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On normal and non-normal holomorphic functions on complex Banach manifolds

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Abstract. L X b a c Ba ac a d. A c c $f : X \to \mathbb{C}$ ca d a a c a $\mathcal{F}_f = \{f \circ \varphi : \varphi \in \mathcal{O}(\Delta, X)\}$ a a a M ($\mathcal{O}(\Delta, X)$ d a c a c d c X). C a ac a a c a d. A c d a a c a d - a c b - a . C a a - c c b - a a b a d.

T a d d a c c b da b a a c c a c b d d d a D a c Ba ac ac V. L $\{x_n\}$ b a c D c d a b da $\xi \in \partial D$ c a $_{n\to\infty} f(x_n) = L$ $L \in \overline{\mathbb{C}}$. S c c d a c $\{x_n\}$ D a d a a c c f a f a ad b a L, d b a d b Ba a d S d .

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1. Introduction

T d a a ca a c c f c d c $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ a a $\mathcal{F} = \{f \circ g : g \in$ A $(\Delta)\}$, A (Δ) b c a Δ, a d a c b c f a $\mathcal{F}$ a a a Y da [16] 1934 a d c d d b N [14] 1937. I 1957, L a d V a [12] d d a c b c c f a c a d a $\mathcal{F}$ a M .

S c a , b c a c a b d d , b a a d c a ab c a d a a a c a ab (c [11, 13], a d [17]).

R c d Ja a 30, 2008; acc d d J 29, 2008.

I a , w a a c c c Ba-
ac a d . A a S c 2, S c 3 w b a c a
a b w a c , a d P- c . I S c-
4 w c a a c c b a - a c . I
S c 5 w fic c d a a c a d a
a c b a - a c . F a S c 6 w a
b da b a afi d b da a a c dfi d a
c b d d d a a c Ba ac ac .

2. Preliminaries

W ad a [4] a d b [3] a d [5] bac d
c a a fi d . T a w a a
a [4].

L X b a c Ba ac a d d d a c Ba ac ac
, b fi , d ; X a d b a c c d Ha d
ac . F ac x X a ac X a x w b d d b $T_x(X)$.
T a b d $T(X)$ X c d d a (y, v) c a $x \in X$
a d $v \in T_x(X)$. W a d ac a c a
d Δ X b $\mathcal{O}(\Delta, X)$.

T fi a K ba a d c c Ba ac a d X
c k_X $T(X)$ dfi d b a

$$k_X(x, v) = \quad \{|a| : \exists \varphi \in \mathcal{O}(\Delta, X),\ \varphi(0) = x,\ \varphi_*(0)a = v\}$$

w $\varphi_*(0)$ a a d c d b φ $T_0(\Delta)$ $T_{\varphi(0)}(X)$.

W a a k_X a c $k_X(x, v) > 0$ a $(x, v) \in T_x(X)$, $v \neq 0$.

T K ba a a c w C^1 c $\gamma : [0, 1] \to X$ X dfi d
b R a a

$$L_k(\gamma) = \overline{\int_0^1} k_X(\gamma(t), \gamma'(t))\, dt$$

a d d c $\widetilde{K}_X(x, y)$ fi a c w C^1
c x y X.

F z_1, w_1 Δ P ca d a c d b

$$d_\Delta(z_1, w_1) = \frac{1}{2} \quad \frac{1 + \left|\frac{z_1 - w_1}{1 - \bar{z}_1 w_1}\right|}{1 - \left|\frac{z_1 - w_1}{1 - \bar{z}_1 w_1}\right|} = \ \mathrm{a} \quad {}^{-1}\left(\left|\frac{z_1 - w_1}{1 - \bar{z}_1 w_1}\right|\right).$$

T K ba a d c b w w x, y X dfi d a w.
C d a fi c $p_0 = x, p_1, \ldots, p_{k-1}, p_k = y$ X c a

$z_1, \dots, z_k,\ w_1, \dots, w_k$ Δ a d a $\varphi_1, \dots, \varphi_k \in \mathcal{O}(\Delta, X)$ a $\varphi_j(z_j) = p_{j-1}$ a d $\varphi_j(w_j) = p_j$, $j = 1, \dots, k$.

T K ba a d c $K_X(x, y)$, b d ﬁ ,

$$K_X(x, y) = \sum_{j=1}^{k} d_\Delta(z_j, w_j)$$

w ﬁ a a b c c a d a .

T *spherical arc length element ds* R a $\overline{\mathbb{C}}$ b

$$ds(z, dz) = \frac{|dz|}{1 + |z|^2}.$$

T ca

$$s(\gamma) = \int_\gamma ds(z, dz)$$

a c γ $\overline{\mathbb{C}}$ d c a c w a .

G d c a, b R a , d ﬁ

$$s(a, b) = \{s(\gamma)\}$$

w ﬁ a a c w C^1 c $\overline{\mathbb{C}}$ w c a w b. T $s(a, b)$ d ﬁ a c w a ca c.

