On square roots of class $C^m$ of nonnegative functions of one variable

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Abstract. We investigate the regularity of functions $g$ such that $g^2 = f$, where $f$ is a given nonnegative function of one variable. Assuming that $f$ is of class $C^{2m}$ ($m > 1$) and vanishes together with its derivatives up to order $2m - 4$ at all its local minimum points, one can find a $g$ of class $C^m$. Under the same assumption on the minimum points, if $f$ is of class $C^{2m+2}$ then $g$ can be chosen such that it admits a derivative of order $m + 1$ everywhere. Counterexamples show that these results are sharp.

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Introduction

In this paper we study the regularity of functions $g$ of one variable whose square is a given nonnegative function $f$.

For a function $f$ of class $C^2$, first results are due to G. Glaeser [6] who proved that $f^{1/2}$ is of class $C^1$ if the second derivative of $f$ vanishes at the zeros of $f$, and to T. Mandai [8] who proved that one can always choose $g$ of class $C^1$. More recently in [1] (and later in [7]), for functions $f$ of class $C^4$, it was proved that one can find $g$ of class $C^1$ and twice differentiable at every point.

F. Broglia and the authors proved in [3] that this result is sharp in the sense that it is not possible to have in general a greater regularity for $g$. They also showed that if $f$ is of class $C^4$ and vanishes at all its (local) minimum points, one can always find $g$ of class $C^2$ and that the result is sharp. Later, in [4] it was proved that for $f$ of class $C^6$ vanishing at all its minimum points one can find $g$ of class $C^2$ and three times differentiable at every point.

In this paper we generalize these results. First we prove that for $f$ of class $C^{2m}$, $m = 1, 2, \ldots, \infty$, vanishing at its (local) minimum points together with all its derivatives up to order $(2m - 4)$ one can find $g$ of class $C^m$ (Theorem 2.2). If the derivatives vanish only up to order $2m - 6$ at all the minimum points, the other assumptions being unchanged, $g$ can be chosen $m$ times differentiable at every point (Theorem 3.1, where $m$ is replaced by $m + 1$).
Counterexamples are given to show that these assumptions cannot be relaxed and that the regularity of \( g \) cannot be improved in general.

1. Precised square roots

In this paper, \( f \) will always be a nonnegative function of one real variable whose regularity will be precised below. Our results being of local character, we may and will assume that the support of \( f \) is contained in \([0, 1]\).

**Definition 1.1.** Assuming \( f \) of class \( C^{2m} \), \( m = 1, 2, \ldots, \infty \), we say that \( g \) is a square root of \( f \) precised up to order \( m \), if \( g \) is a continuous function satisfying \( g^2 = f \) and if, for any (finite) integer \( k \leq m \) and for any point \( x_0 \) which is a zero of \( f \) of order exactly \( 2k \), the function \( x \mapsto (x-x_0)^k g(x) \) keeps a constant sign near \( x_0 \).

It is clear that \( g \) cannot be \( m \) times differentiable at every point if this condition is not fulfilled.

It is easy to show the existence of square roots precised up to order \( m \) and even to describe all of them. Let us consider the closed set

\[
G = \{ x \in \mathbb{R} \mid f(x) = 0, f'(x) = 0, \ldots, f^{(2m)}(x) = 0 \},
\]

with the convention that all derivatives vanish if \( m = \infty \). Its complement is a union of disjoint intervals \( J_\nu \). In \( J_\nu \), the zeros of \( f \) are isolated and of finite order \( \leq 2m \). For a square root precised up to order \( m \), one should have \( |g| = f^{1/2} \) and the restriction of \( g \) to \( J_\nu \) should be one of two well defined functions \( +g_\nu \) and \(-g_\nu \) thanks to the condition on the change of sign. There is a bijection between the set of families \((\epsilon_\nu)\) with \( \epsilon_\nu = \pm 1 \) and the set of square roots precised up to order \( m \): one has just to set \( g(x) = \epsilon_\nu g_\nu(x) \) for \( x \in J_\nu \) and \( g(x) = 0 \) for \( x \in G \).

