

## On a stronger Lazer-McKenna conjecture for Ambrosetti-Prodi type problems

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**Abstract.** We consider an elliptic problem of Ambrosetti-Prodi type involving critical Sobolev exponent on a bounded smooth domain. We show that if the domain has some symmetry, the problem has infinitely many (distinct) solutions whose energy approach to infinity even for a fixed parameter, thereby obtaining a stronger result than the Lazer-McKenna conjecture.

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### 1. Introduction

Elliptic problems of Ambrosetti-Prodi type have the following form:

$$\begin{cases} -\Delta u = g(u) - \bar{s}\varphi_1(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where  $g(t)$  satisfies  $\lim_{t \rightarrow -\infty} \frac{g(t)}{t} = \nu < \lambda_1$ ,  $\lim_{t \rightarrow +\infty} \frac{g(t)}{t} = \mu > \lambda_1$ ,  $\lambda_1$  is the first eigenvalue of  $-\Delta$  with Dirichlet boundary condition and  $\varphi_1 > 0$  is the first eigenfunction. Here  $\mu = +\infty$  and  $\nu = -\infty$  are allowed. It is well-known that the location of  $\mu, \nu$  with respect to the spectrum of  $(-\Delta, H_0^1(\Omega))$  plays an important role in the multiplicity of solutions for problem (1.1). See for example [3, 8, 9, 18–20, 23–26, 31–34]. In the early 1980s, Lazer and McKenna conjectured that if  $\mu = +\infty$  and  $g(t)$  does not grow too fast at infinity, (1.1) has an unbounded number of solutions as  $\bar{s} \rightarrow +\infty$ . See [24].

In this paper, we will consider the following special case:

$$\begin{cases} -\Delta u = u_+^{2^*-1} + \lambda u - \bar{s}\varphi_1, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (1.2)$$

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where  $\Omega$  is a bounded domain in  $\mathbb{R}^N$  with  $C^2$  boundary,  $N \geq 3$ ,  $\lambda < \lambda_1$ ,  $\bar{s} > 0$ ,  $u_+ = \max(u, 0)$  and  $2^* = 2N/(N - 2)$ .

It is easy to see that (1.2) has a negative solution

$$\underline{u}_{\bar{s}} = -\frac{\bar{s}}{\lambda_1 - \lambda} \varphi_1,$$

if  $\lambda < \lambda_1$ . Moreover, if  $\underline{u}_{\bar{s}} + u$  is a solution of (1.2), then  $u$  satisfies

$$\begin{cases} -\Delta u = (u - s\varphi_1)_+^{2^*-1} + \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where  $s = \frac{\bar{s}}{\lambda_1 - \lambda} > 0$ .

Let us recall some recent results on the Lazer-McKenna conjecture related to (1.3). Firstly, Dancer and the second author proved in [12, 13] that for  $N \geq 2$  and  $\lambda \in (-\infty, \lambda_1)$ , the Lazer-McKenna conjecture is true if the critical exponent in (1.3) is replaced by sub-critical one. In the critical case, it was proved in [27, 28, 36] that if  $N \geq 6$  and  $\lambda \in (0, \lambda_1)$ , then (1.3) has unbounded number of solutions as  $s \rightarrow +\infty$ . The solutions constructed for (1.3) concentrate either at the maximum points of the first eigenfunction [27], or at some boundary points of the domain [36] as  $s \rightarrow +\infty$ . On the other hand, Druet proves in [21] that the conditions  $N \geq 6$  and  $\lambda \in (0, \lambda_1)$  are necessary for the existence of the peak-solutions constructed in [27, 36]. More precisely, the result in [21] states that if  $N = 3, 4, 5$ , or  $N \geq 6$  and  $\lambda \leq 0$ , then (1.3) has no solution  $u_s$ , such that the energy of  $u_s$  is bounded as  $s \rightarrow +\infty$ . This result suggests that it is more difficult to find solutions for (1.3) in the lower dimensional cases  $N = 3, 4, 5$ , or in the case  $\lambda \leq 0$  and  $N \geq 6$ .

Note that all the results just mentioned state that (1.3) has more and more solutions as *the parameter*  $s \rightarrow +\infty$ . But for *fixed*  $s > 0$ , it is hard to estimate how many solutions (1.3) has. (In the critical case, for fixed  $s$ , it is even unknown if there is a solution.)

In this paper, we will deal with (1.3) in the lower dimensional cases  $N = 4, 5, 6$ , or  $N \geq 7$  and  $\lambda \leq 0$ , assuming that the domain  $\Omega$  satisfies the following symmetry condition:

(S1): If  $x = (x_1, \dots, x_N) \in \Omega$ ,  
 then, for any  $\theta \in [0, 2\pi]$ ,  $(r \cos \theta, r \sin \theta, x_3, \dots, x_N) \in \Omega$ , where  $r = \sqrt{x_1^2 + x_2^2}$ ;

(S2): If  $x = (x_1, \dots, x_N) \in \Omega$ ,  
 then, for any  $3 \leq i \leq N$ ,  $(x_1, x_2, x_3, \dots, -x_i, \dots, x_N) \in \Omega$ .

The main result of this paper is the following:

**Theorem 1.1.** *Suppose that  $\Omega$  satisfies (S1) and (S2). Assume that one of the following conditions holds:*

- (i)  $N = 4, 5, \lambda < \lambda_1$  and  $s > 0$ ;
- (ii)  $N = 6, \lambda < \lambda_1$  and  $s > |\lambda|s_0$  for some  $s_0 > 0$ , which depends on  $\Omega$  only;
- (iii)  $N \geq 7, \lambda = 0$  and  $s > 0$ .

*Then, (1.3) has infinitely many distinct solutions whose energy can approach to infinity.*

The result in Theorem 1.1 is stronger than the Lazer-McKenna conjecture. Note that in Theorem 1.1, the constant  $s$  is *fixed*. In fact, all the parameters are *fixed*. This gives a striking contrast to the results in [27, 36], where  $s$  is regarded as a parameter which needs to tend to infinity in order to obtain the results there. As far as the authors know, this seems to be the first such result for Ambrosetti-Prodi type problems. We believe Theorem 1.1 should be true in any general domain and hence we pose the following stronger Lazer-McKenna conjecture:

**Stronger Lazer-McKenna Conjecture:** *Let  $s$  be fixed and  $\lambda < \lambda_1$ . Then problem (1.3) has infinitely many solutions.*

We are not able to obtain similar result for the cases  $N = 3$ , and  $N \geq 7$  and  $\lambda < 0$ . But we have the following weaker result for  $N \geq 7$  and  $\lambda < 0$ , which gives a positive answer to the Lazer-McKenna conjecture in this case:

**Theorem 1.2.** *Suppose that  $\Omega$  satisfies (S1) and (S2), and  $N \geq 7, \lambda < \lambda_1$ . Then, the number of distinct solutions for (1.3) is unbounded as  $s \rightarrow +\infty$ .*

Problem (1.3) is a bit delicate in the case  $N = 3$ . When  $s = 0$ , Brezis and Nirenberg [7] proved that (1.3) has a least energy solution if  $\lambda \in (0, \lambda_1)$ , while for  $N = 3$ , this result holds only if  $\lambda \in (\lambda^*, \lambda_1)$  for some  $\lambda^* > 0$  (if  $\Omega$  is a ball,  $\lambda^* = \frac{\lambda_1}{4}$ ). The main reason for this difference is that the function defined in (1.4) does not decay fast enough if  $N = 3$ . Similarly, the main reason that we are not able to prove Theorem 1.1 for  $N = 3$  is that the function defined in (1.7) does not decay fast enough.

In the Lazer and McKenna conjecture, the parameter  $s$  is large. Let us now consider the other extreme case:  $s \rightarrow 0+$ . Using the same argument as in [7], we can show that for  $\lambda \in (\lambda^*, \lambda_1)$ ,  $\lambda^* = 0$  if  $N = 4$ ,  $\lambda^* > 0$  if  $N = 3$ , (1.3) has a least energy solution if  $s > 0$  is small. We can obtain more in the case  $N = 3$ .

**Theorem 1.3.** *Suppose that  $\Omega$  satisfies (S1) and (S2), and  $N = 3, \lambda < \lambda_1$ . Then, the number of the solutions for (1.3) is unbounded as  $s \rightarrow 0+$ .*

Note that the result in Theorem 1.3 is not trivial, because if  $\lambda < \lambda^*$ , we can not find even one solution by using the method in [7]. Moreover, we show that (1.3) has more and more solutions as  $s \rightarrow 0+$  for all  $\lambda < \lambda_1$  if  $N = 3$ .

The readers can refer to [6, 10, 11, 17] for results on the Lazer-McKenna conjecture for other type of nonlinearities.

In Theorems 1.1-1.3, we have assumed that  $N \geq 3$ . When  $N = 2$ , M. del Pino and Munoz [17] proved the Lazer-McKenna conjecture when the right hand nonlinearity is  $e^u$  (which is still subcritical in  $\mathbb{R}^2$ ). The authors believe that when  $N = 2$ , results similar to Theorems 1.1-1.3 may be true if the right hand nonlinearity is of the *critical type*, i.e.,  $h(u)e^{u^2}$ . When  $N = 1$ , the critical exponent is  $\frac{N+2}{N-2} = -3$ . In this case, some form of Lazer-McKenna conjecture may be true if the right hand nonlinearity is  $-u^{-3}$ . We refer to [1] and [2] for discussions on critical nonlinearities in dimensions  $N = 1, 2$ .

Before we close this section, let us outline the proof of Theorems 1.1 and 1.2 and discuss the conditions imposed in these two theorems.

For any  $\bar{x} \in \mathbb{R}^N, \mu > 0$ , denote

$$U_{\mu, \bar{x}}(y) = (N(N - 2))^{\frac{N-2}{4}} \frac{\mu^{(N-2)/2}}{(1 + \mu^2|y - \bar{x}|^2)^{(N-2)/2}}. \tag{1.4}$$

Then,  $U_{\mu, \bar{x}}$  satisfies  $-\Delta U_{\mu, \bar{x}} = U_{\mu, \bar{x}}^{2^*-1}$ . In this paper, we will use the following notation:  $U = U_{1,0}$ .

Let

$$\varepsilon = \frac{s^{\frac{2}{N-2}}}{k^2}, \quad \mu = \frac{\Lambda}{\varepsilon}, \quad \Lambda \in [\delta, \delta^{-1}]$$

and  $k \geq k_0$ , where  $\delta > 0$  is a small constant, and  $k_0 > 0$  is a large constant, which is to be determined later.

Using the transformation  $u(y) \mapsto \varepsilon^{-\frac{N-2}{2}} u(\frac{y}{\varepsilon})$ , we find that (1.3) becomes

$$\begin{cases} -\Delta u = \left(u - s\varepsilon^{\frac{N-2}{2}}\varphi_1(\varepsilon y)\right)_+^{2^*-1} + \lambda\varepsilon^2 u, & \text{in } \Omega_\varepsilon, \\ u = 0, & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{1.5}$$

where  $\Omega_\varepsilon = \{y : \varepsilon y \in \Omega\}$ . Let

$$\Phi_\varepsilon(y) = \varepsilon^{\frac{N-2}{2}} \varphi_1(\varepsilon y).$$

For  $\xi \in \Omega_\varepsilon$ , we define  $W_{\Lambda, \xi}$  as the unique solution of

$$\begin{cases} -\Delta W - \lambda\varepsilon^2 W = U_{\Lambda, \xi}^{2^*-1} & \text{in } \Omega_\varepsilon, \\ W = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases} \tag{1.6}$$

Let  $y = (y', y'') \in \mathbb{R}^N$ , where  $y' = (y_1, y_2)$ , and  $y'' = (y_3, \dots, y_N)$ . Define

$$\begin{aligned} H_S &= \left\{ u : u \in H^1(\Omega_\varepsilon), u \text{ is even in } y_h, h = 3, \dots, N, u(r \cos \theta, r \sin \theta, y'') \right. \\ &= \left. u\left(r \cos\left(\theta + \frac{2\pi j}{k}\right), r \sin\left(\theta + \frac{2\pi j}{k}\right), y''\right), j = 1, \dots, k-1 \right\}, \end{aligned}$$

and

$$\mathbf{x}_j = \left( \frac{r}{\varepsilon} \cos \frac{2(j-1)\pi}{k}, \frac{r}{\varepsilon} \sin \frac{2(j-1)\pi}{k}, 0 \right), \quad j = 1, \dots, k,$$

where  $0$  is the zero vector in  $\mathbb{R}^{N-2}$ .

Let

$$W_{r,\Lambda}(y) = \sum_{j=1}^k W_{\Lambda, \mathbf{x}_j}. \tag{1.7}$$

We are going to construct a solution for (1.3), which is close to  $W_{r,\Lambda}$  for some suitable  $\Lambda$  and  $r$  and large  $k$ .

