Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5)
Vol. VIII (2009), 51-71

# Stability estimates for certain Faber-Krahn, isocapacitary and Cheeger inequalities

NICOLA FUSCO, FRANCESCO MAGGI AND ALDO PRATELLI

**Abstract.** The first eigenvalue of the $p$-Laplacian on an open set of given measure attains its minimum value if and only if the set is a ball. We provide a quantitative version of this statement by an argument that can be easily adapted to treat also certain isocapacitary and Cheeger inequalities.

**Mathematics Subject Classification (2000):** 49R50(primary); 35J20, 49J40, 26D20 (secondary).

## 1. Introduction and statements

### 1.1. The first eigenvalue of the $p$-Laplacian

It was shown by Faber [11] and Krahn [21] that the first eigenvalue of the Laplacian on a bounded open set $\Omega \subset \mathbb{R}^2$ of given area attains its minimum value if and only if $\Omega$ is a disk. The popularity of this classical result is due in part to its simple, yet interesting, physical interpretation. Indeed, the aforementioned eigenvalue models the principal tone, *i.e.* the lowest frequency of free vibration, of a membrane of shape $\Omega$. The Faber-Krahn result states that the gravest principal tone is obtained in the case of a circular membrane, as conjectured by Lord Rayleigh back to 1877 in his treatise *"The theory of sound"* [29].

Regardless possible physical motivations, the above result was soon generalized to arbitrary dimension $n \geq 2$ by Krahn [22]. Yet another extension is valid, namely the Laplacian can be replaced by the $p$-Laplacian, for any $p \in (1, \infty)$. With modern tools at disposal, such as the theory of radially symmetric decreasing rearrangements, it is rather easy to justify these results.

To be more precise, given an open set $\Omega \subset \mathbb{R}^n$ with finite measure, the first eigenvalue of the $p$-Laplacian on $\Omega$ can be defined via a variational problem as

$$\lambda_p(\Omega) := \inf\left\{\int_\Omega |\nabla f|^p : \int_\Omega |f|^p = 1\,,\ f \in W_0^{1,p}(\Omega)\right\}.$$

The work of F. M. and A. P. was partially supported by the GNAMPA through the 2007 research project *Disuguaglianze geometrico-funzionali in forma ottimale e quantitativa*.
Received October 31, 2007; accepted in revised form July 9, 2008.

The scaling law of $\lambda_p$ is $\lambda_p(r\Omega) = r^{-p}\lambda_p(\Omega)$, $r > 0$. If $B$ denotes the open unit ball in $\mathbb{R}^n$ centered at the origin, and $|E|$ denotes the Lebesgue measure of the Borel set $E \subset \mathbb{R}^n$, then, as an immediate consequence of the classical Pólya-Szegö inequality, we have that

$$|\Omega|^{p/n}\lambda_p(\Omega) \geq |B|^{p/n}\lambda_p(B)\,. \tag{1.1}$$

This inequality is known as the *Faber-Krahn inequality*. The quantity $|\Omega|^{p/n}\lambda_p(\Omega)$ remains constant on rescaling the domain $\Omega$, and of course (1.1) implies that among open sets of given measure, balls minimize the first eigenvalue of the $p$-Laplacian. Since the unique non-negative eigenfunction on the ball is radially symmetric and strictly decreasing, the Brothers-Ziemer Theorem [5], concerning the cases of equality in the Pólya-Szegö inequality, immediately implies that equality holds in (1.1) if and only if $\Omega$ is a ball. Summarizing, there is attainment in the variational problem

$$\inf\{|\Omega|^{p/n}\lambda_p(\Omega) : \Omega \text{ is open in } \mathbb{R}^n, \text{ with } |\Omega| < \infty\}\,,$$

and the set of minimizers consists exactly of all balls of $\mathbb{R}^n$.

The first purpose of the present paper is to refine this last statement into a quantitative estimate. In other words, we aim to prove that the difference between the two sides of (1.1) controls the distance of $\Omega$ from a ball. More rigorously, we introduce the *$p$-Laplacian deficit* of $\Omega$

$$D_p(\Omega) := \frac{|\Omega|^{p/n}\lambda_p(\Omega)}{|B|^{p/n}\lambda_p(B)} - 1\,,$$

as a measure of how much $\Omega$ is far from realizing equality in (1.1). Next, following the terminology of Hall-Hayman-Weitsman [17], for every Borel set $E \subset \mathbb{R}^n$ with finite measure, we define the *Fraenkel asymmetry of $E$* as

$$A(E) := \inf\left\{\frac{|E\Delta(x_0 + rB)|}{|E|} : x_0 \in \mathbb{R}^n\,,\, r^n|B| = |E|\right\}\,.$$

This quantity can be seen as a natural notion of distance of $E$ from the set of all balls of $\mathbb{R}^n$ of measure $|E|$.

Inequality (1.1) and the characterization of its equality cases are equivalently formulated in terms of the set functions $D_p$ and $A$ by stating that: $D_p(\Omega) \geq 0$ for every open set $\Omega$ with finite measure, with $D_p(\Omega) = 0$ if and only if $A(\Omega) = 0$. To make these statements quantitative, we are going to prove (as a corollary of Theorem 1.1 below) the existence of a constant $C(n, p)$, depending on $n$ and $p$ only, such that

$$A(\Omega) \leq C(n, p)\, D_p(\Omega)^{1/(2+p)}\,, \tag{1.2}$$

or, equivalently,

$$|\Omega|^{p/n}\lambda_p(\Omega) \geq |B|^{p/n}\lambda_p(B)\left\{1 + \frac{A(\Omega)^{2+p}}{C(n, p)}\right\}\,.$$

Therefore the stability estimate (1.2) can be formulated as a lower bound on the first eigenvalue of the $p$-Laplacian of $\Omega$. In a similar way, inequalities (1.12) and (1.15) provide lower bounds on the $p$-capacity and the $m$-Cheeger constant of $\Omega$.

Quantitative versions of the Faber-Krahn inequality in the spirit of (1.2) have already been studied in the literature. In the planar case $n = 2$, under the assumption that $\Omega$ is bounded, inequality (1.2) has been proved with the exponent 3 in place of $2 + p$ by Bhattacharya [3]. On assuming convexity of $\Omega$, and without restrictions on dimension $n$, the stability problem has been considered by Melas [25]: under the convexity assumption, the notion of Fraenkel asymmetry is naturally replaced by a sort of "Hausdorff asymmetry", defined by

$$A_H(E) := \inf\left\{d_H(E, x_0 + rB) : x_0 \in \mathbb{R}^n\,, r^n|B| = |E|\right\}\,.$$

Here $d_H(E, F)$ denotes the Hausdorff distance between two bounded sets $E$ and $F$ of $\mathbb{R}^n$. Further results, specific to the case $p = 2$, and dealing with both the Fraenkel asymmetry and the Hausdorff asymmetry, are due to Hansen and Nadirashvili [18]. Besides these various contributions, to the best of our knowledge, an inequality valid for arbitrary values of $n$ and $p$, and on arbitrary domains $\Omega$, such as (1.2), was still missing.

## 1.2. Method of proof

Our approach to (1.2) develops around the classical theory of radially symmetric decreasing rearrangements, combined with a suitable quantitative version of the isoperimetric inequality, namely

$$P(E) \geq n|B|^{1/n}|E|^{(n-1)/n}\left\{1 + \frac{A(E)^2}{C(n)}\right\}\,. \tag{1.3}$$

Here $E$ is an arbitrary Borel set with $0 < |E| < \infty$, and $P(E)$ denotes the distributional perimeter of $E$ (recall that $P(E) = \mathscr{H}^{n-1}(\partial E)$ whenever the boundary of $E$ is at least Lipschitz regular). Inequality (1.3) has been proved on axially symmetric sets by Hall [16], and, in full generality, by the authors in [15]. Let us recall that a weaker version of (1.3), with the 4-th power of the Fraenkel asymmetry in place of its square (that in turn is the best possible exponent), was proved on arbitrary sets $E$ by Hall in [16], combining his result on axially symmetric sets with a symmetrization theorem from [17]. We also spot that many other interesting quantitative versions of the isoperimetric inequality are known, see [14, 26, 27].

