^{2p}u) \\ &= \sum_{j=0}^{r+k} \sum_{\substack{i_1, \dots, i_{j+1}, \ell_1, \dots, \ell_{j+1} \geq 0 \\ i_1 + \dots + i_{j+1} \leq k + 2p + (2p-1)j \\ \ell_1 + \dots + \ell_{j+1} \leq r}} \partial^j A(u) \# \partial_t^{\ell_1} \nabla^{i_1} u \# \dots \# \partial_t^{\ell_{j+1}} \nabla^{i_{j+1}} u. \end{aligned} \tag{3.8}$$

Then, with the same proof as before one shows that a map of the form

$$u \mapsto \partial^j A(u) \# \partial_t^{\ell_1} \nabla^{i_1} u \# \dots \# \partial_t^{\ell_{j+1}} \nabla^{i_{j+1}} u$$

belongs to $C^0(P^{m+1}; L^2)$ whenever $i_1, \dots, i_{j+1}, \ell_1, \dots, \ell_{j+1} \geq 0$ and

$$i_1 + \dots + i_{j+1} + 2p(\ell_1 + \dots + \ell_{j+1}) \leq 2pm + (2p - 1)j. \tag{3.9}$$

Hence the map $u \mapsto \partial_t^r \nabla^k(A(u) \cdot \nabla^{2p}u)$ belongs to $C^0(P^m; L^2)$ for $2pr + k \leq 2p(m - 1)$, which means that $\mathcal{F}_A \in C^0(P^m; P^{m-1})$ as wished.

The map \mathcal{F}_b can be treated in a similar way, so also $\mathcal{F}_b \in C^0(P^m; P^{m-1})$. It remains to prove that $d\mathcal{F}_A, d\mathcal{F}_b \in C^0(P^m; L(P^m; P^{m-1}))$, where $L(P^m; P^{m-1})$ denotes the Banach space of bounded linear maps from P^m into P^{m-1} . We first claim that the Gateaux derivative

$$(u, v) \mapsto d\mathcal{F}_A(u)(v) := \left. \frac{d}{dt} \mathcal{F}_A(u + tv) \right|_{t=0} \tag{3.10}$$

belongs to $C^0(P^m \times P^m; P^{m-1})$. Indeed, $d\mathcal{F}_A(u)(v)$ can be written as

$$B(u, v) \# \nabla^{2p}u + A(u) \cdot \nabla^{2p}v,$$

where B is a tensor depending smoothly on $x, t, u, \dots, \nabla^{2p-1}u$ and linearly on some derivative of v up to the order $2p - 1$, that is, $B(u, v) = \sum_{\ell=0}^{2p-1} B_\ell(u) \cdot \nabla^\ell v$, compare with formula (2.5). The estimates proven for \mathcal{F}_A can be applied to any term of the form $\partial_t^r \nabla^k (B(u, v) \# \nabla^{2p} u)$, since they can be expressed as a sum similar to the right-hand side of identity (3.8). The only difference is that now in every term of such sum one linear occurrence of u is replaced by v . Precisely, writing $u_1 := u, u_2 := v$ every term $\partial^j A(u) \# \partial_t^{i_1} \nabla^{i_1} u \# \dots \# \partial_t^{i_{j+1}} \nabla^{i_{j+1}} u$ has to be replaced by some

$$D(u) \# \partial_t^{i_1} \nabla^{i_1} u_{\tau_1} \# \dots \# \partial_t^{i_{j+1}} \nabla^{i_{j+1}} u_{\tau_{j+1}} \tag{3.11}$$

where exactly one of the indices $\tau_1, \dots, \tau_{j+1}$ is equal to 2, and the others are equal to 1.

An analogous reasoning applies to the term $A(u) \cdot \nabla^{2p} v$. It is then easy to see, since $v \in P^m$ like u , that we can repeat the same estimates used to show the continuity of $u \mapsto \mathcal{F}_A(u)$. This proves in particular that $d\mathcal{F}_A(u) \in L(P^m; P^{m-1})$.