D b $\mathcal{O}(X)$ a c c X. A a $F \subseteq \mathcal{O}(\Delta)$ a d b a Δ c $\{f_n\} \subset F$ a a b c w c c (w c E c d a c) c ac a Δ d ∞ c ac a Δ.

Definition 2.1. A c f $\mathcal{O}(X)$ ca d a a c a $\mathcal{F} = \{f \circ \varphi \,:\, \varphi \in \mathcal{O}(\Delta, X)\}$ a .

A a $F \subseteq \mathcal{O}(\Delta)$ a d b ca c a a $z_0 \in \Delta$ ac b ϵ a b δ c a $s(f(z), f(z_0)) < \epsilon$ $K_\Delta(z, z_0) < \delta$ a d f F.

F w Ga [7] w a d ﬁ a c $\{x_n\}$ X b a P- c $f \in \mathcal{O}(X)$ a c $\{y_n\}$ X c a $K_X(x_n, y_n) \to 0$ a $n \to \infty$ b $s(f(x_n), f(y_n)) \geq \epsilon$ $\epsilon > 0$ ac n.

T P- c wa a d ﬁ d b Ga [6] - c c d Δ. T P- c wa - d c d b Ga [7] w c d a w a a c c .

T w a a a .

Lemma 2.2 (Zalcman's Lemma [15]). *Let $\mathcal{F}$ be a family of analytic functions in Δ. Then $\mathcal{F}$ is not normal in Δ if and only if there exist ( ) a number r with $0 < r < 1$; ( ) points z_n satisfying $|z_n| < r$; ( ) functions $f_n \in \mathcal{F}$; (iv) positive numbers $\rho_n \to 0$ as $n \to \infty$; such that*

$$f_n(z_n + \rho_n \xi) \to g(\xi) \ \text{ as } n \to \infty, \tag{2.1}$$

uniformly on compact subsets of $\mathbb{C}$, where g is a nonconstant entire function in $\mathbb{C}$. The function g may be taken to satisfy the normalization $g^\sharp(z) < g^\sharp(0) = 1$ $(z \in \mathbb{C})$.

H $g^\sharp(z)$ d ca d a

$$g^\sharp(z) = \frac{|g'(z)|}{1 + |g(z)|^2}.$$

Lemma 2.3. *Let f be a normal function on a complex Banach manifold X and suppose k_X is a metric. There exists a constant $c > 1$ such that*

$$(c\mu(f,x)) \le \quad (c\mu(f,y)) \quad ^{2K_X(x,y)} \ \text{for all } x, y \in X.$$

Here $\mu(f,x) :=$ a $\{1, |f(x)|\}$.

T w a d b Za d b [17] ca c a d . T [9, a 39] d d a c Ba ac a d ca .

3. Normality and P-sequences

I c w b a c a a b w a c- , a d P- c .

Theorem 3.1. *Let X be a complex Banach manifold and suppose k_X is a metric. The following statements are equivalent for $f \in \mathcal{O}(X)$:*

(a) *f is normal;*
(b) *there exists a constant $Q > 0$ such that*

$$Q_f(x) := \ _{v \in T_x(X) \setminus \{0\}} \frac{ds(f(x), f_*(x)v)}{k_X(x,v)} < Q \ \text{ for all } x \in X; \tag{3.1}$$

(c) *there exists a constant $L > 0$ such that*

$$s(f(x), f(y)) \le L \cdot K_X(x,y) \ \text{ for all } x, y \in X; \tag{3.2}$$

(d) *f has no P-sequence.*

Proof. (a) ⇒ (b): A a a $\mathcal{F} = \{f \circ \varphi : \varphi \in \mathcal{O}(\Delta, X)\}$ a a a . B Ma ’ [15, a 75], a c a $L > 0$ c a

$$\frac{|(f \circ \varphi)'(0)|}{1 + |f \circ \varphi(0)|^2} < L \quad \text{a} \quad \varphi \in \mathcal{O}(\Delta, X). \tag{3.3}$$

B d k_X $\psi \in \mathcal{O}(\Delta, X)$ c a $\psi(0) = x$, $\psi_*(0)a = v$ $a > 0$ a d $a/2 < k_X(x, v) \le a$. T , (3.3),

$$ds(f(x), f_*(x)v) < 2L \cdot k_X(x, v) \quad \text{a} \quad (x, v) \in T(X).$$

Na , $Q_f \le 2L$.

(b) ⇒ (c): L x a d y b d c X. I ad (3.1) a

$$ds(f(x), f_*(x)v) < Q \cdot k_X(x, v).$$

B a b d ab a a c C^1 c x y X a d b d a

$$s(f(x), f(y)) \le L \cdot \widetilde{K}_X(x, y).$$

S c K_X a d a c k_X ([4, C a 3]) a (3.2).

(c) ⇒ (d): T d a .