A modulus of continuity is a continuous, positive, increasing and concave function defined on an interval \([0, t_0]\) and vanishing at \( 0 \). Any continuous function \( \varphi \) defined on a compact set \( K \) has a modulus of continuity, i.e. a function \( \omega \) as above such that for every \( t_1, t_2 \) with \(|t_2 - t_1| < t_0\), one has \(|\varphi(t_2) - \varphi(t_1)| < \omega(|t_2 - t_1|)\). One says that \( \varphi \in C^k(K) \) if \( \varphi \in C^k(K) \) and if \( \omega \) is a modulus of continuity of \( \varphi^{(k)} \), one says that \( \varphi \in C^{k,\omega}(K) \).

We now state two lemmas taken almost literally from [2, Lemme 4.1, Lemme 4.2 and Corollaire 4.3]. Note that in the rest of this section \( m \) will not be allowed to take the value \( \infty \).

**Lemma 1.2.** Let \( \varphi \in C^{2m}(J) \) be nonnegative, where \( J \) is a closed interval contained in \([-1, 1]\), and let \( M = \sup |\varphi^{(k)}(x)| \) for \( 0 \leq k \leq 2m \) and \( x \in J \). Assume that for some \( j \in \{0, \ldots, m\} \), the inequality \( \varphi^{(2j)}(x) \geq \gamma > 0 \) holds for \( x \in J \) and that \( \varphi \) has a zero of order \( 2j \) at some point \( \xi \in J \).

Let us define \( H \) and \( \psi \) in \( J \) by

\[
\varphi(x) = (x-\xi)^{2j} H(x), \quad \psi(x) = (x-\xi)^{j} H(x)^{1/2}.
\]
Then, $H \in C^{2m-2j}(J)$ and $\psi \in C^{2m-j}(J)$. Moreover, there exists $C_1$, depending only on $m$, such that

$$
\left| \psi^{(k)}(x) \right| \leq C_1 \gamma^{1/2-k} M^k, \quad k = 1, \ldots, 2m - j.
$$

(1.2)

Lemma 1.3. Let $\varphi$ be a nonnegative function of one variable, defined and of class $C^{2m}$ in the interval $[-1, 1]$ such that $|\varphi^{(2m)}(t)| \leq 1$ for $|t| \leq 1$ and that $\max_{0 \leq j \leq m-1} \varphi^{(2j)}(0) = 1$.

(i) There exists a universal positive constant $C_0$, such that

$$
\left| \varphi^{(k)}(t) \right| \leq C_0, \quad \text{for } |t| \leq 1 \text{ and } 0 \leq k \leq 2m.
$$

(1.3)

(ii) There exist universal positive constants $a_j$ and $r_j$, $j = 0, \ldots, m-1$, such that one of the following cases occurs:

(a) One has $\varphi(0) \geq a_0$ and then $\varphi(t) \geq a_0/2$ for $|t| \leq r_0$.

(b) For some $j \in \{1, \ldots, m-1\}$ one has $\varphi^{2j}(t) \geq a_j$ for $|t| \leq r_j$ and $\varphi$ has a local minimum in $[-r_j, r_j]$.

In the following proposition, $G$ is defined by (1.1) and $d(x, G)$ denotes the distance of $x$ from $G$. When $G = \emptyset$, (a) and (b) are always true and condition (1.4) disappears.

Proposition 1.4. Assuming that $f$ is of class $C^{2m}$, the three following properties are equivalent.

(a) There exists $g \in C^m$ such that $g^2 = f$.

(b) Any function $g$ which is a square root of $f$ precised up to order $m$ belongs to $C^m$.

(c) There exists a modulus of continuity $\omega$ such that

$$
\left| \frac{d^k}{dx^k} f^{1/2}(x) \right| \leq d(x, G)^{m-k} \omega(d(x, G)),
$$

(1.4)

for any $x$ such that $f(x) \neq 0$ and any $k \in \{0, \ldots, m\}$.

Proof. It is clear that (b)$\Rightarrow$(a): as said above, precised square roots do exist. Under assumption (a), $g$ and its derivatives up to order $m$ should vanish on $G$. If $\omega$ is a modulus of continuity of $g^{(m)}$ one gets $\left| g^{(m)}(x) \right| \leq \omega(d(x, G))$. Successive integrations prove that the derivatives $g^{(k)}$ are bounded by the right hand side of (1.4). These derivatives being equal, up to the sign, to those of $f^{1/2}$ when $f$ does not vanish, (a) $\Rightarrow$ (c) is proved.

