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On the second boundary value problem for Monge-Ampère type equations and optimal transportation

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Abstract. This paper is concerned with the existence of globally smooth solutions for the second boundary value problem for certain Monge-Ampère type equations and the application to regularity of potentials in optimal transportation. In particular we address the fundamental issue of determining conditions on costs and domains to ensure that optimal mappings are smooth diffeomorphisms. The cost functions satisfy a weak form of the condition (A3), which was introduced in a recent paper with Xi-nan Ma, in conjunction with interior regularity. Our condition is optimal and includes the quadratic cost function case of Caffarelli and Urbas as well as the various examples in our previous work. The approach is through the derivation of global estimates for second derivatives of solutions.

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1. Introduction

This paper is concerned with the global regularity of solutions of the second boundary value problem for equations of Monge-Ampère type and its application to the regularity of potentials in optimal transportation problems with non-quadratic cost functions. In particular we resolve, in the context of global regularity, the fundamental problem of regularity for more general costs than the quadratic cost, as highlighted for example in the recent book, [25, Chapter 4].

The Monge-Ampère equations under consideration have the general form

$$\det\{D^2u - A(\cdot, u, Du)\} = B(\cdot, u, Du), \tag{1.1}$$

where A and B are given $n \times n$ matrix and scalar valued function defined on $\Omega \times \mathbb{R} \times \mathbb{R}^n$, where Ω is a domain in Euclidean n-space, $\mathbb{R}^n$. We use (x, z, p) to denote points in $\Omega \times \mathbb{R} \times \mathbb{R}^n$ so that $A(x, z, p) \in \mathbb{R}^n \times \mathbb{R}^n$, $B(x, z, p) \in \mathbb{R}$ and $(x, z, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$. The equation (1.1) will be elliptic (degenerate elliptic), with respect

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to a solution $u \in C^2(\Omega)$ whenever

$$D^2u - A(\cdot, u, Du) > 0 \quad (\geq 0), \tag{1.2}$$

whence also $B > 0$ (≥ 0).

A particular form of (1.1) arises from the prescription of the Jacobian determinant of a mapping Tu defined by

$$Tu = Y(\cdot, u, Du), \tag{1.3}$$

where Y is a given vector valued function on $\Omega \times \mathbb{R} \times \mathbb{R}^n$, namely

$$\det DY(\cdot, u, Du) = \psi(\cdot, u, Du). \tag{1.4}$$

Assuming that the matrix

$$Y_p = [D_{p_j} Y^i] \tag{1.5}$$

is non-singular, we may write (1.4) in the form (1.1), that is,

$$\det\{D^2u + Y_p^{-1}(Y_x + Y_z \otimes Du)\} = \frac{\psi}{|\det Y_p|}, \tag{1.6}$$

for degenerate elliptic solutions u.

The *second boundary value problem* for equation (1.4) is to prescribe the image

$$Tu(\Omega) = \Omega^*, \tag{1.7}$$

where Ω^* is a given domain in $\mathbb{R}^n$. When Y and ψ are independent of z and ψ is separable in the sense that

$$\psi(x, p) = f(x)/g \circ Y(x, p) \tag{1.8}$$

for positive $f, g \in L^1(\Omega), L^1(\Omega^*)$ respectively, then a necessary condition for the existence of an elliptic solution, for which the mapping T is a diffeomorphism, to the second boundary value problem (1.4), (1.7) is the *mass balance* condition

$$\int_\Omega f = \int_{\Omega^*} g. \tag{1.9}$$

The second boundary value problem (1.4), (1.7) arises naturally in optimal transportation. Here we are given a *cost function* $c : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and the vector field Y is generated by the equation

$$c_x(x, Y(x, p)) = p, \tag{1.10}$$

which we assume to be uniquely solvable for $p \in \mathbb{R}^n$, with non-vanishing determinant, that is

$$\det c_{x,y}(x, y) \neq 0 \tag{1.11}$$

for all $x, y \in \Omega \times \Omega^*$. Using the notation

$$c_{ij\cdots,kl\cdots} = \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\cdots\frac{\partial}{\partial y_k}\frac{\partial}{\partial y_l}\cdots c \tag{1.12}$$