In order now to prove that $d\mathcal{F}_A \in C^0(P^m; L(P^m; P^{m-1}))$ we need to show that

$$\sup_{\|v\|_{P^m} \leq 1} \|d\mathcal{F}_A(\tilde{u})(v) - d\mathcal{F}_A(u)(v)\|_{P^{m-1}} \rightarrow 0 \quad \text{as } \tilde{u} \rightarrow u \text{ in } P^m .$$

Again, this estimate is similar to what we have already done. Indeed, supposing that τ_{j+1} is the only index equal to 2 in (3.11) and assuming that there are no time derivatives for the sake of simplicity, we want to see that, as $\tilde{u} \rightarrow u$ in P^m ,

$$\sup_{\|v\|_{P^m} \leq 1} \|D(\tilde{u}) \# \nabla^{i_1} \tilde{u} \# \dots \# \nabla^{i_j} \tilde{u} \nabla^{i_{j+1}} v - D(u) \# \nabla^{i_1} u \# \dots \# \nabla^{i_j} u \nabla^{i_{j+1}} v\|_{L^2} \rightarrow 0, \tag{3.12}$$

where $i_1 + \dots + i_{j+1} \leq 2pm + (2p - 1)j$ (see formula (3.1) and condition (3.3)).

Adding and subtracting terms, one gets

$$\begin{aligned} & \left| D(\tilde{u}) \# \nabla^{i_1} \tilde{u} \# \dots \# \nabla^{i_j} \tilde{u} \nabla^{i_{j+1}} v - D(u) \# \nabla^{i_1} u \# \dots \# \nabla^{i_j} u \nabla^{i_{j+1}} v \right| \\ & \leq \left\{ |D(\tilde{u}) - D(u)| |\nabla^{i_1} \tilde{u}| \dots |\nabla^{i_j} \tilde{u}| \right. \\ & \quad + |D(u)| |\nabla^{i_1} (\tilde{u} - u)| |\nabla^{i_2} \tilde{u}| \dots |\nabla^{i_j} \tilde{u}| \\ & \quad \left. + \dots + |D(u)| |\nabla^{i_1} u| \dots |\nabla^{i_j} (\tilde{u} - u)| \right\} |\nabla^{i_{j+1}} v|. \end{aligned}$$

Studying now the L^2 norm of this sum, the first term can be bounded as before and it goes to zero as $D(u)$ is continuous from P^m to L^∞ . The L^2 norm of all the other terms, repeating step by step the previous estimates, using Hölder’s inequality and embeddings (4.1)-(4.3), will be estimated by some product

$$C \|u\|_{P^m}^\alpha \|\tilde{u}\|_{P^m}^\beta \|v\|_{P^m}^\gamma \|\tilde{u} - u\|_{P^m}^\sigma \leq C \|u\|_{P^m}^\alpha \|\tilde{u}\|_{P^m}^\beta \|\tilde{u} - u\|_{P^m}^\sigma$$

for a constant C and some nonnegative exponents $\alpha, \beta, \gamma, \sigma$ satisfying $\alpha + \beta + \gamma + \sigma \leq 1$ and $\sigma > 0$. Here we used the fact that $\|v\|_{P^m} \leq 1$.

As $\tilde{u} - u \rightarrow 0$ in P^m , this last product goes to zero in L^2 , hence uniformly for $\|v\|_{P^m} \leq 1$ and inequality (3.12) follows, as claimed. The analysis of the estimates with mixed time/space derivatives is similar and all this argument works analogously for the term $A(u) \cdot \nabla^{2p} v$.

Then, the Gateaux derivative $d\mathcal{F}_A$ is continuous which implies that it coincides with the Fréchet derivative, hence $\mathcal{F}_A \in C^1(P^m; P^{m-1})$.

The map \mathcal{F}_b can be dealt with in the same way and we are done.

4. Parabolic Sobolev embeddings

Proposition 4.1. *Let $u \in P^m(M, T)$. Then for $r, \ell \in \mathbb{N}$ with $2pr + \ell \leq 2mp$, we have*

$$\|\partial_t^r \nabla^\ell u\|_{L^q(M \times [0, T])} \leq C \|u\|_{P^m(M, T)} \text{ if } \frac{1}{q} = \frac{1}{2} - \frac{2pm - \ell - 2pr}{n + 2p} > 0; \tag{4.1}$$

$$\|\partial_t^r \nabla^\ell u\|_{L^q(M \times [0, T])} \leq C \|u\|_{P^m(M, T)} \text{ if } \frac{1}{2} - \frac{2pm - \ell - 2pr}{n + 2p} = 0 \text{ and } 1 \leq q < \infty; \tag{4.2}$$

the function $\partial_t^r \nabla^\ell u$ is continuous and

$$\|\partial_t^r \nabla^\ell u\|_{C^0(M \times [0, T])} \leq C \|u\|_{P^m(M, T)} \text{ if } \frac{1}{2} - \frac{2pm - \ell - 2pr}{n + 2p} < 0, \tag{4.3}$$

where the constant C does not depend on u .