Let us assume (c) and consider any connected component $J_\nu$ of the complement of $G$. Near each zero of $f$ in $J_\nu$, which is of order exactly $2j$ for some $j \in \{1, \ldots, m\}$, the precised square root $g_\nu$ is given (up to the sign) by Lemma 1.2
and so it is of class $C^m$. Moreover, the estimate (1.4) extends by continuity to the points $x \in J_\nu$ where $f$ vanishes and one has

$$\left| g_v^{(k)}(x) \right| \leq d(x, G)^{m-k} \omega(d(x, G))$$

for $x \in J_\nu$ and $k \in \{0, \ldots, m\}$.

If we define $g$ equal to $\epsilon \nu g\nu$ in $J_\nu$ and to 0 in $G$, it remains to prove the existence and the continuity of the derivatives of $g$ at any point $x_0 \in G$. By induction, the estimates above prove, for $k = 0, \ldots, m - 1$, that $g^{k+1}(x_0)$ exists and is equal to 0 and that $g^{k+1}(x) \to 0$ for $x \to x_0$. The proof is complete. \qed

**Corollary 1.5.** Let $f$ be a nonnegative $C^\infty$ function of one variable such that for any $m$ there exists a function $g_m$ of class $C^m$ with $g_2^2 = f$. Then there exists $g$ of class $C^\infty$ such that $g^2 = f$.

Actually, if $g$ is any square root of $f$ precise up to order $\infty$, it is precise up to order $m$ for any $m$ and thus of class $C^m$ for any $m$ by the proposition above.

### 2. Continuously differentiable square roots

We start with an auxiliary result which contains the main argument. The function $f \in C^{2m}$, $m \geq 2$, and the set $G \neq \emptyset$ are as above, and $\Gamma$ is a closed subset of $G$. We will use this lemma for $p = 0$, in which case $\Gamma$ can be disregarded, and for $p = 1$.

**Lemma 2.1.** Assume that $m \neq \infty$ and $f$ and all its derivatives up to order $2m - 4$ (included) vanish at all its local minimum points. Assume moreover that there exist a modulus of continuity $\alpha$ and constants $C > 0$ and $p \geq 0$ such that

$$\left| f^{(2m)}(x) \right| \leq C d(x, \Gamma)^{2p} \alpha(d(x, G)).$$

Then, there exists a constant $\tilde{C}$ such that

$$\left| \frac{d^k}{dx^k} f^{1/2}(x) \right| \leq \tilde{C} d(x, \Gamma)^p d(x, G)^{m-k} \alpha(d(x, G))^{1/2}$$

for any $x$ such that $f(x) \neq 0$ and any $k \in \{0, \ldots, m\}$.

**Proof.** Let $J$ be any connected component of the complement of $G$ and for $x \in J$, let $\tilde{x}$ be (one of) the nearest endpoint(s) of $J$. The distance between $x$ and $\tilde{x}$ is thus equal to $d(x, G)$ and we remark that, for $y$ between $x$ and $\tilde{x}$, we have $d(y, \Gamma) \leq 2d(x, \Gamma)$. Integrating $2m - k$ times the estimate for $f^{(2m)}$ between $\tilde{x}$ and $x$ we get

$$| f^{(k)}(x) | \leq C' d(x, \Gamma)^{2p} d(x, G)^{2m-k} \alpha(d(x, G))$$

for $k = 0, \ldots, 2m$, the constant $C'$ being independent of $J$. 
Next, for $x$ in $J$ such that $f(x) \neq 0$, we define as in $[2]$, 
\[ \rho(x) = \max_{0 \leq k \leq m-1} \left\{ \left[ \frac{f^{(2k)}(x)}{C'd(x, \Gamma)^2 p \alpha(d(x, G))} \right]^{\frac{1}{2m-2k}} \right\}. \]

One has thus $\rho(x) \leq d(x, G)$ and 
\[ |f^{(k)}(x)| \leq C'd(x, \Gamma)^2 p \alpha(d(x, G)) \rho(x)^{2m-k} \]
for $k = 0, \ldots, 2m$. The auxiliary function
\[ \varphi(t) = \frac{f(x + t\rho(x))}{C'd(x, \Gamma)^2 p \alpha(d(x, G)) \rho(x)^{2m}} \]
is defined in $[-1, 1]$ and satisfies the assumptions of Lemma 1.3. Two cases should be considered.