Theorem 1.1 is a direct consequence of the following result:

**Theorem 1.4.** *Under the same conditions as in Theorem 1.1, there is an integer  $k_0 > 0$ , such that for any integer  $k \geq k_0$ , (1.5) has a solution  $u_k$  of the form*

$$u_k = W_{r_k, \Lambda_k}(y) + \omega_k,$$

where  $\omega_k \in H_s$ , and as  $k \rightarrow +\infty$ ,  $r_k \rightarrow r_0 > 0$ ,  $\Lambda_k \rightarrow \Lambda_0 > 0$ ,  $\|\omega_k\|_{L^\infty} \rightarrow 0$ .

On the other hand, if  $N \geq 7$  and  $\lambda < 0$ , we have the following weaker result:

**Theorem 1.5.** *Suppose that  $N \geq 7$  and  $\lambda < \lambda_1$ . Then there is a large constant  $s_0 > 0$ , such that for any  $s > s_0$ , and integer  $k$  satisfying  $s^{\frac{(2-2\theta)(N-4)}{(N-6)(N-2)}} \leq k \leq s^{\frac{(2-\theta)(N-4)}{(N-6)(N-2)}}$ , where  $\theta > 0$  is a fixed small constant, (1.5) has a solution  $u_{k,s}$  of the form*

$$u_{k,s} = W_{r_k, \Lambda_k}(y) + \omega_{k,s},$$

where  $\omega_{k,s} \in H_s$ , and as  $s \rightarrow +\infty$ ,  $r_k \rightarrow r_0 > 0$ ,  $\Lambda_k \rightarrow \Lambda_0 > 0$ ,  $\|\omega_{k,s}\|_{L^\infty} \rightarrow 0$ .

Since  $s^{\frac{(2-\theta)(N-4)}{(N-6)(N-2)}} - s^{\frac{(2-2\theta)(N-4)}{(N-6)(N-2)}} \rightarrow +\infty$  as  $s \rightarrow +\infty$ , Theorem 1.2 is a direct consequence of Theorem 1.5. Let us point out that in the case  $N \geq 7$  and  $\lambda \in (0, \lambda_1)$ , the solutions in Theorem 1.5 are different from those constructed in [27,36], where the energy of the solutions remains bounded as  $s \rightarrow +\infty$ .

It is easy to see that Theorem 1.3 is a direct consequence of the following result:

**Theorem 1.6.** *Suppose that  $N = 3$  and  $\lambda < \lambda_1$ . Then there is a small constant  $s_1 > 0$  and a large constant  $k_0 > 0$  (independent of  $s$ ), such that for any  $s \in (0, s_1)$ , and integer  $k$  satisfying*

$$k_0 \leq k \leq Cs^{-\frac{2\tau}{1-2\tau}}, \tag{1.8}$$

for some  $\tau \in (0, \frac{4}{11})$ , then (1.5) has a solution  $u_{k,s}$  of the form

$$u_{k,s} = W_{r_k, \Lambda_k}(y) + \omega_{k,s},$$

where  $\omega_{k,s} \in H_s$ , and as  $s \rightarrow 0$ ,  $r_k \rightarrow r_0 > 0$ ,  $\Lambda_k \rightarrow \Lambda_0 > 0$ ,  $\|\omega_{k,s}\|_{L^\infty} \rightarrow 0$ .

Let make a few remarks on the conditions imposed on Theorems 1.1 and 1.2. It is easy to see that the first eigenfunction  $\varphi_1 \in H_s$ . In this paper, we denote

$$\bar{\varphi}(r) = \varphi_1(r, 0).$$

The functional corresponding to (1.5) is

$$I(u) = \frac{1}{2} \int_{\Omega_\varepsilon} (|Du|^2 - \lambda \varepsilon^2 u^2) - \frac{1}{2^*} \int_{\Omega_\varepsilon} (u - s\Phi_\varepsilon)_+^{2^*}, \quad u \in H_s.$$

Let  $\Gamma$  be a connected component of the set  $\Omega \cap \{y_3 = \dots = y_N = 0\}$ . Then, by (S1), there are  $r_2 > r_1 \geq 0$ , such that

$$\bar{\Gamma} = \left\{ y : r_1 \leq \sqrt{y_1^2 + y_2^2} \leq r_2, y_3 = \dots = y_N = 0 \right\}.$$

If  $N = 4, 5$ , then  $\frac{N-2}{2} < 2$ . We obtain from Proposition A.3,

$$I(W_{r,\Lambda}) = k \left( A_0 + \frac{A_2 s \bar{\varphi}(r) \varepsilon^{\frac{N-2}{2}}}{\Lambda^{\frac{N-2}{2}}} - \frac{A_3 \varepsilon^{N-2} k^{N-2}}{r^{N-2} \Lambda^{N-2}} + O\left(\varepsilon^{\frac{(N-2)(1+\sigma)}{2}}\right) \right). \quad (1.9)$$

It is easy to see that the function

$$r^{\frac{N-2}{2}} \bar{\varphi}(r), \quad r \in [r_1, r_2], \quad (1.10)$$

has a maximum point  $r_0$ , satisfying  $r_0 \in (r_1, r_2)$ , since  $r_i^{\frac{N-2}{2}} \bar{\varphi}(r_i) = 0, i = 1, 2$ . As a result,

$$\frac{A_2 s \bar{\varphi}(r)}{\Lambda^{\frac{N-2}{2}}} - \frac{A_3}{r^{N-2} \Lambda^{N-2}}, \quad (r, \Lambda) \in (r_1, r_2) \times (\delta, \delta^{-1}),$$

has a maximum point  $(r_0, \Lambda_0)$ , where

$$\Lambda_0 = \left( \frac{2A_3}{A_2 s r_0^{N-2} \bar{\varphi}(r_0)} \right)^{\frac{2}{N-2}},$$

for any fixed  $s > 0$ . Thus,  $I(W_{r,\Lambda})$  has a maximum point in  $(r_1, r_2) \times (\delta, \delta^{-1})$ , if  $k > 0$  is large.

If  $N = 6$ , then  $\frac{N-2}{2} = 2$ . Thus, we find from Proposition A.3,

$$I(W_{r,\Lambda}) = k \left( A_0 + (-\lambda A_1 + A_2 s \bar{\varphi}(r)) \frac{\varepsilon^2}{\Lambda^2} - \frac{A_3 \varepsilon^4 k^4}{r^4 \Lambda^4} + O\left(\varepsilon^{2+\sigma}\right) \right). \quad (1.11)$$

Let

$$g(r) = r^2 (A_2 s \bar{\varphi}(r) - A_1 \lambda), \quad r \in [r_1, r_2]. \quad (1.12)$$

It is easy to see that we can always choose a constant  $s_0 > 0$ , such that if  $s > |\lambda|s_0$ , then  $g(r)$  has a maximum point  $r_0$ , satisfying  $g(r_0) > 0$ ,  $r_0 \in (r_1, r_2)$ . As a result,

$$\frac{-\lambda A_1 + A_2 s \bar{\varphi}(r)}{\Lambda^2} - \frac{A_3}{r^4 \Lambda^4}, \quad (r, \Lambda) \in (r_1, r_2) \times (\delta, \delta^{-1}),$$

has maximum point  $(r_0, \Lambda_0)$ , where

$$\Lambda_0 = \left( \frac{2A_3}{(-\lambda A_1 + A_2 s \bar{\varphi}(r_0))r_0^4} \right)^{\frac{1}{2}},$$

for any fixed  $s > 0$ . Thus,  $I(W_{r,\lambda})$  has a maximum point in  $(r_1, r_2) \times (\delta, \delta^{-1})$ , if  $k > 0$  is large.

If  $N \geq 7$  and  $\lambda = 0$ , then Proposition A.3 gives

$$I(W_{r,\Lambda}) = k \left( A_0 + \frac{A_2 s \bar{\varphi}(r) \varepsilon^{\frac{N-2}{2}}}{\Lambda^{\frac{N-2}{2}}} - \frac{A_3 \varepsilon^{N-2} k^{N-2}}{r^{N-2} \Lambda^{N-2}} + O\left(\varepsilon^{\frac{(N-2)(1+\sigma)}{2}}\right) \right), \quad (1.13)$$

So, we are in the same situation as the case  $N = 4, 5$ .

On the other hand, if  $N \geq 7$ , then  $\frac{N-2}{2} > 2$ . Thus  $\varepsilon^{\frac{N-2}{2}}$  is a higher order term of  $\varepsilon^2$ . Thus if  $\lambda \neq 0$ , then for each fixed  $s > 0$ , we have

$$I(W_{r,\Lambda}) = k \left( A_0 - \frac{\lambda A_1 \varepsilon^2}{\Lambda^2} - \frac{A_3 \varepsilon^{N-2} k^{N-2}}{r^{N-2} \Lambda^{N-2}} + O\left(\varepsilon^{2+\sigma}\right) \right), \quad (1.14)$$

But

$$-\frac{\lambda A_1}{\Lambda^2} - \frac{A_3}{r^{N-2} \Lambda^{N-2}}, \quad (r, \Lambda) \in (r_1, r_2) \times (\delta, \delta^{-1}),$$

does not have a critical point even if  $\lambda < 0$ . So, we don't know whether  $I(W_{r,\Lambda})$  has a critical point. Thus, to obtain a solution for (1.3), we need to let  $s$  change so that

$$\varepsilon^2 \ll s \varepsilon^{\frac{N-2}{2}}, \quad \varepsilon \ll 1. \quad (1.15)$$

If (1.15) holds, then

$$I(W_{r,\Lambda}) = k \left( A_0 + \frac{A_2 \bar{\varphi}(r) s \varepsilon^{\frac{N-2}{2}}}{\Lambda^{\frac{N-2}{2}}} - \frac{A_3 \varepsilon^{N-2} k^{N-2}}{r^{N-2} \Lambda^{N-2}} + O\left((s \varepsilon^{\frac{N-2}{2}})^{1+\sigma}\right) \right). \quad (1.16)$$

So, we are in a similar situation as  $\lambda = 0$ . Note the (1.15) implies

$$k \ll s^{\frac{2(N-4)}{(N-2)(N-6)}}, \quad k \gg s^{\frac{1}{N-2}},$$

which gives an upper bound for  $k$ . Therefore, in this case, we are not able to obtain the existence of infinitely many solutions even if  $s > 0$  is large.

In the case  $N = 3$ , for fixed  $s > 0$ , some estimates which are valid for  $N \geq 4$  may not be true due to the slow decay of the function  $W_{r,\Lambda}$ . Under the condition  $s \leq Ck^{-\frac{1}{2\tau}+1}$  for some  $\tau \in (0, \frac{4}{11})$ , we can recover all these estimates. But the condition  $s \leq Ck^{-\frac{1}{2\tau}+1}$  imposes an upper bound (1.8) for the number of bubbles  $k$ .

The energy of the solutions obtained in Theorems 1.4 and 1.5 is very large because  $k$  must be large. This result is in consistence of the result in [21].

Finally, let us point out that the eigenvalue  $\varphi_1$  is not essential in this paper. We can replace  $\varphi_1$  by any function  $\varphi$ , satisfying  $\varphi > 0$  in  $\Omega$ ,  $\varphi = 0$  on  $\partial\Omega$  and  $\varphi \in H_s$ .

We will use the reduction argument as in [4, 5, 14–16, 29, 30] and [38] to prove the main results of this paper. Unlike those papers, where a parameter always appears in some form, in Theorem 1.4,  $s$  is a fixed positive constant. To prove Theorem 1.4, *the number of the bubbles  $k$  is used as a parameter to carry out the reduction*. Similar idea has been used in [35, 37].

## 2. The reduction

In this section, we will reduce the problem of finding a  $k$ -peak solution for (1.3) to a finite dimension problem.

Let

$$\|u\|_* = \sup_y \left( \sum_{j=1}^k \frac{1}{(1 + |y - \mathbf{x}_j|)^{\frac{N-2}{2} + \tau}} \right)^{-1} |u(y)|, \tag{2.1}$$

and

$$\|f\|_{**} = \sup_y \left( \sum_{j=1}^k \frac{1}{(1 + |y - \mathbf{x}_j|)^{\frac{N+2}{2} + \tau}} \right)^{-1} |f(y)|, \tag{2.2}$$

where  $\tau \in (0, 1)$  is a constant, such that

$$\sum_{j=2}^k \frac{1}{|\mathbf{x}_j - \mathbf{x}_1|^\tau} \leq C. \tag{2.3}$$

Recall that  $\varepsilon = \frac{s^{\frac{2}{N-2}}}{k^2}$ , and

$$\sum_{j=2}^k \frac{1}{|\mathbf{x}_j - \mathbf{x}_1|^\tau} \leq C\varepsilon^\tau k^\tau \sum_{j=2}^k \frac{1}{j^\tau} \leq C\varepsilon^\tau k.$$

In order to achieve (2.3), we need to choose  $\tau$  according to whether  $s > 0$  is fixed or not. We choose  $\tau$  as follows:

$$\tau = \begin{cases} \frac{1}{2}, & \text{in Theorems 1.4 and 1.5;} \\ \text{the number in (1.8),} & \text{in Theorem 1.6.} \end{cases} \tag{2.4}$$



Let

$$Y_{i,1} = \frac{\partial W_{\Lambda, \mathbf{x}_i}}{\partial \Lambda}, \quad Z_{i,1} = -\Delta Y_{i,1} - \lambda \varepsilon^2 Y_{i,1} = (2^* - 1) U_{\Lambda, \mathbf{x}_i}^{2^*-2} \frac{\partial U_{\Lambda, \mathbf{x}_i}}{\partial \Lambda},$$

and

$$Y_{i,2} = \frac{\partial W_{\Lambda, \mathbf{x}_i}}{\partial r}, \quad Z_{i,2} = -\Delta Y_{i,2} - \lambda \varepsilon^2 Y_{i,2} = (2^* - 1) U_{\Lambda, \mathbf{x}_i}^{2^*-2} \frac{\partial U_{\Lambda, \mathbf{x}_i}}{\partial r}.$$

We consider

$$\begin{cases} -\Delta \phi_k - \lambda \varepsilon^2 \phi_k - (2^* - 1) (W_{r, \Lambda} - s \Phi_\varepsilon)_+^{2^*-2} \phi_k = h + \sum_{j=1}^2 \sum_{i=1}^k c_j Z_{i,j}, & \text{in } \Omega_\varepsilon, \\ \phi_k \in H_s, \\ \left\langle \sum_{i=1}^k Z_{i,j}, \phi_k \right\rangle = 0, & j = 1, 2, \end{cases} \quad (2.5)$$

for some number  $c_j$ , where  $\langle u, v \rangle = \int_{\Omega_\varepsilon} uv$ .