(d) ⇒ (a): I (d) d , a $\mathcal{F} = \{f \circ \varphi : \varphi \in \mathcal{O}(\Delta, X)\}$ c - a ac Δ, c a $z_0 \in \Delta$, $\epsilon > 0$, a c $\{z_n\}$ Δ $z_n \to z_0$, a d a c $\{f \circ \varphi_n\} \subseteq \mathcal{F}$ a

$$s(f \circ \varphi_n(z_n), f \circ \varphi_n(z_0)) \ge \epsilon, \quad n = 1, 2, \dots . \tag{3.4}$$

S c $K_\Delta = d_\Delta$, a d $z_n \to z_0$ a a ca [3, P 3.2] $K_\Delta(z_n, z_0) \to 0$, a $z_n \to z_0$. B c ac K ba a c,

$$K_X(\varphi(z_n), \varphi(z_0)) \le K_\Delta(z_n, z_0) \to 0, \text{ a } z_n \to z_0. \tag{3.5}$$

F (3.4), (3.5) a c $\{\varphi_n(z_0)\}$ X a P- c f c c ad c (d).

T , $\mathcal{F} = \{f \circ \varphi : \varphi \in \mathcal{O}(\Delta, X)\}$ ca c a a ac Δ, a d c , b M ’ [15, a 74], $\mathcal{F}$ a . T (a). □

Remark 3.2. Ha [8] a a b d a a c a ac a a a c ac b c a d M d . U a , Ha ’ a c c , b ca a a $\mathcal{F} = \{f \circ \varphi : \varphi \in \mathcal{O}(\Delta, M)\}$ a c a d a a *there exist an* $\epsilon > 0$ *such that for all* $n \in N$ *there exist sequences* $\{z_n\}$ *and* $\{w_n\}$ *in* Δ *with* $K_\Delta(z_n, w_n) < 1/n$ *but* $s(f(\psi(z_n)), f(\psi(w_n))) \ge \epsilon$ *for* ***some*** $\psi \in \mathcal{O}(\Delta, M)$ ([8, a 60]).

Theorem 3.3. *Let X and k_X be given as in Theorem* 3.1 *and let f in $\mathcal{O}(X)$. If $Q_f(x_m) \to \infty$ as $m \to \infty$, then $\{x_m\}$ contains a subsequence which is a P-sequence of f.*

Proof. S c $Q_f(x_m) \to \infty$ a $m \to \infty$, b (3.1) a c $\{v_m\}$, $v_m \in T_{x_m}(X)$, a d $\{r_m\} \subset R$, $r_m \to \infty$ a $m \to \infty$,

$$ds(f(x_m), f_*(x_m)v_m) > r_m \cdot k_X(x_m, v_m). \tag{3.6}$$

B d k_X $\psi_m \in \mathcal{O}(\Delta, X)$ c a $\psi_m(0) = x_m$, $\psi_{m*}(0)a_m = v_m$ $a_m > 0$ a d $a_m/2 < k_X(x_m, v_m) \leq a_m$. H c , a (3.6) ,

$$\frac{|(f \circ \psi_m)'(0)|}{1 + |f \circ \psi_m(0)|^2} > r_m \cdot \frac{k_X(x_m, v_m)}{a_m} > \frac{r_m}{2} \to \infty \text{ a } m \to \infty.$$

B Ma ' [15, a 75], a $\{f \circ \psi_m\} \subseteq \mathcal{O}(\Delta)$ a a d c $\Delta_{\frac{1}{n}} = \{z \in \mathbb{C} : |z| < \frac{1}{n}\}$. T b a ca ada a Za c a L a [15, a 152], a b c $\{f \circ \varphi_n\}$ $\{f \circ \psi_m\}$, $z_n \to 0$, $\rho_n \to 0^+$ a d a c a c g $f \circ \varphi_n(z_n + \rho_n \xi) \to g(\xi)$ a $\mathbb{C}$. Pa a b c c a , ca a a $\{f \circ \varphi_n(0)\}$ c a $\beta \in \overline{\mathbb{C}}$. S $g_n(\xi) = f \circ \varphi_n(z_n + \rho_n \xi)$. T c c $\{g_n\}$ c ca g. L α b a c b , $\alpha \neq \beta$, c a $g(\xi) = \alpha$ a a ξ_0 c a , a , $g^\sharp(\xi_0) \neq 0$. B a H [15, a 9], ac b d ξ_0 a b a b c $\{g_n\}$ a a α. T a c $\{\xi_n\} \subseteq \mathbb{C}$ c a $\xi_n \to \xi_0$ a d $g_n(\xi_n) = \alpha$ n c a . I $s(f \circ \varphi_n(z_n + \rho_n \xi_n), f \circ \varphi_n(0)) \geq s(\alpha, \beta)/2 > 0$ $n \geq N_0$. T P ca c d_Δ c d E c d a c a 0 $d_\Delta(0, z_n + \rho_n \xi_n) \to 0$ a $n \to \infty$. H c $K_X(\varphi_n(z_n + \rho_n \xi_n), \varphi_n(0)) \leq d_\Delta(z_n + \rho_n \xi_n, 0) \to 0$ a $n \to \infty$. W c a a b c $\{\varphi_n(0)\}_{n=N_0}^{\infty}$ $\{x_m\}$ a P- c f. T c . ☐