we have

$$Y_p(x, p) = [c^{i,j}(x, Y(x, p))], \tag{1.13}$$

where $[c^{i,j}]$ is the inverse of $[c_{i,j}]$. The corresponding Monge-Ampère equation can now be written as

$$\det\{D^2u - c_{xx}(\cdot, Y(\cdot, Du))\} = |\det c_{x,y}|\psi, \tag{1.14}$$

that is in the form (1.1) with

$$\begin{aligned} A(x, z, p) &= c_{xx}(x, Y(x, p)), \\ B(x, z, p) &= |\det c_{x,y}(x, Y(x, p))|\psi(x, z, p). \end{aligned} \tag{1.15}$$

In the case of the (quadratic) cost function

$$c(x, y) = x \cdot y, \tag{1.16}$$

we have

$$Y(x, p) = p, \quad Tu = Du, \tag{1.17}$$

and equation (1.14) reduces to the standard Monge-Ampère equation

$$\det D^2u = \psi. \tag{1.18}$$

For this case, global regularity of solutions was proved by Delanoë [4], Caffarelli [2] and Urbas [22], with interior regularity obtained earlier by Caffarelli [1]. In this paper we will prove global estimates and regularity for general cost functions under corresponding geometric conditions. In particular, we will assume that the cost function $c \in C^4(\mathbb{R}^n \times \mathbb{R}^n)$ satisfies the following conditions:

For each $p, q \in \mathbb{R}^n$, there exists unique $y = Y(x, p)$, $x = X(q, y)$ such that

$$\begin{aligned} c_x(x, y) &= p \quad \forall\, x \in \Omega, \\ c_y(x, y) &= q \quad \forall\, y \in \Omega^*. \end{aligned} \tag{A1}$$

$$\det c_{x,y}(x, y) \neq 0, \quad \forall\, x \in \overline{\Omega},\, y \in \overline{\Omega}^*. \tag{A2}$$

$$\begin{aligned} \mathcal{F}(x, p; \xi, \eta) :&= D_{p_i p_j} A_{kl}(x, p)\xi_i\xi_j\eta_k\eta_l \\ &\geq 0 \;\; \forall\; x \in \Omega, p \in \mathbb{R}^n,\; \xi \perp \eta \in \mathbb{R}^n. \end{aligned} \tag{A3w}$$

Conditions (A1) and (A2) are the same conditions as in [15] but condition (A3w) is the degenerate form of condition (A3) in [15],

$$\mathcal{F}(x, p; \xi, \eta) \geq c_0|\xi|^2|\eta|^2 \;\; \forall\; x \in \Omega, p \in \mathbb{R}^n,\; \xi \perp \eta \in \mathbb{R}^n, \tag{A3}$$

where c_0 is a positive constant. As will be seen in our examples, we do not necessarily require c to be defined on all of $\mathbb{R}^n \times \mathbb{R}^n$ and the vectors p and q in conditions (A1) and (A3w) need only lie in the ranges of $c_x(x, y)$ and $c_y(x, y)$ on Ω^* and Ω. Moreover, as done at the outset in [15], we may also write

$$\mathcal{F}(x, p; \xi, \eta) = (c_{ij,rs} - c^{k',l'} c_{ij,k'} c_{l',rs}) c^{r,k} c^{s,l}(x, y)\xi_i \xi_j \eta_k \eta_l \tag{1.19}$$

where y and p are related through (A1). This shows that conditions (A3w), (A3) are also symmetric in x and y.