1. — One has $\varphi(0) \geq a_0$ and then $\varphi(t) \geq a_0/2$ for $|t| \leq r_0$ while the derivatives of $\varphi$ are uniformly bounded by $C_0$. Thus, there exists an universal constant $C''$ such that $\left| \frac{d^k}{dx^k} \varphi^{1/2}(t) \right| \leq C''$ in this interval. We have thus, by the change of variable $t \mapsto x + t\rho(x)$,
\[ \left| \frac{d^k}{dx^k} f^{1/2}(x) \right| \leq C'' d(x, \Gamma)^p \rho(x)^{m-k} \alpha(d(x, G))^{1/2} \]
which implies (2.2).

2. — We are in case (b) of Lemma 1.3: all the derivatives of $\varphi$ are bounded by $C_0$ and for some $j \in \{1, \ldots, m-1\}$ one has $\varphi^{2j}(t) \geq a_j$ for $|t| \leq r_j$ and $\varphi$ has a local minimum at some point $\xi \in [-r_j, r_j]$. Our assumptions imply that $\varphi^{2k}(\xi)$ vanishes for $k \in \{0, \ldots, m-2\}$ so $j$ is necessarily equal to $m - 1$. We can thus set $\varphi(t) = (t - \xi)^{2m-2} H(t)$ and $\psi(t) = (t - \xi)^{m-1} H(t)^{1/2}$ as in Lemma 1.2. There is a universal constant $C'''$ (computed from $C_0$ and $a_{m-1}$) such that $\left| \frac{d^k}{dx^k} \psi(t) \right| \leq C'''$ for $|t| \leq r_{m-1}$. In particular, for $t = 0$, these derivatives coincide up to the sign with those of $\varphi^{1/2}$. The change of variable $t \mapsto x + t\rho(x)$ gives again the estimates (2.2) on the derivatives of $f^{1/2}(x)$. The proof is complete. \hfill $\square$

**Theorem 2.2.** Let $f$ be a nonnegative function of one variable of class $C^{2m}$ with $m \geq 2$ such that, at all its minimum points, $f$ and its derivatives up to the order $(2m - 4)$ vanish. Then any square root of $f$ precised up to order $m$ is of class $C^m$.

**Proof.** The result is evident if $G$ is empty and we can thus assume $G \neq \emptyset$. If $\alpha$ is a modulus of continuity of $f^{(2m)}$, we have $\left| f^{2m}(x) \right| \leq \alpha(d(x, G))$ which is the assumption (2.1) for $p = 0$. By the preceding lemma, we have the estimates
\[ \left| \frac{d^k}{dx^k} f^{1/2}(x) \right| \leq C d(x, G)^{m-k} \alpha(d(x, G))^{1/2} \]
when $f(x) \neq 0$. By Proposition 1.4, this implies that all the square roots precised up to order $m$ are of class $C^m$. The case $m = \infty$ follows now from Corollary 1.5. \hfill $\square$
Remark 2.3. It is certainly not necessary to assume that $f$ vanishes at all its minimum points. For instance, we could also allow nonzero minima at points $\bar{x}_i, i \in \mathbb{N}$, provided that the values $f(\bar{x}_i)$ be not “too small”. With the notations of Lemma 2.1, it suffices to have $f(\bar{x}_i) \geq C\alpha(d(\bar{x}_i, G))\rho(\bar{x}_i)^{2m}$ for some uniform positive constant $C$.

It is clear that the assumption $f \in C^{2m}$ of Theorem 2.2 cannot be weakened to $f \in C^{2m-1.1}$ (take $f(t) = t^{2m} + 1/2 t^{2m-1}|t|$). The two following counterexamples show that in the general case no stronger regularity is possible (Theorem 2.4) and that the vanishing of $2m - 4$ derivatives cannot be replaced by the vanishing of $2m - 6$ derivatives (Theorem 2.5).

Theorem 2.4. For any given modulus of continuity $\omega$ there is a nonnegative function $f$ of class $C^\infty$ on $\mathbb{R}$ such that, at all its minimum points, $f$ and all its derivatives up to the $(2m - 4)$-th one vanish, but there is no function $g$ of class $C^m,\omega$ such that $g^2 = f$.