As said, the framework in which we shall apply (1.3) is provided by the theory of radially symmetric decreasing rearrangements. Given a Borel function $f : \mathbb{R}^n \to [0, \infty)$ with $|\{f > t\}| < \infty$ for every $t > 0$, its *radially symmetric decreasing rearrangement* $f^\star : \mathbb{R}^n \to [0, \infty)$ is defined for $x \in \mathbb{R}^n$ by

$$f^\star(x) := \sup\{t > 0 : |\{f > t\}| > |B||x|^n\}\,.$$

In this way $\{f^\star > t\}$ is a ball centered at the origin, with measure equal to $|\{f > t\}|$. Therefore the operation of rearrangement preserves every $L^q$-norm. If furthermore $\nabla f \in L^p(\mathbb{R}^n, \mathbb{R}^n)$, then $\nabla f^\star \in L^p(\mathbb{R}^n, \mathbb{R}^n)$ and we have the *Pölya-Szegó inequality*

$$\int_{\mathbb{R}^n} |\nabla f|^p \geq \int_{\mathbb{R}^n} |\nabla f^\star|^p .$$

A classical proof (see [5, 30]) of this inequality is based on the Fleming-Rishel Coarea Formula [12] and on the isoperimetric inequality, applied to the level sets of $f$. As we shall see below, on replacing in this argument the standard isoperimetric inequality by its quantitative version (1.3), one comes to prove that

$$\int_0^\infty A(\{f > t\})^2 \frac{\mu(t)^{p/n'}}{|\mu'(t)|^{p-1}}\, dt \leq C(n, p) \left( \int_{\mathbb{R}^n} |\nabla f|^p - \int_{\mathbb{R}^n} |\nabla f^\star|^p \right), \quad (1.4)$$

where $\mu'(t)$ is the classical derivative, that exists for a.e. $t > 0$, of the non-increasing function $\mu(t) := |\{f > t\}|$. We will apply (1.4) to a non negative optimal function $f$ for the variational problem defining $\lambda_p(\Omega)$. By construction, $\int |\nabla f|^p = \lambda_p(\Omega)$. Furthermore $f^\star \in W_0^{1,p}(B)$ if, say, $|\Omega| = |B|$, and $1 = \int f^p = \int (f^\star)^p$. Therefore $\int |\nabla f^\star|^p \geq \lambda_p(B)$ and inequality (1.4) implies

$$\int_0^\infty A(\{f > t\})^2 \frac{\mu(t)^{p/n'}}{|\mu'(t)|^{p-1}}\, dt \leq C(n, p) D_p(\Omega) . \quad (1.5)$$

This shows that the $p$-Laplacian deficit controls, in a weighted form, the Fraenkel asymmetry of the level sets of $f$.

In order to come to (1.2), what is missing is a link between the Fraenkel asymmetry of $\Omega$ and the Fraenkel asymmetry of the level sets of the optimizer $f$. This link shall be provided in the form of the following inequality, in which $A(\Omega)$ is controlled in terms of the Fraenkel asymmetry of a generic level set of $f$, the height of the considered level set and the $p$-Laplacian deficit of $\Omega$. Namely, one has

$$A(\Omega) \leq C(n, p)\{t + A(\{f > t\}) + D_p(\Omega)\} . \quad (1.6)$$

The proof shall then be concluded by selecting via (1.5) a height $t$ such that $A(\{f > t\})$, and $t$ itself, are comparable with $D_p(\Omega)$.

### 1.3. Faber-Krahn type inequalities

The purely variational nature of our approach allows to treat, with basically the same effort, other interesting problems. The first situation we describe is that of the following family of Faber-Krahn type inequalities: let $n \geq 2$ and $p \in (1, \infty)$ be given as above, consider a further parameter $q$, obeying

$$\begin{cases} 1 \leq q < p^\star , \text{ if } 1 < p < n \text{ and } p^\star := np/(n-p) , \\ 1 \leq q < \infty , \text{ if } p \geq n , \end{cases} \quad (1.7)$$

and define the variational problems

$$\lambda_{p,q}(\Omega) := \inf\left\{\int_\Omega |\nabla f|^p : \int_\Omega |f|^q = 1\,,\ f \in W_0^{1,p}(\Omega)\right\}. \tag{1.8}$$

When $q = p$ we recover the case of the first eigenvalue of the $p$-Laplacian, but there are in fact other relevant cases: for example, when $n = 2$, $\lambda_{2,1}(\Omega)^{-1}$ is proportional to the *torsional rigidity* of $\Omega$ (see [28, page 87]).

By the Pólya-Szegö inequality, if $|\Omega| = |B|$ then $\lambda_{p,q}(\Omega) \geq \lambda_{p,q}(B)$. This last inequality can be set in a scale invariant form by noticing that $\lambda_{p,q}(r\Omega) = r^{-np\gamma}\lambda_{p,q}(\Omega)$ for every $r > 0$, where

$$\gamma := \frac{1}{n} + \frac{1}{q} - \frac{1}{p}\,. \tag{1.9}$$

Thus we come to the Faber-Krahn type inequality

$$|\Omega|^{p\gamma}\lambda_{p,q}(\Omega) \geq |B|^{p\gamma}\lambda_{p,q}(B)\,,$$

in which equality holds if and only if $\Omega$ is a ball. The related notion of deficit is of course given by

$$D_{p,q}(\Omega) := \frac{|\Omega|^{p\gamma}\lambda_{p,q}(\Omega)}{|B|^{p\gamma}\lambda_{p,q}(B)} - 1\,,$$

and the following theorem contains (1.2) as a particular case (*i.e.*, $q = p$):

**Theorem 1.1.** *Whenever $\Omega \subset \mathbb{R}^n$ is an open set with finite measure, then*

$$A(\Omega) \leq K(n, p, q) D_{p,q}(\Omega)^{1/(2+p)}\,, \tag{1.10}$$

*or, equivalently,*

$$|\Omega|^{p\gamma}\lambda_{p,q}(\Omega) \geq |B|^{p\gamma}\lambda_{p,q}(B)\left\{1 + \frac{A(\Omega)^{2+p}}{K(n, p, q)^{2+p}}\right\}.$$

*Here $K(n, p, q)$ denotes a constant depending only on $n$, $p$ and $q$, which is bounded if $p$ is bounded from above and $\gamma$ is bounded from below by a strictly positive constant.*

Notice that, by the definition (1.9), saying that $\gamma$ is bounded from below means that either $p > n$ or $p \leq n$ and $q$ is “far” from $p^\star$.

### 1.4. Isocapacitary inequalities

We now briefly introduce another family of inequalities that can be established in quantitative form along the same lines outlined above. For $n \geq 2$ and $1 < p < n$, define the *p-capacity* of an open set of finite measure $\Omega \subset \mathbb{R}^n$ as

$$\mathrm{Cap}_p(\Omega) := \inf\left\{\int_{\mathbb{R}^n} |\nabla f|^p : f \geq 1_\Omega\,,\ f \in L^{p^\star}(\mathbb{R}^n)\,,\ \nabla f \in L^p(\mathbb{R}^n, \mathbb{R}^n)\right\}.$$

We remark that slightly different definitions of $p$-capacity are commonly used, depending on the context of interest; for example the one adopted in [23, Section 11.15], and stemming from physical considerations, differs from the one used in [10, Section 4.7] in view of applications to Geometric Measure Theory. The main motivation of our choice relies in the fact that we are going to work with the Fraenkel asymmetry of $\Omega$.