We need the following result, whose proof is standard.

**Lemma 2.1.** *Let  $f$  satisfy  $\|f\|_{**} < \infty$  and let  $u$  be the solution of*

$$-\Delta u - \lambda \varepsilon^2 u = f \quad \text{in } \Omega_\varepsilon, \quad u = 0 \quad \text{on } \partial \Omega_\varepsilon,$$

where  $\lambda < \lambda_1$ . Then we have

$$|u(y)| \leq C \int_{\Omega_\varepsilon} \frac{|f(z)|}{|z - y|^{N-2}} dz.$$

Next, we need the following lemma to carry out the reduction.

**Lemma 2.2.** *Assume that  $\phi_k$  solves (2.5) for  $h = h_k$ . If  $\|h_k\|_{**}$  goes to zero as  $k$  goes to infinity, so does  $\|\phi_k\|_*$ .*

*Proof.* We argue by contradiction. Suppose that there are  $k \rightarrow +\infty$ ,  $h = h_k$ ,  $\Lambda_k \in [\delta, \delta^{-1}]$ , and  $\phi_k$  solving (2.5) for  $h = h_k$ ,  $\Lambda = \Lambda_k$ , with  $\|h_k\|_{**} \rightarrow 0$ , and  $\|\phi_k\|_* \geq c' > 0$ . We may assume that  $\|\phi_k\|_* = 1$ . For simplicity, we drop the subscript  $k$ .

By Lemma 2.1,

$$\begin{aligned} |\phi(y)| &\leq C \int_{\Omega_\varepsilon} \frac{1}{|z - y|^{N-2}} W_{r, \Lambda}^{2^*-2} |\phi(z)| dz \\ &\quad + C \int_{\Omega_\varepsilon} \frac{1}{|z - y|^{N-2}} \left( |h(z)| + \left| \sum_{j=1}^2 \sum_{i=1}^k c_j Z_{i,j}(z) \right| \right) dz \end{aligned} \quad (2.6)$$

Using Lemma B.4 and B.5, there is a strictly positive number  $\theta$  such that

$$\left| \int_{\Omega_\varepsilon} \frac{1}{|z-y|^{N-2}} W_{r,\Lambda}^{2^*-2} \phi(z) dz \right| \leq C \|\phi\|_* \sum_{j=1}^k \frac{1}{(1+|y-\mathbf{x}_j|)^{\frac{N-2}{2}+\tau+\theta}}. \quad (2.7)$$

It follows from Lemma B.3 that

$$\begin{aligned} \left| \int_{\Omega_\varepsilon} \frac{1}{|z-y|^{N-2}} h(z) dz \right| &\leq C \|h\|_{**} \int_{\mathbb{R}^N} \frac{1}{|z-y|^{N-2}} \sum_{j=1}^k \frac{1}{(1+|z-\mathbf{x}_j|)^{\frac{N+2}{2}+\tau}} dz \\ &\leq C \|h\|_{**} \sum_{j=1}^k \frac{1}{(1+|y-\mathbf{x}_j|)^{\frac{N-2}{2}+\tau}}, \end{aligned} \quad (2.8)$$

and

$$\begin{aligned} \left| \int_{\Omega_\varepsilon} \frac{1}{|z-y|^{N-2}} \sum_{i=1}^k Z_{i,j}(z) dz \right| &\leq C \sum_{i=1}^k \int_{\mathbb{R}^N} \frac{1}{|z-y|^{N-2}} \frac{1}{(1+|z-\mathbf{x}_i|)^{N+2}} dz \\ &\leq C \sum_{i=1}^k \frac{1}{(1+|y-\mathbf{x}_i|)^{\frac{N-2}{2}+\tau}}. \end{aligned} \quad (2.9)$$

Next, we estimate  $c_j$ . Multiplying (2.5) by  $Y_{1,l}$  and integrating, we see that  $c_j$  satisfies

$$\left\langle \sum_{j=1}^2 \sum_{i=1}^k Z_{i,j}, Y_{1,l} \right\rangle c_j = \left\langle -\Delta\phi - \lambda\varepsilon^2\phi - (2^*-1)W_{r,\Lambda}^{2^*-2}\phi, Y_{1,l} \right\rangle - \langle h, Y_{1,l} \rangle. \quad (2.10)$$

It follows from Lemma B.2 that

$$|\langle h, Y_{1,l} \rangle| \leq C \|h\|_{**} \int_{\mathbb{R}^N} \frac{1}{(1+|z-\mathbf{x}_1|)^{N-2-\beta}} \sum_{j=1}^k \frac{1}{(1+|z-\mathbf{x}_j|)^{\frac{N+2}{2}+\tau}} dz \leq C \|h\|_{**},$$

since  $\beta > 0$  can be chosen as small as desired.

On the other hand,

$$\begin{aligned}
 & \left\langle -\Delta\phi - \lambda\varepsilon^2\phi - (2^* - 1)W_{r,\Lambda}^{2^*-2}\phi, Y_{1,l} \right\rangle \\
 &= \left\langle -\Delta Y_{1,l} - \lambda\varepsilon^2 Y_{1,l} - (2^* - 1)W_{r,\Lambda}^{2^*-2} Y_{1,l}, \phi \right\rangle \tag{2.11} \\
 &= (2^* - 1) \left\langle U_{\Lambda, \mathbf{x}_1}^{2^*-2} \partial_l U_{\Lambda, \mathbf{x}_1} - W_{r,\Lambda}^{2^*-2} Y_{1,l}, \phi \right\rangle,
 \end{aligned}$$

where  $\partial_l = \partial_\Lambda$  if  $l = 1$ ,  $\partial_l = \partial_r$  if  $l = 2$ .

By Lemmas B.1,

$$|\phi(y)| \leq C \|\phi\|_*.$$

We consider the cases  $N \geq 6$  first. Note that  $\frac{4}{N-2} \leq 1$  for  $N \geq 6$ . Using Lemmas A.1 and B.2, noting that

$$|W_{r,\Lambda}^{2^*-2} - W_{\Lambda, \mathbf{x}_1}^{2^*-2}| \leq \sum_{j=2}^k W_{\Lambda, \mathbf{x}_j}^{2^*-2},$$

and

$$\varepsilon \leq \frac{C}{1 + |z - \mathbf{x}_1|},$$

we obtain

$$\begin{aligned}
 & \left| \left\langle U_{\Lambda, \mathbf{x}_1}^{2^*-2} \partial_l U_{\Lambda, \mathbf{x}_j} - W_{r,\Lambda}^{2^*-2} Y_{1,l}, \phi \right\rangle \right| \\
 & \leq C \|\phi\|_* \int_{\Omega_\varepsilon} \frac{1}{(1 + |z - \mathbf{x}_1|)^{N-2-\beta}} \sum_{i=2}^k \frac{1}{(1 + |z - \mathbf{x}_i|)^{4-\beta}} dz \\
 & \quad + C \|\phi\|_* \int_{\Omega_\varepsilon} U_{\Lambda, \mathbf{x}_1}^{2^*-2} \left( \varepsilon^{N-2} + \frac{|\lambda|\varepsilon^2}{(1 + |y - \mathbf{x}_j|)^{N-4-\beta}} \right) \sum_{j=1}^k \frac{1}{(1 + |z - \mathbf{x}_j|)^{\frac{N-2}{2} + \tau}} \\
 & \quad + C \|\phi\|_* \int_{\Omega_\varepsilon} U_{\Lambda, \mathbf{x}_1} \left( \varepsilon^{N-2} + \frac{|\lambda|\varepsilon^2}{(1 + |y - \mathbf{x}_j|)^{N-4-\beta}} \right)^{2^*-2} \sum_{j=1}^k \frac{1}{(1 + |z - \mathbf{x}_j|)^{\frac{N-2}{2} + \tau}} \\
 & \leq C \|\phi\|_* \sum_{j=2}^k \frac{1}{|\mathbf{x}_1 - \mathbf{x}_j|^{1+\sigma}} + o(1) \|\phi\|_* = o(1) \|\phi\|_*. \tag{2.12}
 \end{aligned}$$

For  $N = 3, 4, 5$ , we have  $\frac{4}{N-2} > 1$ . By Lemmas B.1, B.2,

$$\begin{aligned}
& \left| \left\langle U_{\Lambda, \mathbf{x}_1}^{2^*-2} \partial_l U_{\Lambda, \mathbf{x}_j} - W_{r, \Lambda}^{2^*-2} Y_{1, l}, \phi \right\rangle \right| \\
& \leq C \int_{\Omega_\varepsilon} W_{\Lambda, \mathbf{x}_1}^{2^*-3} \sum_{j=2}^k W_{\Lambda, \mathbf{x}_j} |Y_{1, l} \phi| + C \int_{\Omega_\varepsilon} \left( \sum_{j=2}^k W_{\Lambda, \mathbf{x}_j} \right)^{\frac{4}{N-2}} |Y_1 \phi| \\
& \quad + C \|\phi\|_* \int_{\Omega_\varepsilon} U_{\Lambda, \mathbf{x}_1}^{2^*-2} \left( \varepsilon^{N-2} + \frac{|\lambda| \varepsilon^2}{(1+|y-\mathbf{x}_j|)^{N-4-\beta}} \right) \sum_{j=1}^k \frac{1}{(1+|z-\mathbf{x}_j|)^{\frac{N-2}{2}+\tau}} \\
& \quad + C \|\phi\|_* \int_{\Omega_\varepsilon} U_{\Lambda, \mathbf{x}_1} \left( \varepsilon^{N-2} + \frac{|\lambda| \varepsilon^2}{(1+|y-\mathbf{x}_j|)^{N-4-\beta}} \right)^{2^*-2} \sum_{j=1}^k \frac{1}{(1+|z-\mathbf{x}_j|)^{\frac{N-2}{2}+\tau}} \\
& \leq C \|\phi\|_* \int_{\Omega_\varepsilon} \frac{1}{(1+|z-\mathbf{x}_1|)^{4-\beta}} \sum_{j=2}^k \frac{1}{(1+|z-\mathbf{x}_j|)^{N-2-\beta}} \\
& \quad + C \int_{\Omega_\varepsilon} \left( \sum_{j=2}^k U_{\Lambda, \mathbf{x}_j}^{1-\beta} \right)^{\frac{4}{N-2}} |Y_{1, l} \phi| + o(1) \|\phi\|_* \\
& \leq C \|\phi\|_* \int_{\Omega_\varepsilon} \frac{1}{(1+|z-\mathbf{x}_1|)^{N-2-\beta}} \left( \sum_{j=2}^k U_{\Lambda, \mathbf{x}_j}^{1-\beta} \right)^{\frac{4}{N-2}} \sum_{i=1}^k \frac{1}{(1+|y-\mathbf{x}_i|)^{\frac{N-2}{2}+\tau}} \\
& \quad + o(1) \|\phi\|_*.
\end{aligned} \tag{2.13}$$

Let

$$\Omega_j = \left\{ y = (y', y'') \in \Omega_\varepsilon : \left\langle \frac{y'}{|y'|}, \frac{\mathbf{x}_j}{|\mathbf{x}_j|} \right\rangle \geq \cos \frac{\pi}{k} \right\}.$$

If  $y \in \Omega_1$ , then

$$\begin{aligned}
\sum_{j=2}^k U_{\Lambda, \mathbf{x}_j}^{1-\beta} & \leq \frac{1}{(1+|y-\mathbf{x}_1|)^{N-2-\tau-(N-2)\beta-\theta}} \sum_{j=2}^k \frac{1}{|\mathbf{x}_j-\mathbf{x}_1|^{\tau+\theta}} \\
& = o(1) \frac{1}{(1+|y-\mathbf{x}_1|)^{N-2-\tau-(N-2)\beta-\theta}},
\end{aligned}$$

and

$$\sum_{i=1}^k \frac{1}{(1+|y-\mathbf{x}_i|)^{\frac{N-2}{2}+\tau}} \leq \frac{C}{(1+|y-\mathbf{x}_1|)^{\frac{N-2}{2}}}.$$