Remark 3.4. T c T 3.3 a , a a [2, E a (4.3)] . L $X = \Delta$, $f(z) = \frac{i}{1-z} \in \mathcal{O}(\Delta)$, $z_n = \frac{n^2}{1+n^2} - \frac{i}{n+n^3}$, $w_n = \frac{n^2}{1+n^2}$. H $Q_f(w_n) \to \infty$, a d c b T 3.3 c $\{w_n\}$ c a a b c $\{w_m\}$ c a P- c f. S c $K_\Delta(z_m, w_m) \to 0$ a b c $\{z_m\} \subseteq \{z_n\}$ a P- c f , c $Q_f(z_m) \to 0$.

4. Criteria for non-normality

T Za c a L a [18] c a ac a a c c a d a a d a d c a- .

A a b - a c c a d c a P- c ab d .

Theorem 4.1. *Let X and k_X be given as in Theorem 3.1. A function $f \in \mathcal{O}(X)$ is not a normal function on X iff f has a P-sequence.*

Proof. I f a a $\mathcal{F} = \{f \circ \varphi : \varphi \in \mathcal{O}(\Delta, X)\}$ a Δ. B Za c a ' L a, ca d $f \circ \varphi_n \in \mathcal{H}$, $z_n \subseteq \Delta_r$, $z_n \to z_0$, a d $\rho_n \to 0^+$ c a $f \circ \varphi_n(z_n + \rho_n \xi) \to g(\xi)$ c ac a, g a c a c $\mathbb{C}$. S c g c a $\mathbb{C}$ $\zeta \in \mathbb{C}$ c a $g(0) \neq g(\zeta)$. O a c a

$$\underset{n\to\infty}{} s(f \circ \varphi_n(z_n), f \circ \varphi_n(z_n + \rho_n \zeta)) = s(g(0), g(\zeta)) \geq \epsilon \qquad \epsilon > 0,$$

a d $K_X(\varphi_n(z_n), \varphi_n(z_n + \rho_n \zeta)) \leq d_\Delta(z_n, z_n + \rho_n \zeta) \to 0$ a $n \to \infty$. H c a c $\{\varphi_n(z_n)\}$ X a P- c f.

C , f a a P- c $\{x_n\}$, a c $\{y_n\}$ X c a $_{n\to\infty} K_X(x_n, y_n) = 0$ b

$$s(f(x_n), f(y_n)) \geq \epsilon \qquad \epsilon > 0, n = 1, 2, \ldots . \tag{4.1}$$

S a f a a c . B (3.2)

$$s(f(x_n), f(y_n)) \leq L \cdot K_X(x_n, y_n) \qquad \text{a} \quad n \geq 1.$$

I $s(f(x_n), f(y_n)) \to 0$ a $n \to \infty$ c c ad c (4.1). T f a a c . T . □

A c c d - a a c c Δ b La a [10, L a 3]. F c c a c Ba ac a d a .

Theorem 4.2. *Let X and k_X be given as in Theorem 3.1. The following statements are equivalent for $f \in \mathcal{O}(X)$:*

(a) *f is not a normal function;*
(b) *There exist sequences $\{y_m\}$, $\{x_m\}$ in X, and a constant $M > 0$ such that $K_X(y_m, x_m) < M$ for all $m \geq 1$, $_{m\to\infty} f(x_m) = \infty$, and $_{m\to\infty} f(y_m) = a \in \mathbb{C}$.*

Proof. (a)⇒ (b): S c h a a a $\mathcal{F} = \{f \circ \varphi : \varphi \in \mathcal{O}(\Delta, X)\}$ a Δ. A Za c a L a $\mathcal{F}$ r, z_n, ρ_n, $f \circ \varphi_n$, a d g a . S c g c a c $\mathbb{C}$ a a c $\{\xi_m\} \subset \mathbb{C}$ c a $|g(\xi_m)| > m$. I ac , ca , g b d d $\mathbb{C}$. B L ' $g \equiv$ c a , a c ad c w a a g c a c . F d ξ_m c n_m a a

$$\begin{aligned} &(\ ) \ \ |z_{n_m} + \rho_{n_m}\xi_m| < (1+r)/2;\\ &(\ ) \ \ |f \circ \varphi_{n_m}(z_{n_m} + \rho_{n_m}\xi_m)| > m/2. \end{aligned}$$

P $w_{n_m} = z_{n_m} + \rho_{n_m}\xi_m$, $x_m = \varphi_{n_m}(w_{n_m})$, a d $y_m = \varphi_{n_m}(z_{n_m})$.
B c ac K ba a c,

$$K_X(y_m, x_m) = K_X(\varphi_{n_m}(w_{n_m}), \varphi_{n_m}(z_{n_m})) \le d_\Delta(w_{n_m}, z_{n_m}).$$