Once again the Pólya-Szegö inequality implies that $\mathrm{Cap}_p(\Omega) \geq \mathrm{Cap}_p(B)$ whenever $|\Omega| = |B|$. As $\mathrm{Cap}_p(r\Omega) = r^{n-p}\mathrm{Cap}_p(\Omega)$, $r > 0$, one comes to the *isocapacitary inequality*

$$|\Omega|^{(p/n)-1}\mathrm{Cap}_p(\Omega) \geq |B|^{(p/n)-1}\mathrm{Cap}_p(B)\,,$$

in which equality is seen to hold if and only if $\Omega$ is a ball. Introducing the proper notion of deficit

$$E_p(\Omega) := \frac{|\Omega|^{(p/n)-1}\mathrm{Cap}_p(\Omega)}{|B|^{(p/n)-1}\mathrm{Cap}_p(B)} - 1\,,$$

we come to the corresponding quantitative isocapacitary inequality:

**Theorem 1.2.** *Whenever $\Omega \subset \mathbb{R}^n$ is an open set with finite measure, then*

$$A(\Omega) \leq K(n, p) E_p(\Omega)^{1/(2+p)}\,, \tag{1.11}$$

*or, equivalently,*

$$|\Omega|^{(p/n)-1}\mathrm{Cap}_p(\Omega) \geq |B|^{(p/n)-1}\mathrm{Cap}_p(B)\left\{1 + \frac{A(\Omega)^{2+p}}{K(n, p)^{2+p}}\right\}. \tag{1.12}$$

*Here $K(n, p)$ denotes a constant depending only on $n$ and $p$, which, for any $n \in \mathbb{N}$, is bounded from above if $p$ is bounded away from $n$.*

Various stability results are known for isocapacitary inequalities. Hall, Hayman and Weitsman [17] have proved an analogous estimate to (1.11) in the case $p = 2$, with the dimension dependent exponent $n + 1$ in place of $2 + p$ (note that they cover also the case $n = p = 2$, that is left out in our approach), see also [18]. Similar results, specialized to the case of planar domains ($n = 2$) are found in [4]. Once again, as far as we know, inequality (1.11) covers a wide range of previously untreated dimensions and exponents.

### 1.5. Cheeger constants

Given $m \in [1, n')$ and an open set of finite measure $\Omega$, we define the *$m$-Cheeger constant of* $\Omega$ as

$$h_m(\Omega) = \inf\left\{ \frac{P(A)}{|A|^{1/m}} : A \subseteq \Omega \text{ is open} \right\} ,$$

(since $h_{n'}(\Omega) = n|B|^{1/n}$ for every $\Omega$, we do not consider the case $m = n'$). The Cheeger constants are strictly related to the Faber-Krahn type variational problems (1.8), considered in the limit case $p \to 1^+$. As shown by Kawohl and Fridman [13],

$$\lim_{p\to 1^+} \lambda_{p,p}(\Omega) = h_1(\Omega) .$$

Therefore, on passing to the limit as $p \to 1^+$ in the Faber-Krahn inequality for $\lambda_{p,p}$, one comes to a corresponding inequality for Cheeger constants,

$$|\Omega|^{1/n} h_1(\Omega) \geq |B|^{1/n} h_1(B) . \tag{1.13}$$

Note that this method of proof does not lead to a characterization of the equality cases of (1.13). Since from the proof of Theorem 1.1 one can easily see that the constant $K(n, p, p)$ is bounded as $p \to 1^+$, then letting $p \to 1^+$ in (1.10) we derive a quantitative form of (1.13), that is,

$$A(\Omega)^3 \leq C(n) \left( \frac{|\Omega|^{1/n} h_1(\Omega)}{|B|^{1/n} h_1(B)} - 1 \right) .$$

In particular equality holds in (1.13) if and only if $\Omega$ is equivalent to a ball. The same argument works for the other values of $m$.

**Theorem 1.3.** *For every open set $\Omega$ with finite measure we have*

$$\lim_{p\to 1^+} \lambda_{p,p'm/(p'-m)}(\Omega) = h_m(\Omega) . \tag{1.14}$$

*Moreover,*

$$A(\Omega)^3 \leq C(n, m) \left( \frac{|\Omega|^{1/n+1/m-1} h_m(\Omega)}{|B|^{1/n+1/m-1} h_m(B)} - 1 \right) , \tag{1.15}$$

*and in particular $h_m(\Omega) \geq h_m(B)$ whenever $|\Omega| = |B|$.*

We conclude this introductory section by remarking that, in recent years, much work has been done in studying the existence and the properties of the Cheeger sets, that is, the subsets of $\Omega$ realizing the infimum in the definition of $h_m(\Omega)$ (see for instance [1, 6–8, 19, 20]). From this point of view, it is interesting to obtain bounds from below on the Cheeger constant $h_m(\Omega)$ for a given set $\Omega$ with $|\Omega| = |B|$; of course, the inequality (1.15) can be seen as a better and more precise estimate than the well known bound $h_m(\Omega) \geq h_m(B)$.

## 2. Proofs

This section is entirely devoted to the proofs of Theorem 1.1, Theorem 1.2 and Theorem 1.3. We carry on the proofs of the first two results at the same time, to stress the fact that the underlying argument is basically the same, while the proof of Theorem 1.3 is presented separately. In the following $\Omega$ shall always denote an open set with finite measure in $\mathbb{R}^n$, $n \geq 2$. The parameters $p$ and $q$ are fixed so that $1 < p < \infty$ and $q$ satisfies (1.7) when dealing with Theorem 1.1, while we ask $1 < p < n$ when proving Theorem 1.2. In the proofs of Theorems 1.1 and 1.2 several constants will appear, which have been numbered so to help the reader to keep trace of them. Notice that all these constants are bounded whenever $p$ and $q$ are in the range stated in Theorems 1.1 and 1.2.

**Step one.** We show that for every open set $\Omega$ with finite measure there exists a non negative minimizer in the variational problems defining $\lambda_{p,q}(\Omega)$ and $\mathrm{Cap}_p(\Omega)$.

*The Faber-Krahn case.* Let $\{f_h\}_{h\in\mathbb{N}}$ be a minimizing sequence for $\lambda_{p,q}(\Omega)$, *i.e.* $f_h \in W_0^{1,p}(\Omega)$, $\int_\Omega |f_h|^q = 1$ and $\int_\Omega |\nabla f_h|^p \to \lambda_{p,q}(\Omega)$. A straightforward application of the Direct Method suffices to prove the existence of a minimizer, once we show that the sequence $\{f_h\}_h$ is compact in $L^q$. To this end, thanks to the uniform bound on the $L^p$ norms of the gradients and to the restrictions on $p$ and $q$ set in (1.7), it will suffice to show that no $L^q$-mass is concentrated by $\{f_h\}_h$ at infinity, *i.e.* that for every $\varepsilon > 0$ there exists $R > 0$ such that

$$\sup_{h\in\mathbb{N}} \|f_h\|_{L^q(\mathbb{R}^n\setminus RB)} \leq \varepsilon\,.$$

When $1 < p < n$ we just note that

$$\|f_h\|_{L^q(\mathbb{R}^n\setminus RB)} \leq \|f_h\|_{L^{p^\star}(\mathbb{R}^n\setminus RB)}|\Omega\setminus RB|^{1/q-1/p^\star}\,,$$

so that, by the Sobolev inequality,

$$\sup_{h\to\infty} \|f_h\|_{L^q(\mathbb{R}^n\setminus RB)} \leq C(n,p)|\Omega\setminus RB|^{1/q-1/p^\star}\,,$$

The assertion follows since $|\Omega| < \infty$. The case $p \geq n$ is, of course, treated similarly. Once a minimizer $f$ is shown to exist, $|f|$ is a non negative minimizer.