So, we obtain

$$\begin{aligned} & \int_{\Omega_1} \frac{1}{(1 + |z - \mathbf{x}_1|)^{N-2-\beta}} \left( \sum_{j=2}^k U_{\Lambda, \mathbf{x}_j}^{1-\beta} \right)^{\frac{4}{N-2}} \sum_{i=1}^k \frac{1}{(1 + |y - \mathbf{x}_i|)^{\frac{N-2}{2}+\tau}} \\ &= o(1) \int_{\Omega_1} \frac{1}{(1 + |z - \mathbf{x}_1|)^{N + \frac{N+2}{2} - \frac{4(\tau+\theta)}{N-2} - 4\beta}} = o(1), \end{aligned}$$

since  $\frac{N+2}{2} - \frac{4(\tau+\theta)}{N-2} - 4\beta > 0$ , if  $\beta > 0$  and  $\theta > 0$  are small.  
 If  $y \in \Omega_l, l \geq 2$ , then

$$\sum_{j=2}^k U_{\Lambda, \mathbf{x}_j}^{1-\beta} \leq \frac{C}{(1 + |y - \mathbf{x}_l|)^{N-2-\tau-(N-2)\beta}},$$

and

$$\sum_{i=1}^k \frac{1}{(1 + |y - \mathbf{x}_i|)^{\frac{N-2}{2}+\tau}} \leq \frac{C}{(1 + |y - \mathbf{x}_l|)^{\frac{N-2}{2}}}.$$

As a result,

$$\begin{aligned} & \int_{\Omega_l} \frac{1}{(1 + |z - \mathbf{x}_1|)^{N-2}} \left( \sum_{j=2}^k U_{\Lambda, \mathbf{x}_j}^{1-\beta} \right)^{\frac{4}{N-2}} \sum_{i=1}^k \frac{1}{(1 + |y - \mathbf{x}_i|)^{\frac{N-2}{2}+\tau}} \\ & \leq C \int_{\Omega_l} \frac{1}{(1 + |z - \mathbf{x}_1|)^{N-2}} \frac{1}{(1 + |y - \mathbf{x}_l|)^{4-4\beta - \frac{4\tau}{N-2} + \frac{N-2}{2}}} \\ & \leq \frac{C}{|\mathbf{x}_l - \mathbf{x}_1|^{\frac{N+2}{2} - \frac{4\tau}{N-2} - \theta - 4\beta}}, \end{aligned}$$

where  $\theta > 0$  is a fixed small constant.

Since  $\tau = \frac{1}{2}$  for  $N \geq 4$ , and  $\tau < \frac{1}{2}$  for  $N = 3$ , we find that for  $\theta > 0$  and  $\beta > 0$  small,  $\frac{N+2}{2} - \frac{4\tau}{N-2} - \theta - 4\beta > \tau$ . Thus

$$\begin{aligned} & \int_{\Omega_\epsilon} \frac{1}{(1 + |z - \mathbf{x}_1|)^{N-2}} \left( \sum_{j=2}^k U_{\Lambda, \mathbf{x}_j}^{1-\beta} \right)^{\frac{4}{N-2}} \sum_{i=1}^k \frac{1}{(1 + |y - \mathbf{x}_i|)^{\frac{N-2}{2}+\tau}} \\ & \leq o(1) + C \sum_{l=2}^k \frac{1}{|\mathbf{x}_l - \mathbf{x}_1|^{\frac{N+2}{2} - \frac{4\tau}{N-2} - \theta}} = o(1). \end{aligned}$$

So, we have proved

$$\left| \left\langle U_{\Lambda, \mathbf{x}_1}^{2^*-2} \partial_l U_{\Lambda, \mathbf{x}_j} - W_{r, \Lambda}^{2^*-2} Y_1, \phi \right\rangle \right| = o(1) \|\phi\|_*.$$





where we use the inequality

$$\sum_{j=1}^k a_j b_j \leq \left( \sum_{j=1}^k a_j^p \right)^{\frac{1}{p}} \left( \sum_{j=1}^k b_j^q \right)^{\frac{1}{q}}, \quad \frac{1}{p} + \frac{1}{q} = 1, a_j, b_j \geq 0, j = 1, \dots, k,$$

and

$$\sum_{j=1}^k \frac{1}{(1 + |y - \mathbf{x}_j|)^\tau} \leq C + \sum_{j=2}^k \frac{C}{|\mathbf{x}_1 - \mathbf{x}_j|^\tau} \leq C.$$

which follows from Lemma B.1.

For  $N = 3, 4, 5$ , similarly to the case  $N \geq 6$ , we have

$$\begin{aligned} & |\bar{N}(\phi)| \\ & \leq C \|\phi\|_*^2 \left( \sum_{j=1}^k \frac{1}{(1 + |y - \mathbf{x}_j|)^{N-2-\beta}} \right)^{\frac{6-N}{N-2}} \left( \sum_{j=1}^k \frac{1}{(1 + |y - \mathbf{x}_j|^{\frac{N-2}{2}+\tau}} \right)^2 \\ & \quad + C \|\phi\|_*^{2^*-1} \sum_{j=1}^k \frac{1}{(1 + |y - \mathbf{x}_j|^{\frac{N+2}{2}+\tau}} \\ & \leq C \|\phi\|_*^2 \left( \sum_{j=1}^k \frac{1}{(1 + |y - \mathbf{x}_j|^{\frac{N-2}{2}+\tau}} \right)^{2^*-1} + C \|\phi\|_*^{2^*-1} \sum_{j=1}^k \frac{1}{(1 + |y - \mathbf{x}_j|^{\frac{N+2}{2}+\tau}} \\ & \leq C \|\phi\|_*^2 \sum_{j=1}^k \frac{1}{(1 + |y - \mathbf{x}_j|^{\frac{N+2}{2}+\tau}}. \end{aligned} \tag{2.21}$$

□

Next, we estimate  $l_k$ .

**Lemma 2.6.** *We have*

$$\|l_k\|_{**} \leq C \left( s\varepsilon^{\frac{N-2}{2}} \right)^{\frac{1}{2}+\sigma} + C|\lambda|\varepsilon^{1+\sigma},$$

where  $\sigma > 0$  is a fixed small constant.

*Proof.* Recall

$$\Omega_j = \left\{ y = (y', y'') \in \Omega_\varepsilon : \left\langle \frac{y'}{|y'|}, \frac{\mathbf{x}_j}{|\mathbf{x}_j|} \right\rangle \geq \cos \frac{\pi}{k} \right\}.$$

From the symmetry, we can assume that  $y \in \Omega_1$ . Then,

$$|y - \mathbf{x}_j| \geq |y - \mathbf{x}_1|, \quad \forall y \in \Omega_1.$$



Thus, for  $y \in \Omega_1$ , by Lemma A.1,

$$\begin{aligned}
 |I_k| &\leq \frac{C}{(1 + |y - \mathbf{x}_1|)^{4-\beta}} \sum_{j=2}^k \frac{1}{(1 + |y - \mathbf{x}_j|)^{N-2-\beta}} \\
 &\quad + C \left( \sum_{j=2}^k \frac{1}{(1 + |y - \mathbf{x}_j|)^{N-2-\beta}} \right)^{2^*-1} \\
 &\quad + C \sum_{j=1}^k \frac{1}{(1 + |y - \mathbf{x}_j|)^{4-\beta}} \left( \varepsilon^{N-2} + \frac{|\lambda|\varepsilon^2}{(1 + |y - \mathbf{x}_j|)^{N-4-\beta}} \right) \\
 &\quad + CW_{r,\Lambda}^{2^*-1-\frac{1}{2}-\frac{2\sigma}{N-2}} s^{\frac{1}{2}+\frac{2\sigma}{N-2}} \varepsilon^{\frac{N-2}{4}+\sigma}.
 \end{aligned} \tag{2.22}$$

Here, we have used the inequality: for any bounded  $a > 0$  and  $b > 0$ ,  $\alpha \in (0, 1]$ :

$$|(a - b)_+^{2^*-1} - a^{2^*-1}| \leq Ca^{2^*-1-\alpha}b^\alpha.$$

Let us estimate the first term of (2.22). Using Lemma B.2, we obtain

$$\begin{aligned}
 &\frac{1}{(1 + |y - \mathbf{x}_1|)^{4-\beta}} \frac{1}{(1 + |y - \mathbf{x}_j|)^{N-2-\beta}} \\
 &\leq C \left( \frac{1}{(1 + |y - \mathbf{x}_1|)^{\frac{N+2}{2}+\tau}} + \frac{1}{(1 + |y - \mathbf{x}_j|)^{\frac{N+2}{2}+\tau}} \right) \frac{1}{|\mathbf{x}_j - \mathbf{x}_1|^{\frac{N+2}{2}-\tau-2\beta}} \\
 &\leq C \frac{1}{(1 + |y - \mathbf{x}_1|)^{\frac{N+2}{2}+\tau}} \frac{1}{|\mathbf{x}_j - \mathbf{x}_1|^{\frac{N+2}{2}-\tau-2\beta}}, \quad j > 1.
 \end{aligned} \tag{2.23}$$

Since  $\frac{N+2}{2} - \tau - 2\beta > 1$ , we find

$$\begin{aligned}
 &\frac{1}{(1 + |y - \mathbf{x}_1|)^{4-\beta}} \sum_{j=2}^k \frac{1}{(1 + |y - \mathbf{x}_j|)^{N-2-\beta}} \\
 &\leq C \frac{1}{(1 + |y - \mathbf{x}_1|)^{\frac{N+2}{2}+\tau}} (k\varepsilon)^{\frac{N+2}{2}-\tau-2\beta} \\
 &\leq C \left( s\varepsilon^{\frac{N-2}{2}} \right)^{\frac{1}{2}+\sigma} \frac{1}{(1 + |y - \mathbf{x}_1|)^{\frac{N+2}{2}+\tau}}.
 \end{aligned} \tag{2.24}$$

Here we have used

$$(k\varepsilon)^{\frac{N+2}{2}-\tau-2\beta} = O \left( \left( s\varepsilon^{\frac{N-2}{2}} \right)^{\frac{1}{2}+\sigma} \right), \tag{2.25}$$

for some small  $\sigma > 0$ .

In fact, if  $s > 0$  is fixed (as in Theorem 1.4), then  $k = \frac{1}{\sqrt{\varepsilon}}$  and  $\tau = \frac{1}{2}$ . As a result,

$$(k\varepsilon)^{\frac{N+2}{2}-\tau-2\beta} = O\left(\varepsilon^{\frac{N+2}{4}-\frac{\tau}{2}-\beta}\right) = O\left(\varepsilon^{\frac{N-2}{4}+\sigma}\right).$$

So, we obtain (2.25).

If  $N \geq 7$ , then  $\tau = \frac{1}{2}$ , and

$$s^{\frac{(2-2\theta)(N-4)}{(N-6)(N-2)}} \leq k \leq s^{\frac{(2-\theta)(N-4)}{(N-6)(N-2)}}. \quad (2.26)$$

But

$$(k\varepsilon)^{\frac{N+2}{2}-\tau-2\beta} = \left(\frac{s^{\frac{2}{N-2}}}{k}\right)^{\frac{N+2}{2}-\tau-2\beta} = \frac{s^{\frac{N+1-4\beta}{N-2}}}{k^{\frac{N+1-4\beta}{2}}}$$

and

$$\left(s\varepsilon^{\frac{N-2}{2}}\right)^{\frac{1}{2}+\sigma} = \left(\frac{s^2}{k^{N-2}}\right)^{\frac{1}{2}+\sigma}$$

Thus, we see that (2.25) is equivalent to

$$s^{\frac{3-4\beta}{N-2}-2\sigma} \leq Ck^{\frac{3}{2}-2\beta-(N-2)\sigma}. \quad (2.27)$$

Using (2.26), we find (2.27) holds.

For  $N = 3$ ,  $k = \frac{s}{\sqrt{\varepsilon}}$ . Thus,

$$(k\varepsilon)^{\frac{5}{2}-\tau-2\beta} = (s\varepsilon^{\frac{1}{2}})^{\frac{5}{2}-\tau-2\beta} \leq C(s\varepsilon^{\frac{1}{2}})^{\frac{1}{2}+\sigma}.$$

So, we obtain (2.25).

Now, we estimate the second term of (2.22).