Proof. Of course we can write

$$P^m(M, T) = L^2([0, T]; H^{2mp}(M)) \cap H^1([0, T]; H^{2p(m-1)}(M)) \cap \dots \cap H^m([0, T]; L^2(M)).$$

By standard interpolation theory, see e.g. [8, Theorem 2.3], we have the continuous immersion

$$P^m(M, T) \hookrightarrow H^s([0, T]; H^{2p(m-s)}(M)), \quad \text{for all } s \in [0, m].$$

We shall now assume that $\frac{1}{2} - \frac{2pm - \ell - 2pr}{n + 2p} > 0$ and prove inequality (4.1). For $0 \leq \sigma < \frac{1}{2}$ and for any Hilbert space X we have the Sobolev embedding

$$H^\sigma([0, T]; X) \hookrightarrow L^q([0, T]; X) \quad \text{for } \frac{1}{q} = \frac{1}{2} - \sigma.$$

Then, for $\ell, r \in \mathbb{N}$ with $2pr + \ell \leq 2pm$ and for any $s \in (m - \frac{\ell}{2p} - \frac{n}{4p}, m - \frac{\ell}{2p}] \cap [r, r + \frac{1}{2})$, also using the standard Sobolev embeddings on M , for every $u \in P^m(M, T)$ one gets

$$\begin{aligned} \partial_t^r \nabla^\ell u \in H^{s-r}([0, T]; H^{2p(m-s)-\ell}(M)) &\hookrightarrow L^q([0, T]; H^{2p(m-s)-\ell}(M)) \\ &\hookrightarrow L^q([0, T]; L^{\tilde{q}}(M)), \end{aligned}$$

with

$$\frac{1}{q} = \frac{1}{2} - s + r \quad \text{and} \quad \frac{1}{\tilde{q}} = \frac{1}{2} - \frac{2p(m-s) - \ell}{n}.$$

We now choose $s = \frac{rn+2pm-\ell}{n+2p}$ and claim that $s \in (m - \frac{\ell}{2p} - \frac{n}{4p}, m - \frac{\ell}{2p}] \cap [r, r + \frac{1}{2})$. Then

$$\frac{1}{q} = \frac{1}{\tilde{q}} = \frac{1}{2} - \frac{2pm - \ell - 2pr}{n + 2p},$$

hence for such $q \in \mathbb{R}$ we have

$$u \in L^q([0, T]; L^q(M)) \simeq L^q(M \times [0, T]),$$

and embedding (4.1) is proven. As for the claim, the inequalities $s \geq r$ and $s \leq m - \frac{\ell}{2p}$ easily follow from the inequality $2pr + \ell \leq 2pm$, while inequality $s < r + \frac{1}{2}$ is equivalent to $\frac{1}{2} - \frac{2pm-\ell-2pr}{n+2p} > 0$. This means $\frac{1}{q} > 0$ which implies $s > m - \frac{\ell}{2p} - \frac{n}{4p}$.

The proof of inequality (4.2) is analogous.

Finally, if $\frac{1}{2} - \frac{2pm-\ell-2pr}{n+2p} < 0$, using that for $\sigma > \frac{1}{2}$ one has $H^\sigma([0, T]; X) \hookrightarrow C^0([0, T]; X)$ and that for $\sigma > \frac{n}{2}$ one has $H^\sigma(M) \hookrightarrow C^0(M)$, for every $u \in P^m(M, T)$ we infer

$$\partial_t^r \nabla^\ell u \in H^{s-r}([0, T]; H^{2p(m-s)-\ell}(M)) \hookrightarrow C^0([0, T]; C^0(M)) \simeq C^0(M \times [0, T]),$$

for $s = \frac{rn+2pm-\ell}{n+2p} \in (r + \frac{1}{2}, m - \frac{\ell}{2p} - \frac{n}{4p})$. This proves embedding (4.3). □

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