Proof. Let $\chi \in C^\infty(\mathbb{R})$ be the even function with support in $[-2, 2]$ defined by $\chi(t) = 1$ for $t \in [0, 1]$ and by $\chi(t) = \exp\{1/(t-2)e^{t/(t-1)}\}$ for $t \in (1, 2)$. We note that the logarithm of $\chi$ is a concave function on $(1, 2)$. For every $(a, b) \in [0, 1] \times [0, 1]$, $(a, b) \neq (0, 0)$, and every $m \geq 1$ the function $t \mapsto \log(at^{2m} + bt^{2m-2})$ is concave on $(0, +\infty)$ and thus the function

$$t \mapsto \chi^2(t)(at^{2m} + bt^{2m-2})$$

has only one local maximum point and no local minimum points in $(1, 2)$, for its logarithmic derivative vanishes exactly once. Set

$$\rho_n = \frac{1}{n^2}, \quad t_n = 2\rho_n + \sum_{j=n+1}^\infty 5\rho_j,$$

$$I_n = [t_n - 2\rho_n, t_n + 2\rho_n], \quad \alpha_n = \frac{1}{2n}$$

and

$$\epsilon_n = \omega^{-1}(\alpha_n), \quad \beta_n = \alpha_n\epsilon_n^2.$$

Notice that the $I_n$’s are closed and disjoint and that, for $n \geq 4$, one has

$$\epsilon_n \leq \alpha_n \leq \rho_n.$$

Define

$$f = \sum_{n=4}^\infty \chi^2(\frac{t-I_n}{\rho_n})\alpha_n(t-t_n)^{2m} + \beta_n(t-t_n)^{2m-2}).$$

Clearly, $f$ is of class $C^\infty$; this is obvious at every point except perhaps at the origin, but for small $t \in I_n$ and a suitable positive constant $C_k$ one has that

$$|f^{(k)}(t)| \leq C_k\rho_n^{2m-2-k}\alpha_n.$$
that converges to 0 as \( t \) goes to 0 (which implies that \( n \) goes to infinity). Moreover, \( f \) takes the value 0 at all its local minimum points, which are the points \( t_n \) and the points between \( I_n \) and \( I_{n+1} \).

We argue by contradiction and look for functions \( g \) of class \( C^{m, \omega} \) such that \( g^2 = f \); but any such \( g \) must be of the form

\[
g = \sum_{n=1}^{\infty} \sigma_n \chi \left( \frac{t - t_n}{\rho_n} \right) (t - t_n)^{m-1} \sqrt{\beta_n + \alpha_n(t - t_n)^2} \quad (2.5)
\]

for some choice of the signs \( \sigma_n = \pm 1 \). In order to evaluate \( g^{(m)} \), let us calculate first \( (\sqrt{\beta_n + \alpha_n(t - t_n)^2})^{(h)} \) for \( h = 1, \ldots, m \). To this end, we will use Faà di Bruno’s formula (see [5]), with \( F(x) = x^{1/2} \) and \( \psi(t) \) given by \( \psi(t) = \beta + \alpha t^2 \):

\[
(F \circ \psi)^{(h)} = \sum_{j=1}^{h} (F^{(j)} \circ \psi) \sum_{p(h, j)} h! \prod_{i=1}^{h} \frac{\psi^{(j)}(\mu_i)}{(\mu_i!)^{(j)}},
\]

where:

\[
p(h, j) = \left\{ (\mu_1, \ldots, \mu_h) : \mu_i \geq 0, \sum_{i=1}^{h} \mu_i = j, \sum_{i=1}^{h} \mu_i = h \right\}.
\]

Now obviously we have:

\[
F^{(j)}(x) = (x^{1/2})^{(j)} = 2^{-j} (2j - 3)! (-1)^{j+1} x^{1/2-j},
\]

where, for \( n \) odd, \( n!! = 1 \cdot 3 \cdots n \) and, for \( n \) even, \( n!! = 2 \cdot 4 \cdots n \). Moreover, in our case, the only nonzero terms are those with \( i = 1 \) or \( i = 2 \) and \( \mu_1 = 2j - h, \mu_2 = h - j \), with \( \left\lfloor \frac{h+1}{2} \right\rfloor \leq j \leq h \). So we have:

\[
\left( \sqrt{\beta + \alpha t^2} \right)^{(h)} = \sum_{j=\left\lfloor \frac{h+1}{2} \right\rfloor}^{h} \frac{h! 2^{j-h} (2j - 3)! (-1)^{j+1} (\beta + \alpha t^2)^{1/2-j} \alpha^j t^{2j-h}}{(2j-h)! (h-j)!} \quad (2.6)
\]