Using Lemma B.2 again, we find for  $y \in \Omega_1$ ,

$$\begin{aligned} & \frac{1}{(1 + |y - \mathbf{x}_j|)^{N-2-\beta}} \leq \frac{1}{(1 + |y - \mathbf{x}_1|)^{\frac{N-2-\beta}{2}}} \frac{1}{(1 + |y - \mathbf{x}_j|)^{\frac{N-2-\beta}{2}}} \\ & \leq \frac{C}{|\mathbf{x}_j - \mathbf{x}_1|^{\frac{N-2}{2}-\beta-\frac{N-2}{N+2}\tau}} \left( \frac{1}{(1 + |y - \mathbf{x}_1|)^{\frac{N-2}{2}+\frac{N-2}{N+2}\tau}} + \frac{1}{(1 + |y - \mathbf{x}_j|)^{\frac{N-2}{2}+\frac{N-2}{N+2}\tau}} \right) \\ & \leq \frac{C}{|\mathbf{x}_j - \mathbf{x}_1|^{\frac{N-2}{2}-\beta-\frac{N-2}{N+2}\tau}} \frac{1}{(1 + |y - \mathbf{x}_1|)^{\frac{N-2}{2}+\frac{N-2}{N+2}\tau}}. \end{aligned} \quad (2.28)$$

Suppose that  $N \geq 5$ . Then  $\frac{N-2}{2} - \beta - \frac{N-2}{N+2}\tau > 1$  since  $\tau < 1$ . Then (2.28) gives for  $y \in \Omega_1$

$$\begin{aligned} & \left( \sum_{j=2}^k \frac{1}{(1 + |y - \mathbf{x}_j|)^{N-2-\beta}} \right)^{2^*-1} \\ & \leq C (k\varepsilon)^{\frac{N+2}{2}-\tau-(2^*-1)\beta} \frac{1}{(1 + |y - \mathbf{x}_1|)^{\frac{N+2}{2}+\tau}} \tag{2.29} \\ & = C \left( s\varepsilon^{\frac{N-2}{2}} \right)^{\frac{1}{2}+\sigma} \frac{1}{(1 + |y - \mathbf{x}_1|)^{\frac{N+2}{2}+\tau}}. \end{aligned}$$

If  $N = 3, 4$ , then (2.28) gives

$$\begin{aligned} & \left( \sum_{j=2}^k \frac{1}{(1 + |y - \mathbf{x}_j|)^{N-2-\beta}} \right)^{2^*-1} \\ & \leq C \left( k\varepsilon^{\frac{N-2}{2}-\frac{N-2}{N+2}\tau-\beta} \right)^{2^*-1} \frac{1}{(1 + |y - \mathbf{x}_1|)^{\frac{N+2}{2}+\tau}} \tag{2.30} \\ & = C k^{\frac{N+2}{N-2}} \varepsilon^{\frac{N+2}{2}-\tau-(2^*-1)\beta} \frac{1}{(1 + |y - \mathbf{x}_1|)^{\frac{N+2}{2}+\tau}}. \end{aligned}$$

If  $N = 4$ , then

$$k^{\frac{N+2}{N-2}} \varepsilon^{\frac{N+2}{2}-\tau-(2^*-1)\beta} = k^3 \varepsilon^{3-\frac{1}{2}-(2^*-1)\beta} \leq C \varepsilon^{1-(2^*-1)\beta} \leq C \varepsilon^{\frac{1}{2}+\sigma}.$$

Hence for  $N = 4$ ,

$$\left( \sum_{j=2}^k \frac{1}{(1 + |y - \mathbf{x}_j|)^2} \right)^{2^*-1} \leq \sum_{i=1}^k \frac{C \varepsilon^{\frac{N-2}{4}+\sigma}}{(1 + |y - \mathbf{x}_i|)^{\frac{N+2}{2}+\tau}}.$$

For  $N = 3$ , we have

$$k^5 \varepsilon^{\frac{5}{2}-\tau-(2^*-1)\beta} = k^{2\tau+2(2^*-1)\beta} s^{5-2\tau-2(2^*-1)\beta}.$$

But

$$\left( s\varepsilon^{\frac{1}{2}} \right)^{\frac{1}{2}+\sigma} = \frac{s^{1+2\sigma}}{k^{\frac{1}{2}+\sigma}}.$$

So,  $k^5 \varepsilon^{\frac{5}{2}-\tau-(2^*-1)\beta} \leq C(s\varepsilon^{\frac{1}{2}})^{\frac{1}{2}+\sigma}$  is equivalent to

$$k \leq Cs^{-\frac{8-4\tau-4\sigma-4(2^*-1)\beta}{1+4\tau+2\sigma+4(2^*-1)\beta}} \tag{2.31}$$

Since  $k \leq Cs^{-\frac{2\tau}{1-2\tau}}$ , we see that (2.31) is valid if

$$\frac{8-4\tau}{1+4\tau} > \frac{2\tau}{1-2\tau}.$$

Thus, if  $\tau \in (0, \frac{4}{11})$ , (2.31) holds. Hence for  $N = 3$ , we also have

$$\left( \sum_{j=2}^k \frac{1}{(1+|y-\mathbf{x}_j|)^2} \right)^{2^*-1} \leq \sum_{i=1}^k \frac{C(s\varepsilon^{\frac{N-2}{2}})^{\frac{1}{2}+\sigma}}{(1+|y-\mathbf{x}_i|)^{\frac{N+2}{2}+\tau}}.$$

Note that for  $y \in \Omega_1$ ,

$$W_{r,\Lambda}(y) \leq \frac{C}{(1+|y-\mathbf{x}_1|)^{N-2-\tau-\beta}}.$$

We claim that

$$\left( \frac{N+2}{N-2} - \frac{1}{2} - \frac{2\sigma}{N-2} \right) (N-2-\tau) \geq \frac{N+2}{2} + \tau, \tag{2.32}$$

if  $N \geq 3$ .

In fact, (2.32) is equivalent to

$$\tau < \frac{4(N-2)}{3N+2},$$

which is true, since  $\tau = \frac{1}{2}$  if  $N \geq 4$ ,  $\tau < \frac{4}{11}$  if  $N = 3$ .

Thus, we obtain

$$s^{\frac{1}{2}+\frac{2\sigma}{N-2}} \varepsilon^{\frac{N-2}{4}+\sigma} W_{r,\Lambda}^{\frac{N+2}{N-2}-\frac{1}{2}-\frac{2\sigma}{N-2}} \leq Cs^{\frac{1}{2}+\frac{2\sigma}{N-2}} \varepsilon^{\frac{N-2}{4}+\sigma} \frac{C}{(1+|y-\mathbf{x}_j|)^{\frac{N+2}{2}+\tau}}.$$

Finally,

$$\begin{aligned} \sum_{j=1}^k \frac{1}{(1+|y-\mathbf{x}_j|)^4} \frac{|\lambda|\varepsilon^2}{(1+|y-\mathbf{x}_j|)^{N-4-\beta}} &= \sum_{j=1}^k \frac{|\lambda|\varepsilon^2}{(1+|y-\mathbf{x}_j|)^{N-\beta}} \\ &\leq C|\lambda|\varepsilon^2 \sum_{j=1}^k \frac{1}{(1+|y-\mathbf{x}_j|)^{\frac{N+2}{2}+\tau}}, \end{aligned}$$

and

$$\begin{aligned} & \sum_{j=1}^k \frac{1}{(1 + |y - \mathbf{x}_j|)^4} \varepsilon^{N-2} \leq C \varepsilon^{N-2-\frac{N-6}{2}-\tau} \sum_{j=1}^k \frac{1}{(1 + |y - \mathbf{x}_j|)^{\frac{N+2}{2}+\tau}} \\ & = C \varepsilon^{\frac{N+2}{2}-\tau} \sum_{j=1}^k \frac{1}{(1 + |y - \mathbf{x}_j|)^{\frac{N+2}{2}+\tau}} \leq C(k\varepsilon)^{\frac{N+2}{2}-\tau} \sum_{j=1}^k \frac{1}{(1 + |y - \mathbf{x}_j|)^{\frac{N+2}{2}+\tau}} \\ & \leq C(s\varepsilon^{\frac{N-2}{2}})^{\frac{1}{2}+\sigma} \sum_{j=1}^k \frac{1}{(1 + |y - \mathbf{x}_j|)^{\frac{N+2}{2}+\tau}}. \end{aligned}$$

Combining all the above estimates, we obtain the result. □

Now, we are ready to prove Proposition 2.4.

*Proof of Proposition 2.4.* Let us recall that

$$\varepsilon = \frac{s^{\frac{2}{N-2}}}{k^2}.$$

Let

$$E_N = \left\{ u : u \in C(\Omega_\varepsilon), \|u\|_* \leq \sqrt{s\varepsilon}^{\frac{N-2}{4}}, \int_{\Omega_\varepsilon} \sum_{i=1}^k Z_{i,j} u = 0, j = 1, 2 \right\}$$

Then, (2.19) is equivalent to

$$\phi = A(\phi) =: L(\bar{N}(\phi)) + L(l_k).$$

Now we prove that  $A$  is a contraction map from  $E_N$  to  $E_N$ . Using Lemma 2.5, we have

$$\begin{aligned} \|A\phi\|_* & \leq C\|\bar{N}(\phi)\|_{**} + C\|l_k\|_{**} \leq C\|\phi\|_*^{\min(2^*-1,2)} + C\|l_k\|_{**} \\ & \leq C(\sqrt{s\varepsilon}^{\frac{N-2}{4}})^{\min(2^*-1,2)} + C\|l_k\|_{**} \quad (2.33) \\ & \leq C(\sqrt{s\varepsilon}^{\frac{N-2}{4}})^{1+\sigma} + C\|l_k\|_{**}. \end{aligned}$$

Thus, by Lemma 2.6, we find that  $A$  maps  $E_N$  to  $E_N$ .

Next, we show that  $A$  is a contraction map.

$$\|A(\phi_1) - A(\phi_2)\|_* = \|L(\bar{N}(\phi_1)) - L(\bar{N}(\phi_2))\|_* \leq C\|\bar{N}(\phi_1) - \bar{N}(\phi_2)\|_{**}.$$

Using

$$|\bar{N}'(t)| \leq \begin{cases} C|t|^{2^*-2}, & N \geq 6; \\ C\left(W^{\frac{6-N}{N-2}}|\phi| + |\phi|^{2^*-2}\right), & N = 3, 4, 5, \end{cases}$$

we can prove that

$$\begin{aligned} \|A(\phi_1) - A(\phi_2)\|_* &\leq C \|\bar{N}(\phi_1) - \bar{N}(\phi_2)\|_{**} \\ &\leq C \left( \|\phi_1\|_*^{\min(1, 2^*-2)} + \|\phi_2\|_*^{\min(1, 2^*-2)} \right) \|\phi_1 - \phi_2\|_* \\ &\leq \frac{1}{2} \|\phi_1 - \phi_2\|_*. \end{aligned}$$

Thus,  $A$  is a contraction map.

It follows from the contraction mapping theorem that there is a unique  $\phi \in E_N$ , such that

$$\phi = A(\phi).$$

Moreover, it follows from (2.33) that

$$\|\phi\|_* \leq C(\sqrt{s\varepsilon}^{\frac{N-2}{4}})^{1+\sigma} + C\|I_k\|_{**}.$$

So, the estimate for  $\|\phi\|_*$  follows from Lemma 2.6. □

### 3. Proof of the main results

Let

$$F(r, \Lambda) = I(W_{r,\Lambda} + \phi),$$

where  $\phi$  is the function obtained in Proposition 2.4, and let

$$I(u) = \frac{1}{2} \int_{\Omega_\varepsilon} (|Du|^2 - \lambda\varepsilon^2 u^2) - \frac{1}{2^*} \int_{\Omega_\varepsilon} (u - s\Phi_\varepsilon)_+^{2^*}.$$

Using the symmetry, we can check that if  $(r, \Lambda)$  is a critical point of  $F(\Lambda)$ , then  $W_{r,\Lambda} + \phi$  is a solution of (1.3).

**Proposition 3.1.** *We have*

$$\begin{aligned} F(r, \Lambda) = k \left( A_0 + \frac{A_2 \bar{\varphi}(r) s \varepsilon^{\frac{N-2}{2}}}{\Lambda^{\frac{N-2}{2}}} - \frac{A_3 k^{N-2} \varepsilon^{N-2}}{r^{N-2} \Lambda^{N-2}} \right. \\ \left. + O\left( (s\varepsilon^{\frac{N-2}{2}})^{1+\sigma} + (k\varepsilon)^{(N-2)(1+\sigma)} \right) \right), \quad N = 3, 4; \end{aligned}$$

and

$$\begin{aligned} F(r, \Lambda) = k \left( A_0 - \frac{A_1 \lambda \varepsilon^2}{\Lambda^2} + \frac{A_2 \bar{\varphi}(r) s \varepsilon^{\frac{N-2}{2}}}{\Lambda^{(N-2)/2}} - \frac{A_3 k^{N-2} \varepsilon^{N-2}}{r^{N-2} \Lambda^{N-2}} \right. \\ \left. + O\left( |\lambda| \varepsilon^{2+\sigma} + \left( s \varepsilon^{\frac{N-2}{2}} \right)^{1+\sigma} + (k\varepsilon)^{(N-2)(1+\sigma)} \right) \right), \quad N \geq 5. \end{aligned}$$

where the constant  $A_i > 0, i = 0, 1, 2$  are positive constants, which are given in Proposition A.3.