B a a $d_\Delta(w_{n_m}, z_{n_m}) \le d_\Delta(0, z_{n_m}) + d_\Delta(0, w_{n_m})$. F a $z \in \Delta$ P ca d a c

$$d_\Delta(0,z) = \frac{1}{2} \quad \frac{1+|z|}{1-|z|}.$$

S c $|z_{n_m}| < r$, a d $|w_{n_m}| < (1+r)/2$, w a

$$d_\Delta(w_{n_m}, z_{n_m}) \le \frac{1}{2}\Big(\quad ((1+r)/(1-r)) + \quad ((3+r)/(1-r))\Big).$$

P ab w $K_X(y_m, x_m) \le M$ w

$$2M = \quad ((1+r)/(1-r)) + \quad ((3+r)/(1-r)) < \infty.$$

B ( ), w a a

$$\underset{m\to\infty}{} f(x_m) = \underset{m\to\infty}{} f \circ \varphi_{n_m}(z_{n_m} + \rho_{n_m}\xi_m) = \infty.$$

B (2.1), w a a

$$\underset{m\to\infty}{} f(y_m) = \underset{m\to\infty}{} f \circ \varphi_{n_m}(z_{n_m} + \rho_{n_m}0) = g(0) \in \mathbb{C}.$$

(b)⇒ (a): A , a c ad c , a f a . B L a 2.3,

$$(c\mu(f, x_m)) \le \quad (c\mu(f, y_m)) \quad ^{2K_X(x_m, y_m)} \qquad \text{a} \quad m \ge 1.$$

T - a d d ab a d a $m \to \infty$ w - a d d d a b w c a (c a $\{1, |a|\}$) ^{2r} a d w a a c ad c . T c ad c . ☐

5. Further results

Theorem 5.1. *Let X and k_X be given as in Theorem* 3.1. *If $f \in \mathcal{O}(X)$ is a normal function and if h is a non-normal holomorphic function on X such that each sequence $\{x_m\}$ of points in X contains a subsequence $\{x_{m_k}\}$ on which at most one of f or h is unbounded, then $f + h$ is a non-normal function.*

Proof. S c h a X a a $\mathcal{H} = \{h \circ \varphi : \varphi \in \mathcal{O}(\Delta, X)\}$ a Δ. B Za c a L a, ca d $h \circ \varphi_n \in \mathcal{H}$, $z_n \subseteq \Delta_r$, a d $\rho_n \to 0^+$ c a $h \circ \varphi_n(z_n + \rho_n \xi) \to g(\xi)$ c ac a, g a c a c $\mathbb{C}$. B

$$\rho_n \frac{|h_*(\varphi_n(z_n + \rho_n\xi))\varphi_n'(z_n + \rho_n\xi)|}{1 + |h(\varphi_n(z_n + \rho_n\xi))|^2} \to g^\sharp(\xi).$$

W c a a a c $\{\xi_m\} \subset \mathbb{C}$ c a $|g(\xi_m)| > m$ a d $g'(\xi_m) \neq 0$. L .

1. I $g'(\xi) \neq 0$ a d g b d d $\mathbb{C}$ b L ' $g \equiv$ c a , a c ad c a a g c a c .
2. L $\{a_m\}$ b $g'(\xi)$. S a $|g(\xi)| < L < \infty$ $\mathbb{C} \setminus \{\xi \in \mathbb{C} : g'(\xi) = 0\}$. T $g'(\xi)$ a a d. S c c a c c a ca a a, c c d a $|g(a_m)| < L$. T g b d d $\mathbb{C}$, c c a b L ' . B c ad c a a g c a c , a d a d.

F d ξ_m c n_m a a

( ) $|\lambda_{n_m}| < (1 + r)/2$;

( ) $g^\sharp(\xi_m)/\rho_{n_m} > m/2$;

( ) $\rho_{n_m}|h_*(x_m)v_m|/(1 + |h(x_m)|^2) > g^\sharp(\xi_m)/2$;

( ) $|h(x_m)| > m/2$.

H $\lambda_{n_m} = z_{n_m} + \rho_{n_m}\xi_m$, $x_m = \varphi_{n_m}(\lambda_{n_m})$, a d $v_m = \varphi_{n_m}'(\lambda_{n_m})$. I ( ), ( ), a d ( ) d, a a

$$\frac{|h_*(x_m)v_m|}{1 + |h(x_m)|^2} > \frac{g^\sharp(\xi_m)}{2\rho_{n_m}} > \frac{(4 - (1 + r)^2)m}{16} \cdot \frac{1}{1 - |\lambda_{n_m}|^2}.$$

B d a c d c a K ba a c

$$\frac{1}{1 - |\lambda_{n_m}|^2} = k_\Delta(\lambda_{n_m}, 1) \geq k_X(\varphi_{n_m}(\lambda_{n_m}), \varphi_{n_m}'(\lambda_{n_m}) \cdot 1).$$