We calculate now \( g^{(m)}(t) \) for \( t \in \tilde{I}_n := [t_n - \rho_n, t_n + \rho_n] \), with \( g \) given by (2.5).

We note that on \( \tilde{I}_n \) one has \( g(t) = \sigma_n (t - t_n)^{m-1} \sqrt{\beta_n + \alpha_n(t - t_n)^2} \), and so, for \( t \in \tilde{I}_n \):

\[
g^{(m)}(t) = \sigma_n \sum_{h=1}^{m} \frac{(m)!}{h!(m-h)!} (t-t_n)^{h-1} (m-1)! \left( \sqrt{\beta_n + \alpha_n(t - t_n)^2} \right)^{(h)} \quad (2.7)
\]
Now, set $t'_n = t_n + \lambda \varepsilon_n$, with $\lambda$ to be chosen later, $1/2 \leq \lambda \leq 1$, so that, thanks to (2.4), $t'_n \in \tilde{I}_n$. Taking (2.6) and (2.7) into account, we have:

$$g^{(m)}(t'_n) = \sigma_n \alpha_n^{1/2} \sum_{h=1}^{m} \frac{(m)!}{h!(m-h)!} \frac{(m-1)!}{(h-1)!} \times \sum_{j=[\frac{h+1}{2}]}^{h} \frac{h!2^{j-h}(2j-3)!!(1+\lambda^2\lambda^{j-1})^{1/2-j}}{(2j-h)(h-j)!} = \sigma_n \alpha_n^{1/2} \mathcal{K}_m(\lambda).$$

Since $\mathcal{K}_m(\lambda)$ is a nonzero polynomial of degree $2m-1$ in $\frac{\lambda}{(1+\lambda^2)^{1/2}}$, we can choose a value $\lambda_0$, $1/2 \leq \lambda_0 \leq 1$, in such a way that $\mathcal{K}_m(\lambda_0) \neq 0$. But now since $g^{(m)}(t_n) = 0$ we have that

$$\frac{|g^{(m)}(t_n + \lambda_0 \varepsilon_n) - g^{(m)}(t_n)|}{\omega(\lambda_0 \varepsilon_n)} = \frac{|g^{(m)}(t_n + \lambda_0 \varepsilon_n)|}{\omega(\lambda_0 \varepsilon_n)} = \frac{\alpha_n^{1/2} |\mathcal{K}_m(\lambda_0)|}{\omega(\lambda_0 \varepsilon_n)} \geq \frac{\alpha_n^{1/2} |\mathcal{K}_m(\lambda_0)|}{\omega(\epsilon_n)} \geq \frac{|\mathcal{K}_m(\lambda_0)|}{\alpha_n^{1/2}}$$

that goes to infinity as $n \to \infty$. \hfill \Box

**Theorem 2.5.** There is a nonnegative function $f$ of class $C^\infty$ on $\mathbb{R}$ such that, at all its minimum points, $f$ and all its derivatives up to the $(2m-6)$-th one vanish, but there is no function $g$ of class $C^m$ such that $g^2 = f$.

**Proof.** Let $\chi$ be a function of class $C^\infty$ as in Theorem 2.4 and define $\rho_n, t_n, I_n$ and $\alpha_n$ as in (2.3); define also

$$\varepsilon_n = \alpha_n, \quad \beta_n = \alpha_n \varepsilon_n^2$$

and

$$f = \sum_{n=4}^{\infty} \chi^2 \left( \frac{t - t_n}{\rho_n} \right) \left( \alpha_n(t - t_n)^{2m-2} + \beta_n(t - t_n)^{2m-4} \right).$$

The function $f$ is obviously of class $C^\infty$ and satisfies our hypotheses. Again, any function $g$ of class $C^{m-1}$ such that $g^2 = f$ is of the form

$$g = \sum_{n=1}^{\infty} \sigma_n \chi \left( \frac{t - t_n}{\rho_n} \right) (t - t_n)^{m-2} \sqrt{\beta_n + \alpha_n(t - t_n)^2}$$

for some choice of the signs $\sigma_n = \pm 1$. 