*Proof.* There is  $t \in (0, 1)$ , such that

$$\begin{aligned}
 F(r, \Lambda) &= I(W_{r,\Lambda}) + \langle I'(W_{r,\Lambda}), \phi \rangle + \frac{1}{2} D^2 I(W_{r,\Lambda} + t\phi)(\phi, \phi) \\
 &= I(W_{r,\Lambda}) - \int_{\Omega_\varepsilon} l_k \phi + \int_{\Omega_\varepsilon} \left( |D\phi|^2 + \varepsilon^2 \mu \phi^2 - (2^* - 1) (W_{r,\Lambda} - s\Phi_\varepsilon + t\phi)_+^{2^*-2} \phi^2 \right) \\
 &= I(W_{r,\Lambda}) - (2^* - 1) \int_{\Omega_\varepsilon} \left( (W_{r,\Lambda} - s\Phi_\varepsilon + t\phi)_+^{2^*-2} - (W_{r,\Lambda} - s\Phi_\varepsilon)_+^{2^*-2} \right) \phi^2 \\
 &\quad + \int_{\Omega_\varepsilon} N(\phi)\phi \\
 &= I(W_{r,\Lambda}) - (2^* - 1) \int_{\Omega_\varepsilon} \left( (W_{r,\Lambda} - s\Phi_\varepsilon + t\phi)_+^{2^*-2} - (W_{r,\Lambda} - s\Phi_\varepsilon)_+^{2^*-2} \right) \phi^2 \\
 &\quad + O\left( \int_{\Omega_\varepsilon} |\bar{N}(\phi)| |\phi| \right).
 \end{aligned} \tag{3.1}$$

But

$$\begin{aligned}
 &\int_{\Omega_\varepsilon} |\bar{N}(\phi)| |\phi| \\
 &\leq C \|\bar{N}(\phi)\|_{**} \|\phi\|_* \int_{\Omega_\varepsilon} \sum_{j=1}^k \frac{1}{(1 + |y - \mathbf{x}_j|)^{\frac{N+2}{2} + \tau}} \sum_{i=1}^k \frac{1}{(1 + |y - \mathbf{x}_i|)^{\frac{N-2}{2} + \tau}}.
 \end{aligned} \tag{3.2}$$

Using Lemma B.2, we find

$$\begin{aligned}
 &\sum_{j=1}^k \frac{1}{(1 + |y - \mathbf{x}_j|)^{\frac{N+2}{2} + \tau}} \sum_{i=1}^k \frac{1}{(1 + |y - \mathbf{x}_i|)^{\frac{N-2}{2} + \tau}} \\
 &= \sum_{j=1}^k \frac{1}{(1 + |y - \mathbf{x}_j|)^{N+2\tau}} + \sum_{j=1}^k \sum_{i \neq j} \frac{1}{(1 + |y - \mathbf{x}_j|)^{\frac{N+2}{2} + \tau}} \frac{1}{(1 + |y - \mathbf{x}_i|)^{\frac{N-2}{2} + \tau}} \\
 &\leq \sum_{j=1}^k \frac{1}{(1 + |y - \mathbf{x}_j|)^{N+2\tau}} + C \sum_{j=1}^k \frac{1}{(1 + |y - \mathbf{x}_j|)^{N+\frac{1}{2}\tau}} \sum_{i=2}^k \frac{1}{|\mathbf{x}_i - \mathbf{x}_1|^{\frac{3}{2}\tau}} \\
 &\leq C \sum_{j=1}^k \frac{1}{(1 + |y - \mathbf{x}_j|)^{N+\frac{1}{2}\tau}},
 \end{aligned}$$

Thus, we obtain

$$\int_{\Omega_\varepsilon} |\bar{N}(\phi)| |\phi| \leq Ck \|\bar{N}(\phi)\|_{**} \|\phi\|_* \leq Ck \|\phi\|_*^2 \leq Ck \left( |\lambda| \varepsilon^{2+\sigma} + \left( s\varepsilon^{\frac{N-2}{2}} \right)^{1+\sigma} \right).$$

Now

$$\begin{aligned} & (W_{r,\Lambda} - s\Phi_\varepsilon + t\phi)_+^{2^*-2} - (W_{r,\Lambda} - s\Phi_\varepsilon)_+^{2^*-2} \\ &= \begin{cases} O\left(|\phi|^{2^*-2}\right), & N \geq 6; \\ O\left(W_{r,\Lambda}^{\frac{6-N}{N-2}}|\phi| + |\phi|^{2^*-2}\right), & N = 3, 4, 5. \end{cases} \end{aligned}$$

Thus, we have

$$\begin{aligned} & \left| \int_{\Omega_\varepsilon} \left( (W_{r,\Lambda} - s\Phi_\varepsilon + t\phi)^{2^*-2} \right) - \left( (W_{r,\Lambda} - s\Phi_\varepsilon)^{2^*-2} \right) \phi^2 \right| \\ & \leq C \|\phi\|_*^{2^*} \int_{\Omega_\varepsilon} \left( \sum_{j=1}^k \frac{1}{(1 + |y - \mathbf{x}_j|)^{\frac{N-2}{2} + \tau}} \right)^{2^*}, \end{aligned}$$

if  $N \geq 6$ . If  $N = 3, 4, 5$ , noting that  $N - 2 > \frac{N-2}{2} + \tau$ , we obtain

$$\begin{aligned} & \left| \int_{\Omega_\varepsilon} \left( (W_{r,\Lambda} - s\Phi_\varepsilon + t\phi)^{2^*-2} \right) - \left( (W_{r,\Lambda} - s\Phi_\varepsilon)^{2^*-2} \right) \phi^2 \right| \\ & \leq C \int_{\Omega_\varepsilon} W_{r,\Lambda}^{\frac{6-N}{N-2}} |\phi|^3 + C \int_{\Omega_\varepsilon} |\phi|^{2^*} \leq \|\phi\|_*^3 \int_{\Omega_\varepsilon} \left( \sum_{j=1}^k \frac{1}{(1 + |y - \mathbf{x}_j|)^{\frac{N-2}{2} + \tau}} \right)^{2^*}. \end{aligned}$$

Let  $\bar{\eta} > 0$  be small. Using Lemma B.2, if  $y \in \Omega_1$ , then

$$\begin{aligned} & \sum_{j=2}^k \frac{1}{(1 + |y - \mathbf{x}_j|)^{\frac{N-2}{2} + \tau}} \\ & \leq \sum_{j=2}^k \frac{1}{(1 + |y - \mathbf{x}_1|)^{\frac{N-2}{4} + \frac{1}{2}\tau}} \frac{1}{(1 + |y - \mathbf{x}_j|)^{\frac{N-2}{4} + \frac{1}{2}\tau}} \\ & \leq C \frac{1}{(1 + |y - \mathbf{x}_1|)^{\frac{N-2}{2} + \frac{1}{2}\bar{\eta}}} \sum_{j=2}^k \frac{1}{|\mathbf{x}_j - \mathbf{x}_1|^{\tau - \frac{1}{2}\bar{\eta}}} \leq C\varepsilon^{-\bar{\eta}} \frac{1}{(1 + |y - \mathbf{x}_1|)^{\frac{N-2}{2} + \frac{1}{2}\bar{\eta}}}. \end{aligned}$$

As a result,

$$\left( \sum_{j=1}^k \frac{1}{(1 + |y - \mathbf{x}_j|)^{\frac{N-2}{2} + \tau}} \right)^{2^*} \leq C\varepsilon^{-2^*\bar{\eta}} \frac{1}{(1 + |y - \mathbf{x}_1|)^{N+2^*\frac{1}{2}\bar{\eta}}}, \quad y \in \Omega_1.$$

Thus

$$\int_{\Omega_\varepsilon} \left( \sum_{j=1}^k \frac{1}{(1 + |y - \mathbf{x}_j|)^{\frac{N-2}{2} + \tau}} \right)^{2^*} \leq Ck\varepsilon^{-2^*\bar{\eta}}.$$



So, we have proved

$$\begin{aligned} & \left| \int_{\Omega_\varepsilon} \left( (W_{r,\Lambda} - s\Phi_\varepsilon + t\phi)^{2^*-2} \right) - \left( (W_{r,\Lambda} - s\Phi_\varepsilon)^{2^*-2} \right) \phi^2 \right| \\ & \leq Ck\varepsilon^{-2^*\bar{\eta}} \|\phi\|_*^{\min(3,2^*)} \leq Ck\varepsilon^{-2^*\bar{\eta}} \left( |\lambda|\varepsilon^{1+\sigma} + \left( s\varepsilon^{\frac{N-2}{2}} \right)^{\frac{1}{2}+\sigma} \right)^{\min(3,2^*)} \\ & \leq Ck \left( |\lambda|\varepsilon^{2+\sigma} + \left( s\varepsilon^{\frac{N-2}{2}} \right)^{1+\sigma} \right) \end{aligned} \tag{3.3}$$

Combining (3.1), (3.2) and (3.3), we find

$$F(r, \Lambda) = I(W_{r,\Lambda}) + kO \left( |\lambda|\varepsilon^{2+\sigma} + \left( s\varepsilon^{\frac{N-2}{2}} \right)^{1+\sigma} \right). \tag{3.4}$$

□

*Proof of Theorems 1.4, 1.5 and 1.6.* We just need to prove that  $F(r, \Lambda)$  has a critical point.

Firstly, we consider the cases  $N \neq 6$ . It follows from (3.4) and Proposition A.3 that

$$\begin{aligned} F(r, \Lambda) = k \left( A_0 + \frac{A_2\bar{\varphi}(r)s\varepsilon^{\frac{N-2}{2}}}{\Lambda^{(N-2)/2}} - \frac{A_3k^{N-2}\varepsilon^{N-2}}{r^{N-2}\Lambda^{N-2}} \right. \\ \left. + O \left( (k\varepsilon)^{(N-2)(1+\sigma)} + \left( s\varepsilon^{\frac{N-2}{2}} \right)^{1+\sigma} \right) \right). \end{aligned}$$

Let

$$\bar{F}(r, \Lambda) = \frac{A_2\bar{\varphi}(r)}{\Lambda^{(N-2)/2}} - \frac{A_3}{r^{N-2}\Lambda^{N-2}}, \quad (r, \Lambda) \in [r_1, r_2] \times [\delta, \delta^{-1}].$$

Then,  $\bar{F}(r, \Lambda)$  has a maximum point at  $(r_0, \Lambda_0)$ , where

$$\Lambda_0 = \left( \frac{2A_3}{A_2r_0^{N-2}\bar{\varphi}(r_0)} \right)^{\frac{2}{N-2}},$$

and  $r_0$  is a maximum point of  $r^{\frac{N-2}{2}}\bar{\varphi}(r) = r^{\frac{N-2}{2}}\varphi_1(r, 0)$ . So, if  $\delta > 0$  is small,  $(r_0, \Lambda_0)$  is an interior point of  $[r_1, r_2] \times [\delta, \delta^{-1}]$ . Thus, if  $k > 0$  is large,  $F(r, \Lambda)$  attains its maximum in the interior of  $[r_1, r_2] \times [\delta, \delta^{-1}]$ . As a result,  $F(r, \Lambda)$  has a critical point in  $[r_1, r_2] \times [\delta, \delta^{-1}]$ .

If  $N = 6$ , then

$$\begin{aligned} F(r, \Lambda) = k \left( A_0 + \frac{-\lambda A_1\varepsilon^2 + A_2\bar{\varphi}(r)s\varepsilon^2}{\Lambda^2} - \frac{A_3k^4\varepsilon^4}{r^4\Lambda^4} \right. \\ \left. + O \left( (k\varepsilon)^{4(1+\sigma)} + (s\varepsilon^2)^{1+\sigma} \right) \right). \end{aligned}$$

Let

$$\bar{F}(r, \Lambda) = \frac{-\lambda A_1 s^{-1} + A_2 \bar{\varphi}(r)}{\Lambda^2} - \frac{A_3}{r^4 \Lambda^4}, \quad (r, \Lambda) \in [r_1, r_2] \times [\delta, \delta^{-1}].$$

It is easy to see that there is an  $s_0 > 0$ , such that if  $s > |\lambda|s_0$ , then

$$\tilde{\varphi}(r) =: r^{\frac{N-2}{2}} \left( -\lambda A_1 s^{-1} + A_2 \bar{\varphi}(r) \right), \quad r \in [r_1, r_2]$$

has a maximum point  $r_0 \in (r_1, r_2)$  and  $\tilde{\varphi}(r_0) > 0$ . Then,  $\bar{F}(r, \Lambda)$  has a maximum point at  $(r_0, \Lambda_0)$ , where

$$\Lambda_0 = \left( \frac{2A_3}{r_0^4 \tilde{\varphi}(r_0)} \right)^{\frac{1}{2}}.$$

So, we can prove that  $F(r, \Lambda)$  has a critical point in  $[r_1, r_2] \times [\delta, \delta^{-1}]$ . □

### A. Appendix

In this section, we will expand  $I(W_{r,\Lambda})$ . We always assume that  $d(\bar{\mathbf{x}}_j, \partial\Omega) \geq c_0 > 0$ , where  $\bar{\mathbf{x}}_j = \varepsilon \mathbf{x}_j$ . Denote

$$\bar{\varphi}(r) = \varphi_1(r, 0).$$

First, let us recall that  $W_{\Lambda,\xi}$  is the solution of

$$\begin{cases} -\Delta W - \lambda \varepsilon^2 W = U_{\Lambda,\xi}^{2^*-1} & \text{in } \Omega_\varepsilon, \\ W = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases} \quad (\text{A.1})$$

Let

$$\psi_{\Lambda,\xi} = U_{\Lambda,\xi} - W_{\Lambda,\xi}.$$

Then,

$$\begin{cases} -\Delta \psi_{\Lambda,\xi} - \lambda \varepsilon^2 \psi_{\Lambda,\xi} = -\lambda \varepsilon^2 U_{\Lambda,\xi} & \text{in } \Omega_\varepsilon, \\ \psi_{\Lambda,\xi} = U_{\Lambda,\xi}, & \text{on } \partial\Omega_\varepsilon. \end{cases} \quad (\text{A.2})$$

To calculate  $I(W_{r,\Lambda})$ , we need to estimate  $\psi_{\Lambda,\xi}$ .