*The isocapacitary case.* Let $\{f_h\}_{h\in\mathbb{N}}$ be a minimizing sequence for $\mathrm{Cap}_p(\Omega)$, so that $f_h \geq 1_\Omega$, $f_h \in L^{p^\star}(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} |\nabla f_h|^p \to \mathrm{Cap}_p(\Omega)$. Without loss of generality we may assume that $f_h$ converges weakly in $L^{p^\star}(\mathbb{R}^n)$ to a function $f \in L^{p^\star}(\mathbb{R}^n)$, with $\nabla f \in L^p(\mathbb{R}^n,\mathbb{R}^n)$ and, in fact, $\int_{\mathbb{R}^n} |\nabla f|^p \leq \mathrm{Cap}_p(\Omega)$. The convex constraint $f_h \geq 1_\Omega$ is stable under weak convergence, therefore $f \geq 1_\Omega$. Thus $f$ is optimal in $\mathrm{Cap}_p(\Omega)$.

**Step two.** First, in the proof of Theorems 1.1 and 1.2, it can be assumed that $|\Omega| = |B|$. Indeed, writing for simplicity $\delta(\Omega)$ to denote $D_{p,q}(\Omega)$ or $E_p(\Omega)$ depending whether we are in the Faber-Krahn or in the isocapacitary case, $\Omega \mapsto A(\Omega)$ and $\Omega \mapsto \delta(\Omega)$ are both scaling invariant set functions.

Second, if for a certain $\delta > 0$ there exists an exponent $\beta$ and a constant $C(\delta)$ such that

$$A(\Omega) \le C(\delta)\delta(\Omega)^{\beta}\,, \qquad \text{whenever } \delta(\Omega) < \delta, \tag{2.1}$$

then, on arbitrary sets $\Omega$, we have

$$A(\Omega) \le \max\left\{C(\delta), \frac{2}{\delta^{\beta}}\right\}\delta(\Omega)^{\beta}\,.$$

Indeed assume (2.1) has been proved, and consider $\Omega$ such that $\delta(\Omega) \ge \delta$. Then, as $A(\Omega)$ is always bounded by 2, we find $A(\Omega) \le 2 \le (2/\delta^{\beta})\delta(\Omega)^{\beta}$.

In view of these two remarks, in the following we shall always assume that $|\Omega| = |B|$, and that $D_{p,q}(\Omega)$ and $E_p(\Omega)$ are bounded by a certain parameter $\delta$ on which restrictions shall be imposed in the course of the proof.

**Step three.** We come now to derive estimates of the form (1.6). Let $f$ be a non negative optimal function for $\lambda_{p,q}(\Omega)$ or $\mathrm{Cap}_p(\Omega)$, depending on the context. We have to separate once again the exposition depending if we work with $\lambda_{p,q}$ or with $\mathrm{Cap}_p$, because in the first case the level sets $\{f > t\}$ that are supposed to be close to $\Omega$ are those for $t$ close to 0, in the second case are those with $t$ close to 1 (the difference between the two cases is actually more substantial: in the first case the relevant level sets are contained in $\Omega$, while in the second one they contain $\Omega$ instead).

*The Faber-Krahn case.* We show that for every $t > 0$ and a suitable $K_1(n, p, q)$ to be specified later, we have

$$A(\Omega) \le K_1(n, p, q)\Big(t + D_{p,q}(\Omega) + A(\{f > t\})\Big)\,, \qquad \forall t > 0\,, \tag{2.2}$$

with the convention that $A(\emptyset) := 0$. The inequality is trivial for $t \ge |B|^{-1/q}$ provided $K_1(n, p, q) \ge 2|B|^{1/q}$. Indeed in this case $A(\Omega) \le 2 \le K_1(n, p, q)|B|^{-1/q} \le K_1(n, p, q)t$. Notice that the inequality $K_1(n, p, q) \ge 2|B|^{1/q}$ is in turn ensured as soon as

$$K_1(n, p, q) \ge K_2 := 2\max\{\omega_n\,, n \in \mathbb{N}\} < +\infty\,,$$

denoting by $\omega_n$ the volume of $n-$dimensional unit ball. Notice that $K_2$ is independent of $n$, $p$ and $q$.

We are then left to consider the case $0 < t < |B|^{-1/q}$, in which $\{f > t\}$ is non empty, as the norm constraint $\|f\|_{L^q(\Omega)} = 1$ implies $\|f\|_{L^\infty(\Omega)} \ge |B|^{-1/q}$.

Let $x_0 \in \mathbb{R}^n$. Of course we have

$$\begin{aligned}
|B|A(\Omega) &\le |\Omega\Delta(x_0 + B)| = 2|(x_0 + B) \setminus \Omega| \\
&\le 2|(x_0 + B) \setminus \{f > t\}| \\
&\le 2|[x_0 + (B \cap \{f^\star \le t\})] \setminus \{f > t\}| + 2|(x_0 + \{f^\star > t\}) \setminus \{f > t\}| \\
&\le 2\Big(|B \cap \{f^\star \le t\}| + |(x_0 + \{f^\star > t\})\Delta\{f > t\}|\Big)\,.
\end{aligned}$$

As $x_0 + \{f^\star > t\}$ is a ball with the same measure as $\{f > t\}$, an optimization over $x_0$ leads to

$$A(\Omega) \le K_3(n)\Big(|B \cap \{f^\star \le t\}| + A(\{f > t\})\Big)\,, \tag{2.3}$$

where

$$K_3(n) = 2\max\left\{\frac{1}{\omega_n},\ 1\right\}\,.$$

In order to prove (2.2), we will first show that

$$|B \cap \{f^\star \le t\}| \le K_4(n,p,q)\{t + D_{p,q}(\Omega)\} \qquad \forall t \in (0, |B|^{-1/q})\,. \tag{2.4}$$

This is done as follows. Consider the comparison function $f_t$, defined by

$$f_t(x) := \max\{f^\star(x) - t, 0\}\,,$$

and let $r > 0$ be such that $\{f_t > 0\} = \{f^\star > t\} = (1-r)B$. Since $f_t \in W_0^{1,p}((1-r)B)$,

$$\begin{aligned}\frac{\lambda_{p,q}(B)}{(1-r)^{n\gamma p}} = \lambda_{p,q}((1-r)B) &\le \frac{\int_{(1-r)B}|\nabla f_t|^p}{\left(\int_{(1-r)B}|f_t|^q\right)^{p/q}} \le \frac{\int_B |\nabla f^\star|^p}{\left(\int_{(1-r)B}|f_t|^q\right)^{p/q}}\\ &\le \frac{\int_\Omega |\nabla f|^p}{\left(\int_{(1-r)B}|f_t|^q\right)^{p/q}} = \frac{\lambda_{p,q}(B)(1+D_{p,q}(\Omega))}{\left(\int_{(1-r)B}(f^\star - t)^q\right)^{p/q}}\,,\end{aligned}$$

*i.e.*

$$\left(\int_{(1-r)B}(f^\star - t)^q\right)^{1/q} \le (1-r)^{n\gamma}(1+D_{p,q}(\Omega))^{1/p}\,. \tag{2.5}$$

As $(1-r)^{n\gamma} \le 1 - \min\{n\gamma, 1\}r$, we have

$$\begin{aligned}(1-r)^{n\gamma}(1+D_{p,q}(\Omega))^{1/p} &\le \Big(1 - \min\{1,n\gamma\}r\Big)\left(1 + \frac{D_{p,q}(\Omega)}{p}\right)\\ &\le 1 + \frac{D_{p,q}(\Omega)}{p} - \min\{1,n\gamma\}\,r\,.\end{aligned} \tag{2.6}$$