In our paper [15], we also introduced a notion of convexity of domains with respect to cost functions, namely Ω is c-convex, with respect to Ω^*, if the image $c_y(\cdot, y)(\Omega)$ is convex in $\mathbb{R}^n$ for each $y \in \Omega^*$, while analogously Ω^* is c^*-convex, with respect to Ω, if the image $c_x(x, \cdot)(\Omega^*)$ is convex for each x in Ω. For global regularity we need to strengthen these conditions in the same way that convexity is strengthened to uniform convexity. Namely we define Ω to be uniformly c-convex, with respect to Ω^*, if Ω is c-convex, with respect to Ω^*, $\partial\Omega \in C^2$ and there exists a positive constant δ_0 such that

$$[D_i \gamma_j(x) - c^{l,k} c_{ij,l}(x, y)\gamma_k(x)]\tau_i \tau_j(x) \geq \delta_0 \tag{1.20}$$

for all $x \in \partial\Omega$, $y \in \Omega^*$, unit tangent vector τ and outer unit normal γ. By pulling back with the mappings $c_y(\cdot, y)$, we see that this is equivalent to the condition that the image domains $c_y(\cdot, y)(\Omega)$ be uniformly convex with respect to $y \in \Omega^*$. Similarly we call Ω^* uniformly c^*-convex, with respect to Ω, when $c^*(x, y) = c(y, x)$. Note that if Ω is connected with boundary $\partial\Omega \in C^2$, then Ω is c-convex if and only if (1.20) holds for $\delta_0 = 0$.

It is also convenient to have a notion of boundedness relative to a cost function. Namely we say that Ω is c-bounded, with respect to Ω^* if there exists some function $\varphi \in C^2(\Omega)$, satisfying

$$[D_{ij}\varphi - c^{l,k} c_{ij,l}(\cdot, y) D_k \varphi]\xi_i \xi_j \geq \delta_1 |\xi|^2, \tag{1.21}$$

in Ω, for all $y \in \Omega^*$, for some constant $\delta_1 > 0$. Clearly for the quadratic cost function (1.16), c-boundedness is equivalent to the usual boundedness (take $\varphi(x) = |x|^2$).

We can now formulate our main estimate.

Theorem 1.1. *Let c be a cost function satisfying hypotheses* (A1), (A2), (A3w), *with respect to bounded C^4 domains Ω, $\Omega^* \in \mathbb{R}^n$ which are respectively uniformly c-convex, c^*-convex with respect to each other. Assume also that either Ω is c-bounded with respect to Ω^* or A depends only on p or condition (A3) holds. Let ψ be a positive function in $C^2(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$. Then any elliptic solution $u \in C^3(\overline{\Omega})$ of the second boundary value problem* (1.14), (1.7) *satisfies the a priori estimate*

$$|D^2 u| \leq C, \tag{1.22}$$

where C depends on c, ψ, Ω , Ω^* and $\sup_\Omega |u|$.

As we will indicate later, the smoothness assumption on the solution and the data may be reduced. Further regularity also follows from the theory of linear elliptic equations for example if c, Ω, Ω^*, ψ are C^∞ then the solution $u \in C^\infty(\overline{\Omega})$.The dependence of the estimate (1.22) on $\sup_\Omega |u|$ may be removed if ψ is independent of u as in (1.8).

As a consequence of Theorem 1.1, we may conclude existence theorems for classical solutions by the method of continuity.

Theorem 1.2. *Suppose in addition to the above hypotheses that the function ψ satisfies* (1.8), (1.9). *Then there exists a unique (up to additive constants) elliptic solution $u \in C^3(\overline{\Omega})$ of the second boundary value problem* (1.14)*,* (1.7).

From Theorem 1.2, we also obtain an existence result for classical solutions of the Monge-Kantorovich problem in optimal transportation. As above we let $c \in C^4(\mathbb{R}^n \times \mathbb{R}^n)$ be a cost function and Ω, Ω^* be two bounded domains in $\mathbb{R}^n$ satisfying the hypotheses of Theorem 1.1. Let $f > 0, \in C^2(\overline{\Omega})$, $g > 0, \in C^2(\overline{\Omega^*})$ be positive densities satisfying the mass balance condition (1.9). Then the corresponding optimal transportation problem is to find a measure preserving mapping $T_0 : \Omega \to \Omega^*$ which maximizes the cost functional

$$C(T) = \int_\Omega f(x)c(x, T(x))dx \tag{1.23}$$

among all measure preserving mappings T from Ω to Ω^*. A mapping $T : \Omega \to \Omega^*$ is called measure preserving if it is Borel measurable and for any Borel set $E \subset \Omega^*$,

$$\int_{T^{-1}(E)} f = \int_E g. \tag{1.24}$$

The reader is referred to the texts [16, 25] and the lecture notes [6, 23] for further information about optimal transportation.