Decompose  $\psi_{\Lambda,\xi}$  as follows

$$\psi_{\Lambda,\xi} = \psi_{\Lambda,\xi,1} + \psi_{\Lambda,\xi,2},$$

where  $\psi_{\Lambda,\xi,1}$  is the solution of

$$\begin{cases} -\Delta \psi_{\Lambda,\xi,1} - \lambda \varepsilon^2 \psi_{\Lambda,\xi,1} = -\lambda \varepsilon^2 U_{\Lambda,\xi} & \text{in } \Omega_\varepsilon, \\ \psi_{\Lambda,\xi,1} = 0, & \text{on } \partial\Omega_\varepsilon, \end{cases} \quad (\text{A.3})$$

and  $\psi_{\Lambda,\xi,2}$  is the solution of

$$\begin{cases} -\Delta\psi_{\Lambda,\xi,2} - \lambda\varepsilon^2\psi_{\Lambda,\xi,2} = 0, & \text{in } \Omega_\varepsilon, \\ \psi_{\Lambda,\xi} = U_{\Lambda,\xi}, & \text{on } \partial\Omega_\varepsilon. \end{cases} \tag{A.4}$$

Since

$$U_{\Lambda,\xi} \leq C\varepsilon^{N-2}, \quad \text{on } \partial\Omega_\varepsilon,$$

it is easy to see that

$$|\psi_{\Lambda,\xi,2}| \leq C\varepsilon^{N-2}. \tag{A.5}$$

Let  $\bar{\psi}_{\Lambda,\xi,\varepsilon}$  be the solution of

$$\begin{cases} -\Delta\psi - \lambda\varepsilon^2\psi = U_{\Lambda,\xi} & \text{in } \Omega_\varepsilon, \\ \psi = 0, & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{A.6}$$

Then, we can check that

$$|\bar{\psi}_{\Lambda,\xi,\varepsilon}(y)| \leq \frac{C \ln^m(2 + |y - \xi|)}{(1 + |y - \xi|)^{N-4}}, \tag{A.7}$$

where  $m = 1$  if  $N = 4$ , otherwise,  $m = 0$ . Thus, we have

**Lemma A.1.** *We have*

$$\psi_{\Lambda,\xi} = -\lambda\varepsilon^2\bar{\psi}_{\Lambda,\xi,\varepsilon} + O(\varepsilon^{N-2}).$$

where  $\bar{\psi}_{\Lambda,\xi,\varepsilon}$  is the solution of (A.6). Moreover,

$$|W_{\Lambda,\xi}| \leq C |\ln \varepsilon|^m U_{\Lambda,\xi},$$

where  $m = 1$  if  $N = 4$ , otherwise,  $m = 0$ .

*Proof.* We only need to show

$$|W_{\Lambda,\xi}| \leq C |\ln \varepsilon|^m U_{\Lambda,\xi},$$

which follows from (A.7) and  $\varepsilon \leq \frac{C}{1+|y-\xi|}$ . □

**Proposition A.2.** *We have*

$$I(W_{\Lambda,x_j}) = A_0 + \frac{A_2\bar{\varphi}(r)s\varepsilon^{\frac{N-2}{2}}}{\Lambda^{\frac{N-2}{2}}} + O\left((s\varepsilon^{\frac{N-2}{2}})^{1+\sigma}\right), \quad N = 3, 4,$$

and

$$I(W_{\Lambda,x_j}) = A_0 - \frac{A_1\lambda\varepsilon^2}{\Lambda^2} + \frac{A_2\bar{\varphi}(r)s\varepsilon^{\frac{N-2}{2}}}{\Lambda^{\frac{N-2}{2}}} + O\left(|\lambda|\varepsilon^{2+\sigma} + \left(s\varepsilon^{\frac{N-2}{2}}\right)^{1+\sigma}\right), \quad N \geq 5;$$

where

$$A_0 = \frac{1}{2} \int_{\mathbb{R}^N} |DU|^2 - \frac{1}{2^*} \int_{\mathbb{R}^N} U^{2^*}, \quad A_2 = \int_{\mathbb{R}^N} U^{2^*-1},$$

$$A_1 = \frac{1}{2} \int_{\mathbb{R}^N} U^2, \quad N \geq 5,$$

and  $\sigma$  is some positive constant.

*Proof.* Write

$$I(u) = \tilde{I}(u) - \frac{1}{2^*} \int_{\Omega_\varepsilon} \left( (u - s\Phi_\varepsilon)_+^{2^*} - |u|^{2^*} \right),$$

where

$$\tilde{I}(u) = \frac{1}{2} \int_{\Omega_\varepsilon} |Du|^2 - \frac{1}{2} \lambda \varepsilon^2 \int_{\Omega_\varepsilon} u^2 - \frac{1}{2^*} \int_{\Omega_\varepsilon} |u|^{2^*}.$$

By Lemma A.1, we have

$$\begin{aligned} \tilde{I}(W_{\Lambda, \mathbf{x}_j}) &= \frac{1}{2} \int_{\Omega_\varepsilon} U_{\Lambda, \mathbf{x}_j}^{2^*-1} W_{\Lambda, \mathbf{x}_j} - \frac{1}{2^*} \int_{\Omega_\varepsilon} W_{\Lambda, \mathbf{x}_j}^{2^*} \\ &= A_0 + \frac{1}{2} \int_{\Omega_\varepsilon} U_{\Lambda, \mathbf{x}_j}^{2^*-1} \psi_{\Lambda, \mathbf{x}_j} + O\left(\int_{\Omega_\varepsilon} U_{\Lambda, \mathbf{x}_j}^{2^*-1-\sigma} \psi_{\Lambda, \mathbf{x}_j}^{1+\sigma}\right) \quad (\text{A.8}) \\ &= A_0 + \frac{1}{2} \int_{\Omega_\varepsilon} U_{\Lambda, \mathbf{x}_j}^{2^*-1} \psi_{\Lambda, \mathbf{x}_j} + O\left(|\lambda| \varepsilon^{2(1+\sigma)} + \varepsilon^{(N-2)(1+\sigma)}\right). \end{aligned}$$

On the other hand,

$$\begin{aligned} &\int_{\Omega_\varepsilon} (W_{\Lambda, \mathbf{x}_j} - s\Phi_\varepsilon)_+^{2^*} - \int_{\Omega_\varepsilon} (W_{\Lambda, \mathbf{x}_j})^{2^*} \\ &= -2^* \int_{R^N} U^{2^*-1} s \varepsilon^{\frac{N-2}{2}} \Lambda_j^{-\frac{N-2}{2}} \bar{\varphi}(r) + O\left((s \varepsilon^{\frac{N-2}{2}})^{1+\sigma}\right). \end{aligned} \quad (\text{A.9})$$

For  $N = 3, 4$ , by Lemma A.1 and (A.7),

$$\int_{\Omega_\varepsilon} U_{\Lambda, \mathbf{x}_j}^{2^*-1} \psi_{\Lambda, \mathbf{x}_j} = O(\varepsilon^{N-2} + \varepsilon^2) = O\left((s \varepsilon^{\frac{N-2}{2}})^{1+\sigma}\right). \quad (\text{A.10})$$

Here, we have used  $\varepsilon = \frac{s^2}{k^2} = \frac{1}{k} s \sqrt{\varepsilon} = (s \sqrt{\varepsilon})^{1+\sigma}$  if  $N = 3$ . So, the result for  $N = 3, 4$  follows from (A.8)–(A.10).

Suppose that  $N \geq 5$ . Let  $\psi_{\Lambda, \xi}$  be the solution of

$$\begin{cases} -\Delta \psi = U_{\Lambda, \xi} & \text{in } R^N, \\ \psi(|y|) \rightarrow 0, & \text{as } |y| \rightarrow +\infty. \end{cases} \quad (\text{A.11})$$

Then,

$$|\bar{\psi}_{\Lambda,\xi}| \leq \frac{C}{(1 + |y - \xi|)^{N-4}},$$

and

$$|\bar{\psi}_{\Lambda,\xi} - \bar{\psi}_{\Lambda,\xi,\varepsilon}| \leq \frac{C\varepsilon^2 \ln^m(2 + |y - \xi|)}{(1 + |y - \xi|)^{N-6}},$$

where  $m = 1$  if  $N = 6$ , otherwise,  $m = 0$ . Thus,

$$\begin{aligned} \int_{\Omega_\varepsilon} U_{\Lambda,\mathbf{x}_j}^{2^*-1} \psi_{\Lambda,\mathbf{x}_j} &= -\lambda\varepsilon^2 \int_{\mathbb{R}^N} U_{\Lambda,\mathbf{x}_j}^{2^*-1} \bar{\psi}_{\Lambda,\mathbf{x}_j} + O\left(\varepsilon^{N-2} + |\lambda|\varepsilon^4 |\ln \varepsilon|\right) \\ &= -\lambda\varepsilon^2 \int_{\mathbb{R}^N} U^2 + O\left(\varepsilon^{N-2} + |\lambda|\varepsilon^4 |\ln \varepsilon|\right). \end{aligned} \tag{A.12}$$

So we obtain the result for  $N \geq 5$ . □

**Proposition A.3.** *We have*

$$\begin{aligned} I(W_{r,\Lambda}) &= k \left( A_0 + \frac{A_2 \bar{\varphi}(r) s \varepsilon^{\frac{N-2}{2}}}{\Lambda^{\frac{N-2}{2}}} - \frac{A_3 k^{N-2} \varepsilon^{N-2}}{r \Lambda^{N-2}} \right. \\ &\quad \left. + O\left((k\varepsilon)^{(N-2)(1+\sigma)} + (s\varepsilon^{\frac{N-2}{2}})^{1+\sigma}\right) \right), \quad N = 3, 4; \end{aligned}$$

and

$$\begin{aligned} I(W_{r,\lambda}) &= k \left( A_0 - \frac{A_1 \lambda \varepsilon^2}{\Lambda^2} + \frac{A_2 \bar{\varphi}(r) s \varepsilon^{\frac{N-2}{2}}}{\Lambda^{(N-2)/2}} - \frac{A_3 k^{N-2} \varepsilon^{N-2}}{r^{N-2} \Lambda^{N-2}} \right. \\ &\quad \left. + O\left((k\varepsilon)^{(N-2)(1+\sigma)} + |\lambda|\varepsilon^{2+\sigma} + (s\varepsilon^{\frac{N-2}{2}})^{1+\sigma}\right) \right), \quad N \geq 5. \end{aligned}$$

*Proof.* By using the symmetry, we have

$$\begin{aligned} &\int_{\Omega_\varepsilon} |DW_{r,\Lambda}|^2 - \lambda\varepsilon^2 \int_{\Omega_\varepsilon} W_{r,\Lambda}^2 = \sum_{j=1}^k \sum_{i=1}^k \int_{\Omega_\varepsilon} U_{\Lambda,\mathbf{x}_i}^{2^*-1} W_{\Lambda,\mathbf{x}_j} \\ &= k \left( \int_{\Omega_\varepsilon} U_{\Lambda,\mathbf{x}_1}^{2^*} + \int_{\Omega_\varepsilon} U_{\Lambda,\mathbf{x}_1}^{2^*-1} \psi_{\Lambda,\mathbf{x}_1} + \sum_{i=2}^k \int_{\Omega_\varepsilon} U_{\Lambda,\mathbf{x}_1}^{2^*-1} U_{\Lambda,\mathbf{x}_i} \right. \\ &\quad \left. + O\left(\sum_{i=2}^k \frac{1}{|\mathbf{x}_i - \mathbf{x}_1|^{N-2+\sigma}}\right) \right) \tag{A.13} \\ &= k \left( \int_{\mathbb{R}^N} U^{2^*} + \int_{\Omega_\varepsilon} U_{\Lambda,\mathbf{x}_1}^{2^*-1} \psi_{\Lambda,\mathbf{x}_1} + \sum_{i=2}^k \frac{B_0}{\Lambda^{N-2} |\mathbf{x}_i - \mathbf{x}_1|^{N-2}} \right. \\ &\quad \left. + O\left(\sum_{i=2}^k \frac{1}{|\mathbf{x}_i - \mathbf{x}_1|^{N-2+\sigma}}\right) \right), \end{aligned}$$

where  $B_0 > 0$  is a constant.