On the other hand, by triangular inequality and also keeping in mind that $t < |B|^{-1/q}$,

$$\begin{aligned}\left(\int_{(1-r)B}(f^\star - t)^q\right)^{1/q} &\ge \left(\int_{(1-r)B}(f^\star)^q\right)^{1/q} - t|(1-r)B|^{1/q}\\ &\ge \left(\int_B (f^\star)^q - \int_{B\setminus(1-r)B}(f^\star)^q\right)^{1/q} - t|B|^{1/q}\\ &\ge \big(1 - t^q|B|\big)^{1/q} - t|B|^{1/q} \ge 1 - t^q|B| - t|B|^{1/q}\\ &\ge 1 - 2t|B|^{1/q}\,.\end{aligned} \tag{2.7}$$

From (2.5), (2.6) and (2.7) we deduce

$$r \leq \frac{1}{\min\{1, n\gamma\}} \left(\frac{1}{p} D_{p,q}(\Omega) + 2|B|^{1/q} t\right) \leq \frac{K_2}{\min\{1, n\gamma\}} \big(D_{p,q}(\Omega) + t\big) .$$

Since $|B \cap \{f^\star \leq t\}| = (1-(1-r)^n)|B| \leq n|B|r$, then (2.4) follows with

$$K_4(n, p, q) := \frac{n|B|K_2}{\min\{1, n\gamma\}} .$$

From (2.3) and (2.4) we derive that the searched inequality (2.2) is true for

$$K_1(n, p, q) \geq \max\Big\{K_3(n) \max\big\{K_4(n, p, q), 1\big\},\ K_2\Big\},$$

which in turn is ensured for

$$K_1(n, p, q) := \frac{c(n)}{\min\{1, n\gamma\}}$$

for a suitable constant $c(n)$.

*The isocapacitary case.* The optimal function $f$ for $\mathrm{Cap}_p(\Omega)$ satisfies of course $0 \leq f \leq 1$ on $\mathbb{R}^n$. Let us show that, provided $E_p(\Omega) \leq 1$, then, for every $t \in (1/2, 1)$,

$$A(\Omega) \leq K_5(n, p)\Big((1-t) + E_p(\Omega) + A(\{f > t\})\Big) \tag{2.8}$$

for a constant $K_5(n, p)$ to be found.

Let $x_0 \in \mathbb{R}^n$. Then,

$$\begin{aligned}|B|A(\Omega) &\leq 2|\Omega \setminus (x_0 + B)| \leq 2|\{f > t\} \setminus (x_0 + B)| \\ &\leq 2|\{f > t\} \setminus (x_0 + \{f^\star > t\})| + 2|\{f^\star > t\} \setminus B| .\end{aligned}$$

A minimization over $x_0$ leads to

$$A(\Omega) \leq \frac{2}{|B|} \Big\{|\{f > t\}|A(\{f > t\}) + |\{f^\star > t\} \setminus B|\Big\} . \tag{2.9}$$

Note that we cannot argue as in the Faber-Krahn case since in the present case we do not have the trivial bound $|\{f > t\}| \leq |B|$. Note also that the role of $|B \cap \{f^\star \leq t\}|$ is now played by $|\{f^\star > t\} \setminus B|$. We claim the following estimate for this last quantity:

$$|\{f^\star > t\} \setminus B| \leq K_6(n, p)\{(1-t) + E_p(\Omega)\} . \tag{2.10}$$

The proper comparison function used in this case is given by

$$f_t(x) := \min\{1, f^\star(x)/t\} .$$

If $r > 0$ is such that $\{f^\star > t\} = (1+r)B$, then $f_t \ge 1_{(1+r)B}$, with $f_t \in L^{p^\star}(\mathbb{R}^n)$ and $\nabla f_t \in L^p(\mathbb{R}^n, \mathbb{R}^n)$. Therefore, by the scaling law of the $p$-capacity

$$\begin{aligned}(1+r)^{n-p}\,\mathrm{Cap}_p(B) = \mathrm{Cap}_p((1+r)B) &\le \int_{\mathbb{R}^n} |\nabla f_t|^p \\ &\le \frac{1}{t^p}\int_{\mathbb{R}^n} |\nabla f^\star|^p \le \frac{1}{t^p}\int_{\mathbb{R}^n} |\nabla f|^p \\ &= \frac{\mathrm{Cap}_p(B)(1+E_p(\Omega))}{t^p}\,,\end{aligned}$$

*i.e.*

$$1 + r \le \frac{(1+E_p(\Omega))^{1/(n-p)}}{(1-(1-t))^{p/(n-p)}}\,.$$

Let us recall that $E_p(\Omega) \le 1$ and that $0 < s := (1-t) < 1/2$. By the elementary inequalities

$$\begin{aligned}(1+E)^{1/(n-p)} &\le 1 + \frac{\max\{1, 2^{-1+1/(n-p)}\}}{n-p}\,E\,, \quad \forall E \in (0,1)\,,\\ \frac{1}{(1-s)^{p/(n-p)}} &\le 1 + \frac{p\,2^{n/(n-p)}}{n-p}\,s\,, \qquad \forall s \in \left(0, \frac{1}{2}\right)\,,\end{aligned} \tag{2.11}$$

we eventually deduce

$$r \le K_7(n,p)\{(1-t) + E_p(\Omega)\}\,, \tag{2.12}$$

where $K_7(n,p)$ is immediately found by (2.11) and it is bounded, for any given $n \in \mathbb{N}$, if $p$ is bounded away from $n$. By (2.12) we know that $r \le 2K_7(n,p)$, and then

$$|\{f^\star > t\} \setminus B| = ((1+r)^n - 1)|B| \le K_8(n,p)r\,.$$

From this estimate and (2.12) we obtain the validity of (2.10) with

$$K_6(n,p) = K_7(n,p)K_8(n,p)\,.$$

By (2.10), one has

$$\begin{aligned}|\{f > t\}| = |\{f^\star > t\}| &\le |B| + K_6(n,p)\{(1-t) + E_p(\Omega)\}\\ &\le \big(|B| + K_6(n,p)\big)\{1 + E_p(\Omega)\} \le 2\big(|B| + K_6(n,p)\big)\,.\end{aligned}$$

Plugging this estimate and (2.10) into (2.9) we finally obtain the validity of (2.8) with

$$K_5(n,p) := \frac{2}{|B|}\big(2|B| + K_6(n,p)\big)\,,$$

a constant which, for any $n \in \mathbb{N}$, is bounded if $p$ is bounded away from $n$.

**Remark 2.1.** Note that (2.4) when applied to $\Omega = B$ provides a simple estimate on how $f_B$, the optimal function for $\lambda_{p,q}(B)$, detaches from the zero boundary value. In a similar way (2.10), when $\Omega = B$, contains a non trivial information on the $t$-level sets of the optimal function for $\mathrm{Cap}_p(B)$ for $t$ close to 1.

**Step four.** Let $f$ be a non negative optimal function for $\lambda_{p,q}(\Omega)$ or $\mathrm{Cap}_p(\Omega)$, respectively. We aim to show that

$$\int_0^\infty A(\{f > t\})^2 \frac{\mu(t)^{p/n'}}{|\mu'(t)|^{p-1}}\, dt \le C(n, p)\, \delta(\Omega)\,, \tag{2.13}$$

where $\mu(t) := |\{f > t\}|$ and for a suitable $C(n, p)$ to be found later.