H c

$$\frac{|h_*(x_m)v_m|}{1+|h(x_m)|^2} > \frac{(4-(1+r)^2)m}{16} \cdot k_X(x_m, v_m).$$

S c $|h(x_m)| \to \infty$, a $m \to \infty$, b c d a b c c a , a a a f b d d $\{x_m\}$, a $|f(x_m)| < M < \infty$ a $m \geq 1$, a d c , m c a , a [1]

$$\frac{|(f+h)_*(x_m)v_m|}{1+|f(x_m)+h(x_m)|^2} > \frac{|h_*(x_m)v_m| - |f_*(x_m)v_m|}{2(1+|f(x_m)|^2)(1+|h(x_m)|^2)} \geq$$

$$\frac{1}{2}\left[\frac{1}{1+|M|^2} \cdot \frac{|h_*(x_m)v_m|}{1+|h(x_m)|^2} - \frac{|f_*(x_m)v_m|}{1+|f(x_m)|^2}\right].$$

B , f a a c X c , b T 3.1, $Q > 0$ c a $Q_f(x_m) < Q$ a $m \geq 1$. T

$$Q_{f+h}(x_m) \geq \frac{1}{2}\left[\frac{(4-(1+r)^2)m}{16(1+|M|^2)} - Q\right] \to \infty \text{ a } m \to \infty.$$

B T 3.1, $f+h$ a a c X.
T c . □

L a d V a [12, a 53] a a a a c a d a b d d c (c c a a) a a c . T a c c , ac a 0 a d 1, d b a ([10, a 191]).
I n d a ca a .

Theorem 5.2. *If under the conditions of Theorem* 3.1*,* $f_1, \ldots, f_l$ *are a finite number of normal holomorphic function on* X *such that each sequence* $\{x_n\}$ *of points in* X *contains a subsequence* $\{x_{n_m}\}$ *on which at most one of* f_j $(1 \leq j \leq l)$ *is unbounded, then* $h := \Sigma_{j=1}^{l} f_j$ *is a normal function.*

Proof. S , c a , a h a a c . B T 4.2, ca n d c $\{x_n\}$ a d $\{y_n\}$ X, a d a c a M c a $K_X(x_m, y_m) < M$ a $m \geq 1$, $_{m\to\infty} h(x_m) = \infty$, a d $_{m\to\infty} h(y_m) = a \in \mathbb{C}$. S c $_{m\to\infty} h(x_m) = \infty$ $\{x_m\}$ c a a b - c a a d d b $\{x_m\}$ c a a f_j, a f_1, b d d

[1] I a a d b $\mathbb{C}$

$$1+|a+b|^2 \leq 1+(|a|+|b|)^2 < 2(1+|a|^2)(1+|b|^2).$$

$\{x_m\}$. S c $_{m\to\infty} h(y_m) = a \in \mathbb{C}$ $\{y_m\}$ c a a b c $\{y_{m_k}\}$ c a :

( ) a a ꝟ f_j $(1 \le j \le l)$ b d d $\{y_{m_k}\}$;

( ) $_{k\to\infty} f_j(y_{m_k}) = \alpha_j \in \mathbb{C}$ $(1 \le j \le l)$.

T ca ( ) c d d b a c a . H c $_{k\to\infty} f_1(x_{m_k}) = \infty$, $_{k\to\infty} f_1(y_{m_k}) = \alpha_1 \in \mathbb{C}$, a d $K_X(x_{m_k}, y_{m_k}) < M$ a $k \ge 1$. B T 4.2 f_1 a a c , a c ad c ꝟ c c a . □

Theorem 5.3. *Under the assumption of Theorem* 3.1 *let* $\{x_n\}$ *and* $\{y_n\}$ *be two sequences of points in* X, *and let* M *be a positive constant such that* $K_X(x_m, y_m) < M$ *for all* $m \ge 1$. *If* $f \in \mathcal{O}(X)$ *is a normal function which omits* $l \in \overline{\mathbb{C}}$ *in* X *but* $_{m\to\infty} s(f(x_m), l) = 0$ *then* $_{m\to\infty} s(f(y_m), l) = 0$.

Proof. A ꞁ a $l \in \mathbb{C}$. S c

$$\frac{1 + |f(x) - l|^2}{1 + |f(x)|^2} < 2(1 + |l|^2)$$

(1) ꝟ a d T 3.1 a $g(x) = 1/(f(x) - l)$ a a c c X. I a a $_{m\to\infty} g(x_m) = \infty$. T , ꝟ a $_{m\to\infty} g(y_m) = \infty$ b T 4.2. H c $_{m\to\infty} s(f(y_m), l) = 0$ a d d.