Let

$$\Omega_j = \left\{ y = (y', y'') \in \Omega_\varepsilon : \left\langle \frac{y'}{|y'|}, \frac{\mathbf{x}_j}{|\mathbf{x}_j|} \right\rangle \geq \cos \frac{\pi}{k} \right\}.$$

Then,

$$|y - \mathbf{x}_i| \geq |y - \mathbf{x}_j|, \quad \forall y \in \Omega_j.$$

We have

$$\begin{aligned} & \frac{1}{2^*} \int_{\Omega_\varepsilon} (W_{r,\Lambda} - s\Phi_\varepsilon)_+^{2^*} = \frac{k}{2^*} \int_{\Omega_1} (W_{r,\Lambda} - s\Phi_\varepsilon)_+^{2^*} \\ & = \frac{k}{2^*} \left( \int_{\Omega_1} (W_{\Lambda, \mathbf{x}_1} - s\Phi_\varepsilon)_+^{2^*} + 2^* \int_{\Omega_1} \sum_{i=2}^k (W_{\Lambda, \mathbf{x}_1} - s\Phi_\varepsilon)_+^{2^*-1} W_{\Lambda, \mathbf{x}_i} \right. \\ & \quad \left. + O \left( \int_{\Omega_1} W_{\Lambda, \mathbf{x}_1}^{2^*-2} \left( \sum_{i=2}^k W_{\Lambda, \mathbf{x}_i} \right)^2 \right) \right) \\ & = \frac{k}{2^*} \left( \int_{\mathbb{R}^N} U^{2^*} - 2^* \int_{\Omega_\varepsilon} U_{\Lambda, \mathbf{x}_1}^{2^*-1} \psi_{\Lambda, \mathbf{x}_1} - \frac{2^* A_2 \bar{\varphi}(r) s \varepsilon^{\frac{N-2}{2}}}{\Lambda^{(N-2)/2}} \right. \\ & \quad \left. + 2^* \int_{\Omega_1} \sum_{i=2}^k U_{\Lambda, \mathbf{x}_1}^{2^*-1} U_{\Lambda, \mathbf{x}_i} + O \left( \int_{\Omega_1} U_{\Lambda, \mathbf{x}_1}^{2^*-2} s \Phi_\varepsilon \sum_{i=2}^k U_{\Lambda, \mathbf{x}_i} \right. \right. \\ & \quad \left. \left. + \int_{\Omega_1} U_{\Lambda, \mathbf{x}_1}^{2^*-2} \left( \sum_{i=2}^k U_{\Lambda, \mathbf{x}_i} \right)^2 + (s \varepsilon^{\frac{N-2}{2}})^{1+\sigma} + |\lambda| \varepsilon^{2+\sigma} \right) \right) \tag{A.14} \\ & = \frac{k}{2^*} \left( \int_{\mathbb{R}^N} U^{2^*} - 2^* \int_{\Omega_\varepsilon} U_{\Lambda, \mathbf{x}_1}^{2^*-1} \psi_{\Lambda, \mathbf{x}_1} - \frac{2^* A_2 \bar{\varphi}(r) s \varepsilon^{\frac{N-2}{2}}}{\Lambda^{\frac{N-2}{2}}} \right. \\ & \quad \left. + \sum_{i=2}^k \frac{2^* B_0}{\Lambda^{N-2} |\mathbf{x}_i - \mathbf{x}_1|^{N-2}} \right. \\ & \quad \left. + O \left( (k\varepsilon)^{(N-2)(1+\sigma)} + (s \varepsilon^{\frac{N-2}{2}})^{1+\sigma} + |\lambda| \varepsilon^{2+\sigma} \right) \right). \end{aligned}$$

Since

$$|\mathbf{x}_j - \mathbf{x}_1| = 2|\mathbf{x}_1| \sin \frac{2(j-1)\pi}{k}, \quad j = 2, \dots, k,$$

we can prove

$$\sum_{j=2}^k \frac{1}{|\mathbf{x}_j - \mathbf{x}_1|^{N-2}} = B_4(\varepsilon k)^{N-2} + O \left( (k\varepsilon)^{(1+\sigma)(N-2)} \right). \tag{A.15}$$

Thus, the result follows from (A.13), (A.14) and (A.15). □

**B. Appendix**

Firstly, we give a few lemmas, whose proof can be found in [35, 37].

**Lemma B.1.** *For any  $\alpha > 0$ ,*

$$\sum_{j=1}^k \frac{1}{(1 + |y - \mathbf{x}_j|)^\alpha} \leq C \left( 1 + \sum_{j=2}^k \frac{1}{|\mathbf{x}_1 - \mathbf{x}_j|^\alpha} \right),$$

where  $C > 0$  is a constant, independent of  $k$ .

For each fixed  $i$  and  $j, i \neq j$ , consider the following function

$$g_{ij}(y) = \frac{1}{(1 + |y - \mathbf{x}_j|)^\alpha} \frac{1}{(1 + |y - \mathbf{x}_i|)^\beta}, \tag{B.1}$$

where  $\alpha \geq 1$  and  $\beta \geq 1$  are two constants. Then, we have

**Lemma B.2.** *For any constant  $0 \leq \sigma \leq \min(\alpha, \beta)$ , there is a constant  $C > 0$ , such that*

$$g_{ij}(y) \leq \frac{C}{|\mathbf{x}_i - \mathbf{x}_j|^\sigma} \left( \frac{1}{(1 + |y - \mathbf{x}_i|)^{\alpha+\beta-\sigma}} + \frac{1}{(1 + |y - \mathbf{x}_j|)^{\alpha+\beta-\sigma}} \right).$$

**Lemma B.3.** *For any constant  $0 < \sigma < N - 2$ , there is a constant  $C > 0$ , such that*

$$\int_{\mathbb{R}^N} \frac{1}{|y - z|^{N-2}} \frac{1}{(1 + |z|)^{2+\sigma}} dz \leq \frac{C}{(1 + |y|)^\sigma}.$$

Let us recall that

$$\varepsilon = \frac{s^{\frac{2}{N-2}}}{k^2}.$$

For the constant  $\tau \in (0, 1)$  defined in (2.4),

$$\sum_{j=2}^k \frac{1}{|\mathbf{x}_j - \mathbf{x}_1|^\tau} \leq C\varepsilon^\tau k^\tau \sum_{j=2}^k \frac{1}{j^\tau} \leq C\varepsilon^\tau k \leq C,$$

and for any  $\theta > 0$ ,

$$\sum_{j=2}^k \frac{1}{|\mathbf{x}_j - \mathbf{x}_1|^{\tau+\theta}} = o(1).$$

**Lemma B.4.** *Suppose that  $N \geq 4$ . There is a small  $\theta > 0$ , such that*

$$\begin{aligned} & \int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-2}} W_{r,\Lambda}^{\frac{4}{N-2}}(z) \sum_{j=1}^k \frac{1}{(1+|z-\mathbf{x}_j|)^{\frac{N-2}{2}+\tau}} dz \\ & \leq C \sum_{j=1}^k \frac{1}{(1+|y-\mathbf{x}_j|)^{\frac{N-2}{2}+\tau+\theta}}, \end{aligned}$$

where  $W_{r,\Lambda}$  is defined in (1.7).

*Proof.* Recall that

$$\Omega_j = \left\{ y = (y', y'') \in \Omega_\varepsilon : \left\langle \frac{y'}{|y'|}, \frac{\mathbf{x}_j}{|\mathbf{x}_j|} \right\rangle \geq \cos \frac{\pi}{k} \right\}.$$

For  $z \in \Omega_1$ , we have  $|z-\mathbf{x}_j| \geq |z-\mathbf{x}_1|$ . Using Lemma B.2, we obtain

$$\begin{aligned} \sum_{j=2}^k \frac{1}{(1+|z-\mathbf{x}_j|)^{N-2-\beta}} & \leq \frac{1}{(1+|z-\mathbf{x}_1|)^{\frac{1}{2}(N-2-\beta)}} \sum_{j=2}^k \frac{1}{(1+|z-\mathbf{x}_j|)^{\frac{1}{2}(N-2-\beta)}} \\ & \leq \frac{C}{(1+|z-\mathbf{x}_1|)^{N-2-\beta-\tau}} \sum_{j=2}^k \frac{1}{|\mathbf{x}_j-\mathbf{x}_1|^\tau} \\ & \leq \frac{C}{(1+|z-\mathbf{x}_1|)^{N-2-\beta-\tau}}, \end{aligned}$$

Thus,

$$W_{r,\Lambda}^{\frac{4}{N-2}}(z) \leq \frac{C}{(1+|z-\mathbf{x}_1|)^{4-\frac{4(\tau+\beta)}{N-2}}}.$$

As a result, for  $z \in \Omega_1$ , using Lemma B.2 again, we find that for  $\theta > 0$  small,

$$W_{r,\Lambda}^{\frac{4}{N-2}}(z) \sum_{j=1}^k \frac{1}{(1+|z-\mathbf{x}_j|)^{\frac{N-2}{2}+\tau}} \leq \frac{C}{(1+|z-\mathbf{x}_1|)^{2+\frac{N-2}{2}+\tau+2-\tau-\frac{4(\tau+\beta)}{N-2}}}.$$

Since  $\theta =: 2-\tau-\frac{4(\tau+\beta)}{N-2} > 0$  if  $N \geq 4$  and  $\beta > 0$  is small, we obtain

$$\begin{aligned} & \int_{\Omega_1} \frac{1}{|y-z|^{N-2}} W_{r,\Lambda}^{\frac{4}{N-2}}(z) \sum_{j=1}^k \frac{1}{(1+|z-\mathbf{x}_j|)^{\frac{N-2}{2}+\tau}} dz \\ & \leq \int_{\Omega_1} \frac{1}{|y-z|^{N-2}} \frac{C}{(1+|z-\mathbf{x}_1|)^{2+\frac{N-2}{2}+\tau+\theta}} dz \leq \frac{C}{(1+|y-\mathbf{x}_1|)^{\frac{N-2}{2}+\tau+\theta}}, \end{aligned}$$



which gives

$$\begin{aligned} & \int_{\Omega_\varepsilon} \frac{1}{|y-z|^{N-2}} W_{r,\Lambda}^{\frac{4}{N-2}}(z) \sum_{j=1}^k \frac{1}{(1+|z-\mathbf{x}_j|)^{\frac{N-2}{2}+\tau}} dz \\ &= \sum_{i=1}^k \int_{\Omega_i} \frac{1}{|y-z|^{N-2}} W_{r,\Lambda}^{\frac{4}{N-2}}(z) \sum_{j=1}^k \frac{1}{(1+|z-\mathbf{x}_j|)^{\frac{N-2}{2}+\tau}} dz \\ &\leq \sum_{i=1}^k \frac{C}{(1+|y-\mathbf{x}_i|)^{\frac{N-2}{2}+\tau+\theta}}. \end{aligned}$$

□

The above proof does not work for  $N = 3$  because

$$2 - \tau - \frac{4\tau}{N-2} < 0 \tag{B.2}$$

if  $N = 3$  and  $\tau = \frac{1}{2}$ . The choice of  $\tau \in (0, 1)$  should ensure

$$\sum_{j=2}^k \frac{1}{|\mathbf{x}_j - \mathbf{x}_1|^\tau} \leq C\varepsilon^\tau k \leq C.$$

The above relation shows that  $\tau$  can be chosen smaller if  $\varepsilon$  becomes smaller, which in turn will make  $2 - \tau - \frac{4\tau}{N-2} > 0$ . Noting that  $\varepsilon = \frac{s^2}{k^2}$ , we find that if  $s \rightarrow 0+$ , then  $\varepsilon = o(\frac{1}{k^2})$ . We have

**Lemma B.5.** *Suppose that  $N = 3$ , the parameter  $s > 0$  and the integer  $k$  satisfy*

$$s \leq Ck^{-\frac{1}{2\tau}+1},$$

*for some  $\tau \in (0, \frac{2}{5})$ . Then, there is a small  $\theta > 0$ , such that*

$$\begin{aligned} & \int_{\mathbb{R}^3} \frac{1}{|y-z|} W_{r,\Lambda}^4(z) \sum_{j=1}^k \frac{1}{(1+|z-\mathbf{x}_j|)^{\frac{1}{2}+\tau}} dz \\ &\leq C \sum_{j=1}^k \frac{1}{(1+|y-\mathbf{x}_j|)^{\frac{1}{2}+\tau+\theta}}. \end{aligned}$$

*Proof.* The proof of this lemma is similar to that of Lemma B.4. We only need to use that for  $\tau < \frac{2}{5}$ ,

$$2 - 5\tau > 0,$$

and

$$\varepsilon^\tau k = s^{2\tau} k^{1-2\tau} \leq C.$$

Thus, we omit the details.

□

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