As explained in the introduction, this is obtained by combining the Coarea Formula with the quantitative isoperimetric inequality. Let $\mu(t) := |\{f > t\}|$, $t > 0$, then $\mu$ is a non-increasing function of $t$. It is shown by the Coarea Formula (see [5, 9]) that the absolutely continuous part of the distributional derivative of $\mu$ has a (non positive) density $\mu'(t)$ satisfying

$$-\mu'(t) = \int_{\{f^\star = t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla f^\star|} \ge \int_{\{f=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla f|}\,, \text{ for a.e. } t \in (0, \|f\|_{L^\infty(\Omega)}). \tag{2.14}$$

Again by the Coarea Formula and the Hölder inequality we have

$$\begin{aligned}
\int_{\mathbb{R}^n} |\nabla f|^p &= \int_0^\infty dt \int_{\{f=t\}} |\nabla f|^{p-1}\, d\mathcal{H}^{n-1} \\
&\ge \int_0^\infty \frac{P(\{f > t\})^p}{\left(\int_{\{f=t\}} |\nabla f|^{-1}\, d\mathcal{H}^{n-1}\right)^{p-1}}\, dt \\
&\ge \int_0^\infty \frac{P(\{f > t\})^p}{|\mu'(t)|^{p-1}}\, dt\,.
\end{aligned} \tag{2.15}$$

Replacing $f$ with $f^\star$, one sees that equality holds both in (2.14) and in (2.15) (as $|\nabla f^\star|$ is constant on each $\{f^\star = t\}$). Therefore

$$\int_{\mathbb{R}^n} |\nabla f^\star|^p = \int_0^\infty \frac{P(\{f^\star > t\})^p}{|\mu'(t)|^{p-1}} dt\,. \tag{2.16}$$

Since $\{f^\star > t\}$ is a ball with the same volume as $\{f > t\}$, it follows from (2.15), (2.16) and the isoperimetric inequality that $\int |\nabla f|^p \ge \int |\nabla f^\star|^p$. On applying instead the *quantitative* isoperimetric inequality (1.3), we find

$$\int_{\mathbb{R}^n} |\nabla f|^p - \int_{\mathbb{R}^n} |\nabla f^\star|^p \ge \frac{p}{C(n)} \int_0^\infty A(\{f > t\})^2 \frac{\left(n|B|^{1/n}\mu(t)^{(n-1)/n}\right)^p}{|\mu'(t)|^{p-1}}\, dt \tag{2.17}$$

where the elementary inequality $a^p - b^p \geq p(a-b)b^{p-1}$, $a \geq b \geq 0$, has also been used. By definition one has

$$\int_{\mathbb{R}^n} |\nabla f|^p - \int_{\mathbb{R}^n} |\nabla f^\star|^p \leq \lambda_{p,q}(\Omega) - \lambda_{p,q}(B) = D_{p,q}(\Omega)\lambda_{p,q}(B)\,,$$
$$\int_{\mathbb{R}^n} |\nabla f|^p - \int_{\mathbb{R}^n} |\nabla f^\star|^p \leq \mathrm{Cap}_p(\Omega) - \mathrm{Cap}_p(B) = E_p(\Omega)\mathrm{Cap}_p(B)\,,$$

depending on which of the two theorems we are proving. By considering a given smooth function $f : B \to [0,1]$ such that $|\{f = 1\}| > 0$, it is immediate to deduce that

$$\lambda_{p,q}(B) \leq C_1(n, p)\,,$$

where, given $n \in \mathbb{N}$, $C_1(n, p)$ is bounded if $p$ is bounded. On the other hand, it is easy to remark that $\mathrm{Cap}_p(B) \leq C_2$ for all $1 < p < n$ and for all $n \geq 2$. From these considerations and (2.17) we immediately obtain (2.13) with

$$C(n, p) = \frac{C(n)C_1(n, p)}{pn^p|B|^{p/n}}$$

for the Faber–Krahn case, and

$$C(n, p) = \frac{C(n)C_2}{pn^p|B|^{p/n}}$$

for the isocapacitary case. Notice that, given $n \in \mathbb{N}$, in Faber–Krahn case the constant $C(n, p)$ is bounded for $p$ bounded, while in the isocapacitary case $C(n, p)$ is always bounded.

We are now in the position to conclude the proof of our theorems.

*Proof of Theorem* 1.1. Without loss of generality we may assume that $|\Omega| = |B|$ and that $D_{p,q}(\Omega) < \delta$ for $\delta < 1$ to be fixed later. We denote by $f$ a non negative optimal function for $\lambda_{p,q}(\Omega)$. From (2.4), if $0 < t < |B|^{-1/q}$,

$$\mu(t) = |\{f > t\}| = |B| - |B \cap \{f^\star \leq t\}| \geq |B| - K_4(n, p, q)\{t + D_{p,q}(\Omega)\}\,.$$

Setting

$$t_0 := \min\left\{|B|^{-1/q}, \frac{|B|}{4K_4(n, p, q)}\right\}, \qquad \delta \leq \frac{|B|}{4K_4(n, p, q)}\,, \tag{2.18}$$

we have

$$\mu(t) \geq \frac{|B|}{2}\,, \qquad \forall t \in (0, t_0)\,.$$

If we plug this into (2.13) we find

$$\int_0^{t_0} \frac{A(\{f > t\})^2}{|\mu'(t)|^{p-1}} dt \le C_3(n, p) D_{p,q}(\Omega)\,, \tag{2.19}$$

with

$$C_3(n, p) := \frac{2^{p/n'}}{|B|^{p/n'}}\, C(n, p)\,.$$

Let us now introduce two parameters $\sigma > 0$ and $\alpha > 0$, and define

$$I_1 := \left\{ t \in (0, t_0) : \frac{A(\{f > t\})^2}{|\mu'(t)|^{p-1}} \ge \sigma \right\}, \quad I_2 := \{t \in (0, t_0) : |\mu'(t)| \ge \sigma^{-\alpha}\}\,.$$

Thanks to (2.19) we have $\mathscr{H}^1(I_1) \le C_3(n, p) D_{p,q}(\Omega)/\sigma$, and moreover

$$\mathscr{H}^1(I_2) \le \int_{I_2} |\mu'(t)| \sigma^\alpha dt \le |B| \sigma^\alpha\,;$$

hence, the set $I := I_1 \cup I_2$ satisfies

$$\mathscr{H}^1(I) \le C_4(n, p) \left\{ \frac{D_{p,q}(\Omega)}{\sigma} + \sigma^\alpha \right\},$$

with

$$C_4(n, p) = \max\{C_3(n, p), |B|\}\,.$$

This suggests to chose $\sigma = D_{p,q}(\Omega)^{1/(1+\alpha)}$, and correspondingly

$$\mathscr{H}^1(I) \le 2C_4(n, p)\, D_{p,q}(\Omega)^{\alpha/(1+\alpha)}\,.$$

This last estimate tell us that, if $\delta$ is small enough depending on $n,\ p,\ q$ and $\alpha$, then

$$(0, t_0) \cap (0, 3C_4(n, p) D_{p,q}(\Omega)^{\alpha/(1+\alpha)}) \setminus I \text{ is non empty.}$$

More precisely, this happens if we choose

$$\delta := \min \left\{ \left( \frac{t_0}{2C_4(n, p)} \right)^{(1+\alpha)/\alpha}, \frac{|B|}{4K_4(n, p, q)}, 1 \right\}. \tag{2.20}$$

Furthermore, as $t$ does not belong to $I$, the Fraenkel asymmetry of $A(\{f > t\})$ is controlled by the deficit too, more precisely

$$A(\{f > t\})^2 \le \sigma |\mu'(t)|^{p-1} \le \sigma^{1-\alpha(p-1)} = D_{p,q}(\Omega)^{(1-\alpha(p-1))/(1+\alpha)}\,.$$

On applying (2.2) with such an height $t$ we find that

$$A(\Omega) \leq K_1(n, p, q)\Big(3C_4(n, p)D_{p,q}(\Omega)^{\alpha/(1+\alpha)} + D_{p,q}(\Omega) + D_{p,q}(\Omega)^{(1-\alpha(p-1))/2(1+\alpha)}\Big).$$