I $l = \infty$ c a a d a c c T 4.2. T c . □

6. Boundary behavior of normal functions in a fixed boundary point

I [1] F. Ba a d W. S d d ꝟ : *Given a sequence* $\{z_j\} \subset \Delta$ *converging to some* $\varsigma \in \partial\Delta$ *and a holomorphic mapping* $f \in \mathcal{O}(\Delta, \overline{\mathbb{C}})$ *such that* $_{j\to\infty} s(f(z_j), l) = 0$ *for some* $l \in \overline{\mathbb{C}}$, *under what conditions on* f *and* $\{z_j\}$ *can* f *have the limit* l *along some continuum in* Δ *which is asymptotic at* ς?

I c , ꝟ ꞁ d a a a . F ꝟ d c dꞁ a d ꝟ a .

I a D b a b d d a d c d a a c Ba ac ac $(V, \|\cdot\|)$, a d K_D b K ba a c D.

A d a $D \subseteq V$ ca d *convex (in the real sense)* $(1-t)x + ty \subset D$ a $x, y \in D$ a d $0 \le t \le 1$.

Lemma 6.1. *Let* D *be a bounded and convex (in the real sense) domain in a complex Banach space* $(V, \|\cdot\|)$ *and let* $\xi \in \partial D$. *Then for all* $a \in D$ *the set* $l_\xi(a) = \{y = \xi + t(a - \xi), 0 < t < 1\}$ *is contained in* D.

Proof. W a $B(a,r) := a + B(0,r) \subset D$, $r > 0$, $B(0,r) :=$ $\{y \in V : \|y\| < r\}$. P $U = [t/(1-t)]B(0,r)+\xi$, $t \neq 1$. T $b \in D \cap U$. W a $b = [t/(1-t)]u + \xi$ $u \in B(0,r)$. S c $a - u \in B(a,r) \subset D$ a d D c $y = (1-t)b + t(a-u) = \xi + t(a-\xi) \in D$. T a $l_\xi(a) \subset D$. □

Definition 6.2. *The approach region of aperture* α, *with vertex* $\xi \in \partial D$, *and pole* $a \in D$ *is the set*

$$A_\alpha(\xi, a) = \{x \in D : K_D(x, l_\xi(a)) < \alpha\}$$

where $K_D(x, l_\xi(a)) = \quad \{K_D(x,y), y \in l_\xi(a)\}$.

Lemma 6.3. *Let* D *be a bounded and convex (in the real sense) domain in a complex Banach space* V *and let* $\xi \in \partial D$. *If* $b \in D$, $b \neq a$, *then there exists a positive constant* C *such that*

$$A_\alpha(\xi, a) \subset A_{\alpha+C}(\xi, b) \subset A_{\alpha+2C}(\xi, a).$$

Proof. T a c a $z \in A_\alpha(\xi, a)$ $z \in$ $A_{\alpha+C}(\xi, b)$. S $z \in A_\alpha(\xi, a)$. F d $A_\alpha(\xi, a)$ a $\varepsilon > 0$ a a_1 $l_\xi(a)$ c a $K_D(z, a_1) < \alpha+\varepsilon$. S c $a_1 \in l_\xi(a)$ a $\tau_0 \in (0,1]$ c a $a_1 = \xi+\tau_0(a-\xi)$. S $b_1 = \xi+\tau_0(b-\xi) \in l_\xi(b)$. L φ b a a $\mathcal{O}(\Delta, D)$ a c a b a a d b.[2] S a $\varphi(0) = a$ a d $\varphi(\zeta) = b$ a ab $\zeta \in \Delta$. D $\psi : \Delta \to V$ b

$$\psi(z) = \xi + \tau_0(\varphi(z) - \xi).$$

B L a 6.1 ψ a Δ D; c

$$K_D(a_1, b_1) = K_D(\psi(0), \psi(\zeta)) \leq d_\Delta(0, \zeta).$$

B a a K ba a d a c

$$K_D(z, b_1) \leq K_D(z, a_1) + K_D(a_1, b_1).$$

I c a, a

$$K_D(z, l_\xi(b)) \leq K_D(z, b_1) \leq \alpha + \varepsilon + d_\Delta(0, \zeta).$$

[2] I $D \subset \mathbb{C}^n$ a c a $\phi : [0,1] \to D$ c a $\phi(0) = a$ a d $\phi(1) = b$. A a ca c - a d S -W a a d a d- ad- a ca a ϕ b a a . B c ac a c b d O $[0,1]$ $\mathbb{C}$ c a $\widetilde{\phi}(O) \subset D$, $\widetilde{\phi}$ a $\mathbb{C}$ b a d b ac a a ab x b c a ab z. A a ca R a a d c a a $\varphi \in \mathcal{O}(\Delta, D)$ a c a b a a d b. S c D a d a a Ba ac ac V a a a a d b D ca b d b a a c Γ. I Y d a b ac V a d b Γ c c d c $\Gamma \cap D$, c a a, a d a d a ac c c a b a a d b. T d a ca b a d c a a $\phi \in \mathcal{O}(\Delta, D)$ a c a b a a d b ([3, a 49]).