Now, set $t'_n = t_n + \lambda \varepsilon_n$, with $1/2 \leq \lambda \leq 1$: thanks to (2.4), $t'_n \in \tilde{I}_n$. Taking (2.6) and (2.7) into account we have again that

$$g^{(m)}(t'_n) = \sigma_n \frac{\alpha_n^{1/2}}{\varepsilon_n} \sum_{h=2}^{m} \frac{(m)!}{h!(m-h)!} \frac{(m-2)!}{(h-2)!} \times \sum_{j=\left[\frac{h+1}{2}\right]}^{h} \frac{h!2^{j-h}(2j-3)!(-1)^{j+1}\lambda^{2j-2}(1+\lambda^2)^{1/2-j}}{(2j-h)!(h-j)!} = \sigma_n \frac{1}{\alpha_n^{1/2}} \mathcal{H}_m(\lambda)$$

where $\mathcal{H}_m$ is a polynomial function in $\frac{\lambda}{(1+\lambda^2)^{1/2}}$; for some good choice of $\lambda$, then, this expression goes to infinity as above. \qed

3. Differentiable square roots

**Theorem 3.1.** Let $f$ be a nonnegative function of one variable of class $C^{2m+2}$ ($2 \leq m \leq \infty$) such that, at all its minimum points, $f$ and all its derivatives up to the order $(2m - 4)$ vanish. Then any square root $g$ of $f$ which is precised up to order $m+1$ is of class $C^m$ and its derivative of order $m+1$ exists everywhere.

**Proof.** Since $f$ is also a function of class $C^{2m}$ and $g$ is in particular precised up to order $m$ we already know that $g$ is of class $C^m$.

Let us consider the following closed set

$$\Gamma = \{x \in \mathbb{R} | f(x) = 0, f'(x) = 0, \ldots, f^{(2m+2)}(x) = 0\}. \quad (3.1)$$

If it is empty, the set $G$ is made of isolated points where $f^{(2m+2)}(x) \neq 0$ and, thanks to the condition on the signs, $g$ is of class $C^{m+1}$. So, we may assume $\Gamma \neq \emptyset$ and thus, for the same reason, $g$ is of class $C^{m+1}$ outside $\Gamma$. What remains to prove is that $g^{(m)}$ is differentiable at each point of $\Gamma$.

The function $\Phi$ defined by $\Phi(x) = d(x, \Gamma)^{-2} f^{(2m)}(x)$ outside $\Gamma$ and by $\Phi(x) = 0$ in $\Gamma$ is continuous and vanishes on $G$. If $\alpha$ is a modulus of continuity of $\Phi$, one has thus

$$\left|f^{(2m)}(x)\right| \leq d(x, \Gamma)^2 \alpha(d(x, G)), \quad (3.2)$$

which is the assumption (2.1) of Lemma 2.1 with $p = 1$. Thanks to this lemma, we get

$$\left|g^{(m)}(x)\right| = \left|\frac{d}{dx} f^{1/2}(x)\right| \leq \tilde{C} d(x, \Gamma) \alpha(d(x, G))^{1/2}$$

for $x$ such that $f(x) \neq 0$ and $k \in \{0, \ldots, m\}$. By continuity, the estimate of $g^{(m)}(x)$ is also valid for the isolated zeros of $f$, and it is trivial for $x \in \Gamma$. For $x_0 \in \Gamma$ one has thus $\left|g^{(m)}(x) - g^{(m)}(x_0)\right| / |x - x_0| \leq C \alpha(d(x, G))^{1/2}$ which converges to 0 for $x \to x_0$. This proves that $g^{m+1}(x_0)$ exists and is equal to 0, which ends the proof. \qed
Remark 3.2. We have already proved that, under the assumptions of the theorem, $g$ is not of class $C^{m+1}$ in general (Theorem 2.5 with $m$ replaced by $m + 1$). Counterexamples analogous to those given above show that the hypotheses cannot be relaxed.

References