In conclusion, provided $\alpha = 1/(1+p)$ and $D_{p,q}(\Omega) < \delta$, we have

$$A(\Omega) \leq K_9(n, p, q)D_{p,q}(\Omega)^{1/(2+p)}$$

with $K_9(n, p, q)$ given by

$$K_9(n, p, q) = K_1(n, p, q)\big(3C_4(n, p) + 2\big).$$

As explained in step two, this concludes the proof of (1.10) with

$$K(n, p, q) = \max\left\{K_9(n, p, q),\ \frac{2}{\delta^{1/(2+p)}}\right\}.$$

To conclude the proof of Theorem 1.1, we only have to check that this constant $p$ is bounded whenever $p$ is bounded from above and $\gamma$ is bounded from below by a strictly positive constant. Concerning $K_9(n, p, q)$, this property has been already pointed out during the proof. We have then to exclude that $\delta^{1/(2+p)} \to 0$, which in turn may happen only if $\delta \to 0$. Recalling (2.20) and (2.18), we obtain that $\delta$ is bounded from above for $p$ and $\gamma$ bounded because in this range the constants $K_4(n, p, q)$ and $C(n, p)$ are bounded from above. The proof is then concluded. □

*Proof of Theorem 1.2.* Without loss of generality we can assume that $|\Omega| = |B|$ and that $E_p(\Omega) \leq \delta < 1/2$. Let $f$ be a non negative optimal function in $\mathrm{Cap}_p(\Omega)$, so that $0 \leq f \leq 1$. As $|\{f > t\}| \geq |B|$ for every $t \in (0, 1)$, from (2.13) we find

$$\int_{1/2}^{1} \frac{A(\{f > t\})^2}{|\mu'(t)|^{p-1}}\, dt \leq \frac{C(n, p)}{|B|^{p/n'}} E_p(\Omega).$$

Let $\sigma > 0$ and $\alpha > 0$ be two parameters to be chosen later, and let

$$I_1 := \left\{t \in (1/2, 1) : \frac{A(\{f > t\})^2}{|\mu'(t)|^{p-1}} \geq \sigma\right\}, \quad I_2 := \{t \in (1/2, 1) : |\mu'(t)| \geq \sigma^{-\alpha}\}.$$

Evidently

$$\mathscr{H}^1(I_1) \leq \frac{C(n, p)E_p(\Omega)}{|B|^{p/n'}\sigma}.$$

On the other hand, thanks to (2.10)

$$\int_{1/2}^{1} |\mu'(t)| dt = |\{f^\star > 1/2\} \setminus B| \leq K_6(n, p),$$

so that $\mathscr{H}^1(I_2) \leq K_6(n, p)\sigma^\alpha$. On putting things together,

$$\mathscr{H}^1(I_1 \cup I_2) \leq K_{10}(n, p) E_p(\Omega)^{\alpha/(1+\alpha)},$$

having defined

$$\sigma := E_p(\Omega)^{1/(1+\alpha)}, \qquad K_{10}(n, p) := \frac{C(n, p)}{|B|^{p/n'}} + K_6(n, p).$$

Let $I := I_1 \cup I_2$. If $\delta$ is small enough, then the set

$$\left[\left(\frac{1}{2}, 1\right) \cap \left(1 - 2K_{10}(n, p) E_p(\Omega)^{\alpha/(1+\alpha)}, 1\right)\right] \setminus I$$

is not empty. More precisely, this happens if we set

$$\delta := \min\left\{\left(\frac{1}{2K_{10}(n, p)}\right)^{(1+\alpha)/\alpha}, \frac{1}{2}\right\}.$$

To conclude, having set $\alpha = 1/(1+p)$, it suffices to consider a $t$ belonging to the set above and to apply (2.8) in order to prove (1.11) with the constant

$$K_5(n, p)\big(2K_{10}(n, p) + 2\big)$$

and under the additional assumption that $E_p(\Omega) < \delta$. Once again, as explained in step two, this implies (1.11) in its full generality, where the final constant is

$$K(n, p) := \max\left\{K_5(n, p)\big(2K_{10}(n, p) + 2\big), \frac{2}{\delta^{1/(2+p)}}\right\}.$$

Also in this case, it is easy to check that the constant $K(n, p)$ is bounded whenever $p$ is bounded away from $n$. □

We eventually prove Theorem 1.3.

*Proof of Theorem* 1.3. By the boundedness property of the constant $K(n, p, q)$ appearing in (1.10), we know that

$$\limsup_{p\to 1^+} K\left(n, p, \frac{mp'}{p'-m}\right) = C(n, m),$$

as $mp'/(p'-m) \to m < n' = 1^\star$ when $p \to 1^+$. Therefore in order to prove the theorem it is sufficient to prove (1.14).

We start by showing, in the spirit of the argument in [13], that

$$\limsup_{p\to 1^+} \lambda_{p,q(p)}(\Omega) \leq h_m(\Omega), \tag{2.21}$$

whenever $\lim_{p\to 1^+} q(p) = m$. Indeed, let $A$ be an open set compactly contained in $\Omega$. If $\rho \in C_c^\infty(B)$, with $0 \le \rho \le 1$ and $\int_B \rho = 1$, and $\rho_\varepsilon(z) := \varepsilon^{-n}\rho(z/\varepsilon)$, then the convolution product $f_\varepsilon = (1_A * \rho_\varepsilon)$ is such that $f_\varepsilon \to 1_A$ in $L^m(\mathbb{R}^n)$ and

$$\omega(\varepsilon) = \left| \int_{\mathbb{R}^n} |\nabla f_\varepsilon(x)| dx - P(A) \right| \to 0\,,$$

as $\varepsilon \to 0$. For every $\varepsilon > 0$ we have $|\nabla f_\varepsilon| \le C(\rho)/\varepsilon$, moreover $f_\varepsilon \in C_c^\infty(\Omega)$ as soon as $\varepsilon \le \varepsilon(A, \Omega)$. We deduce that

$$\begin{aligned}
\frac{P(A)}{|A|^{1/m}} &\ge \frac{\int_{\mathbb{R}^n} |\nabla f_\varepsilon| - \omega(\varepsilon)}{|A|^{1/m}} \\
&\ge \left(\frac{\varepsilon}{C(\rho)}\right)^{p-1} \frac{\int_{\mathbb{R}^n} |\nabla f_\varepsilon|^p}{|A|^{1/m}} - \frac{\omega(\varepsilon)}{|A|^{1/m}} \\
&\ge \left(\frac{\varepsilon}{C(\rho)}\right)^{p-1} \lambda_{p,q(p)}(\Omega) \frac{\left(\int_{\mathbb{R}^n} |f_\varepsilon|^{q(p)}\right)^{p/q(p)}}{|A|^{1/m}} - \frac{\omega(\varepsilon)}{|A|^{1/m}}\,.
\end{aligned}$$

We first pass to the limit $p \to 1^+$, and then let $\varepsilon \to 0$. Since $A$ was taken to be an arbitrary open set well contained in $\Omega$, we have proved (2.21).