S c ε a b a a

$$K_D(z, l_\xi(b)) \le \alpha + d_\Delta(0, \zeta).$$

I

$$z \in A_{\alpha+C}(\xi, b)$$

$C = d_\Delta(0, \zeta) < \infty$. □

Remark 6.4. I D ba $\mathbb{C}^n$ a ac $A_\alpha(\xi, a)$ a c a ab ad b a ac K a .

Definition 6.5. A c $f \in \mathcal{O}(D)$ a a *admissible limit* $l \in \overline{\mathbb{C}}$ a $\xi \in \partial D$, $\alpha > 0$ a d c $\{x_n\} \subset A_\alpha(\xi, a)$ c d ξ

$$\lim_{n\to\infty} s(f(x_n), l) = 0.$$

Remark 6.6. F L a 6.3 a d ad b d d c c $a \in D$.

Theorem 6.7. *Let D be a bounded and convex (in the real sense) domain in a complex Banach space $(V, \|\cdot\|)$. Let $\{x_n\}$ be a sequence of points in $l_\xi(a)$ which tends to $\xi \in \partial D$, such that there exists a constant $\varepsilon > 0$ such that $K_D(x_n, x_{n+1}) < \varepsilon$ for all $n \geq 1$. Suppose that the function $f \in \mathcal{O}(D)$ is normal on D, omits $l \in \overline{\mathbb{C}}$ in D but*

$$\lim_{n\to\infty} s(f(x_n), l) = 0.$$

Then f has an admissible limit l at ξ.

Proof. F a , a a a b a c $\{q_n\}$ $l_\xi(a)$ c d ξ a $n \to \infty$ a $\lim_{n\to\infty} s(f(q_n), l) = 0$. I d d, q_n a j_n a d a c a t_n, $t_n \in [0, 1]$, c a $q_n = x_{j_n} + t_n(x_{j_n+1} - x_{j_n})$. F $t \in [0, 1]$ d $g_t \in \mathcal{O}((D \times D) \times (D \times D), D \times D)$ b $g_t((x, y), (w, z)) = (tx + (1-t)y, tw + (1-t)z)$. T b [5, P IV.1.2, a 83] a $x, y, w, z \in D$ a

$$K_{D\times D}((x, y), (w, z)) \geq K_D(tx + (1-t)y, tw + (1-t)z)\,. \tag{6.1}$$

B [5, P V.4.2, a 136]

$$K_{D\times D}((x, y), (w, z)) = \text{a}\,\{K_D(x, y), K_D(w, z)\}.$$

C b (6.1), a

$$K_D(tx + (1-t)y, tw + (1-t)z) \le max[K_D(x, w), K_D(y, z)]$$

a $0 \le t \le 1$ a d c c x, y, w, z D. T a

$$K_D(x_{j_n}, q_n) = K_D(x_{j_n}, x_{j_n} + t_n(x_{j_n+1} - x_{j_n}) \le K_D(x_{j_n}, x_{j_n+1}) < \varepsilon.$$

A $\quad_{n\to\infty} s(f(x_{j_n}), l) = 0$ a d $K_D(x_{j_n}, q_n) < \varepsilon$ a $n \geq 1$,

$$_{n\to\infty} s(f(q_n), l) = 0$$

b C a 5.3.

L $\{y_n\}$ b a c $A_\alpha(\xi, a)$ c c $\xi \in \partial D$. F y_n a $b_n \in l_\xi(a)$ c a $2\alpha > K_D(y_n, b_n)$ a $n \geq 1$. E b_n ca b $b_n = \xi + \tau_n(a - \xi)$ $\tau_n \in [0, 1]$. H c , $\{b_n\}$ a a c b c $\{b_{n_j}\}$. L $b = \xi + \tau_0(a - \xi)$ b a $\{b_{n_j}\}$.

A $\tau_0 \neq 0$ a d a c ad c . C a b r c a $B_r(b) = \{y \in V : \|y - b\| < r\} \subset D$. S c c $\{b_{n_j}\}$ d b a $j \to \infty$ a j_0 c a $\|b_{n_j} - b\| < r/2$ a $j \geq j_0$. B a (4.3) [3, a 52]

$$K_D(b_{n_j}, b) < \text{a}^{-1}\left(\frac{1}{2}\right) \quad \text{a} \quad j \geq j_0.$$

T $j \geq j_0$ a

$$K_D(y_{n_j}, b) < K_D(y_{n_j}, b_{n_j}) + K_D(b_{n_j}, b) < 2\alpha + \text{a}^{-1}\left(\frac{1}{2}\right).$$

T c ad c a a c $\{y_n\}$ d $\xi \in \partial D$, c ba (D, K_D) b d d a b da ([3, a 88]). H c $\tau_0 = 0$ a d c $\{b_n\} \subset l_\xi(a)$ d ξ a $n \to \infty$. A d ab $_{n\to\infty} s(f(b_n), l) = 0$ a d c $K_D(y_n, b_n) < \alpha$ a $n \geq 1$

$$_{n\to\infty} s(f(y_n), l) = 0$$

b C a 5.3. H c f a ad b l a ξ. □

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