We now show that

$$\lambda_{p,mp'/(p'-m)}(\Omega) \ge \left(\frac{p'-m}{p'}\, h_m(\Omega)\right)^p\,, \tag{2.22}$$

whenever $p < n$. Note that (1.14) follows at once from (2.21) and (2.22). We pass to prove (2.22). By the Coarea Formula, whenever $g \in C_c^\infty(\Omega)$, $g \ge 0$, we have

$$\int_{\mathbb{R}^n} |\nabla g| = \int_0^\infty P(\{g > t\}) dt \ge h_m(\Omega) \int_0^\infty |\{g > t\}|^{1/m} dt\,,$$

as the level sets of $g$ are open and compactly contained in $\Omega$. Notice that, by Fubini and Hölder inequality,

$$\begin{aligned}
\int_\Omega g(x)^m\, dx &= \int_0^{+\infty} \left(\int_{\{g>t\}} g(x)^{m-1}\, dx\right) dt \\
&\le \int_0^{+\infty} \left(\int_{\{g>t\}} g(x)^m\, dx\right)^{(m-1)/m} \big|\{g > t\}\big|^{1/m}\, dt \\
&\le \|g\|_{L^m(\Omega)}^{m-1} \int_0^{+\infty} \big|\{g > t\}\big|^{1/m}\, dt\,,
\end{aligned}$$

so that

$$\int_0^{+\infty} \big|\{g > t\}\big|^{1/m}\, dt \ge \|g\|_{L^m(\Omega)}\,.$$

Hence we deduce by density that for every $g \in W_0^{1,1}(\Omega)$, $g \ge 0$, it is

$$\int_{\mathbb{R}^n} |\nabla g| \ge h_m(\Omega)\|g\|_{L^m(\Omega)}\,.$$

We apply this inequality to $g = |f|^\tau$, for $\tau > 1$ to be chosen properly and a generic $f \in \mathrm{W}_0^{1,p}$, and find

$$\begin{aligned} h_m(\Omega)\left(\int_{\mathbb{R}^n} |f|^{m\tau}\right)^{1/m} &\le \tau \int_{\mathbb{R}^n} |f|^{\tau-1}|\nabla f| \\ &\le \tau \|\nabla f\|_{L^p(\mathbb{R}^n)} \left(\int_{\mathbb{R}^n} |f|^{p'(\tau-1)}\right)^{1/p'}\,. \end{aligned} \tag{2.23}$$

We ask $p'(\tau - 1) = m\tau$, therefore finding

$$\tau = \frac{p'}{p' - m}\,,$$

(note that, with this choice of $\tau$ we certainly have $g \in W_0^{1,1}(\Omega)$ since $m < n'$). Then, (2.23) reads as

$$h_m(\Omega) \le \frac{p'}{p' - m} \frac{\|\nabla f\|_{L^p(\mathbb{R}^n)}}{\left(\int_{\mathbb{R}^n} |f|^{mp'/(p'-m)}\right)^{(p'-m)/mp'}}\,,$$

and thus (2.22) follows and the proof of Theorem 1.3 is achieved. □

## References

[1] F. ALTER, V. CASELLES and A. CHAMBOLLE, *A characterization of convex calibrable sets in* $\mathbb{R}^n$, Math. Ann. **332** (2005), 329–366.

[2] M. BELLONI and B. KAWOHL, *A direct uniqueness proof for equations involving the p-Laplace operator*, Manuscripta Math. **109** (2002), 229–231.

[3] T. BHATTACHARYA, *Some observations on the first eigenvalue of the p-Laplacian and its connections with asymmetry*, Electron. J. Differential Equations **35** (2001), 15 pp.

[4] T. BHATTACHARYA and A. WEITSMAN, *Bounds for capacities in terms of asymmetry*, Rev. Mat. Iberoamericana **12** (1996), 593–639.

[5] J. BROTHERS and W. ZIEMER, *Minimal rearrangements of Sobolev functions*, J. Reine Angew. Math. **384** (1988), 153–179.

[6] G. BUTTAZZO, G. CARLIER and M. COMTE, *On the selection of maximal Cheeger sets*, Differential Integral Equations **20** (2007) 991–1004.

[7] G. CARLIER and M. COMTE, *On a weighted total variation minimization problem*, J. Funct. Anal. **250** (2007), 214–226.

[8] V. CASELLES, A. CHAMBOLLE and M. NOVAGA, *Uniqueness of the Cheeger set of a convex body*, Pacific J. Math. **232** (2007), 77–90.

[9] A. CIANCHI and N. FUSCO, *Functions of bounded variation and rearrangements*, Arch. Ration. Mech. Anal. **165** (2002), 1–40.

[10] L. C. EVANS and R. F. GARIEPY, "Measure Theory and Fine Properties of Functions", Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992.

[11] C. FABER, *Beweis dass unter allen homogen Membranen von gleicher Fläche und gleicher Spannung die kreisförmige den tiefsten Grundton gibt*, Sitzungsber. Bayer. Akad. der Wiss. Math.-Phys., Munich (1923), 169–172.

[12] W. H. FLEMING and R. RISHEL, *An integral formula for total gradient variation*, Arch. Math. (Basel) **11** (1960), 218–222.

[13] V. FRIDMAN and B. KAWOHL, *Isoperimetric estimates for the first eigenvalue of the $p$-Laplace operator and the Cheeger constant*, Comment. Math. Univ. Carolin. **44** (2003), 659–667.

[14] B. FUGLEDE, *Stability in the isoperimetric problem for convex or nearly spherical domains in* $\mathbb{R}^n$, Trans. Amer. Math. Soc. **314** (1989), 619–638.

[15] N. FUSCO, F. MAGGI and A. PRATELLI, *The quantitative sharp isoperimetric inequality*, Ann. of Math. **168** (2008), 941–980.

[16] R. R. HALL, *A quantitative isoperimetric inequality in $n$-dimensional space*, J. Reine Angew. Math. **428** (1992), 161–176.

[17] R. R. HALL, W.K. HAYMAN and A. W. WEITSMAN, *On asymmetry and capacity*, J. Anal. Math. **56** (1991), 87–123.

[18] W. HANSEN and N. NADIRASHVILI, *Isoperimetric inequalities in potential theory*, In: "Proceedings from the International Conference on Potential Theory" (Amersfoort, 1991), Potential Anal. **3** (1994), 1–14.

[19] B. KAWOHL and T. LACHAND-ROBERT, *Characterization of Cheeger sets for convex subsets of the plans*, Pacific J. Math. **225** (2006), 103–118.

[20] B. KAWOHL and M. NOVAGA, *The $p-$Laplace eigenvalue problem as $p \to 1$ and Cheeger sets in a Finsler metric*, J. Convex Anal. **15** (2008), 623–634.

[21] E. KRAHN, *Über eine von Rayleigh formulierte Minimaleigenschaft des Kreises*, Math. Ann. **94** (1925), 97–100.

[22] E. KRAHN, *Über Minimaleigenschaft des Kugel in drei und mehr Dimensionen*, Acta Comment Univ. Tartu Math. (Dorpat) **A9** (1926), 1–44.

[23] E. H. LIEB and M. LOSS, "Analysis", Second edition, Graduate Studies in Mathematics, Vol. 14, American Mathematical Society, Providence, RI, 2001.

[24] P. LINDQVIST, *On the equation* $\mathrm{div}\,(|\nabla u|^{p-1}\nabla u)+\lambda|u|^{p-2}u=0$, Proc. Amer. Math. Soc. **109** (1990), 157–164.

[25] A. MELAS, *The stability of some eigenvalue estimates*, J. Differential Geom. **36** (1992), 19–33.

[26] R. OSSERMAN, *The isoperimetric inequality*, Bull. Amer. Math. Soc. **84** (1978), 1182–1238.

[27] R. OSSERMAN, *Bonnesen-style isoperimetric inequalities*, Amer. Math. Monthly **86** (1979), 1–29.

[28] G. PÓLYA and G. SZEGÖ, "Isoperimetric Inequalities in Mathematical Physics", Annals of Mathematics Studies, Vol. 27, Princeton University Press, Princeton, NJ, 1951.

[29] J. W. STRUTT (Lord Rayleigh), "The Theory of Sound", MacMillan, New York, 1877, 1894; Dover, New York, 1945.
[30] G. TALENTI, *Best constants in Sobolev inequality*, Ann. Mat. Pura Appl. IV **110** (1976), 353–372.

Dipartimento di Matematica e Applicazioni
Via Cintia
80126 Napoli, Italia
nicola.fusco@unina.it

Dipartimento di Matematica
Viale Morgagni, 67/A
50134 Firenze, Italia
maggi@math.unifi.it

Dipartimento di Matematica
Via Ferrata, 1
27100 Pavia, Italia
aldo.pratelli@unipv.it