Theorem 1.3. *Under the above hypotheses, there exists a unique diffeomorphism $T \in [C^2(\overline{\Omega})]^n$ maximizing the functional* (1.23)*, given by*

$$T(x) = Y(x, Du(x)), \tag{1.25}$$

where u is an elliptic solution of the boundary problem (1.7), (1.14).

The solution u of (1.7), (1.14) is called a potential. As indicated above, we also have from elliptic regularity theory that if c, Ω, Ω^*, f, g are C^∞ smooth, then the resultant optimal mapping $T \in [C^\infty(\overline{\Omega})]^n$. Note that in [15] and elsewhere the cost functions and potentials are the negatives of those here and the optimal transportation problem is written (in its usual form), as a minimization problem.

The plan of this paper is as follows. In Section 2, we prove that boundary conditions of the form (1.7) are oblique with respect to functions for which the Jacobian DT is non-singular and we estimate the obliqueness for elliptic solutions of the boundary value problem (1.14), (1.7) under hypotheses (A1) and (A2) (Theorem 2.1). Here the twin assumptions of Ω and Ω^* being uniformly c and c^*-convex with respect to each other are critical. In Section 3, we prove that second derivatives of solutions of equation (1.14) can be estimated in terms of their boundary values under hypothesis (A3w) (Theorem 3.1). This estimation is already immediate from [15] when the non-degenerate condition (A3) is satisfied. The argument is carried out for equations of the general form (1.1) (with symmetric A), in the presence of a global barrier (corresponding to c-boundedness), which is not necessary in the optimal transportation case, with $A = A(p)$ (Theorem 3.2), where it is avoided by duality. This estimation also arises in the treatment of the classical Dirichlet problem [19]. The proof of the global second derivative estimates in Theorem 1.1 for solutions of the boundary value problem (1.14), (1.7) is completed in Section 4. Here the procedure is similar to that in [13] and [24]. We remark here that this global estimate also extends to the more general prescribed Jacobian equation (1.6) [19]. In Section 5, we commence the proof of the existence result, Theorem 1.2, by adapting the method of continuity [7] and establish the result under a stronger uniform c-convexity hypothesis (employed in earlier versions of this paper). Section 6 is devoted to the applications to optimal transportation and the derivation of Theorem 1.3 from Theorem 1.2, which implies the global regularity of the potential functions in [15], under condition (A3w). In Section 7, we finally complete the proof of Theorem 1.2 in its full generality, by showing that there exists a smooth function satisfying the ellipticity condition (1.2), together with the boundary condition (1.7) (at least for approximating domains). In the last section, we discuss our results in the light of examples, most of which are already given in [15]. Note that when the cost function c is given by $c(x, y) = c'(x - y)$ for some $c' \in C^2(\Omega \times \Omega^*)$, then the matrix A depends only on p, namely

$$A(p) = D^2 c'\big[(Dc')^{-1}(p)\big]. \tag{1.26}$$

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2. Obliqueness

In this section, we prove that the boundary condition (1.7) implies an oblique boundary condition and estimate the obliqueness. First we recall that a boundary condition of the form

$$G(x, u, Du) = 0 \quad \text{on } \partial\Omega \tag{2.1}$$

for a second order partial differential equation in a domain Ω is called oblique if

$$G_p \cdot \gamma > 0 \tag{2.2}$$

for all $(x, z, p) \in \partial\Omega \times \mathbb{R} \times \mathbb{R}^n$, where γ denotes the unit outer normal to $\partial\Omega$. Let us now assume that φ and φ^* are C^2 defining functions for Ω and Ω^* respectively, with $\varphi, \varphi^* < 0$ near $\partial\Omega, \partial\Omega^*$, $\varphi = 0$ on $\partial\Omega$, $\varphi^* = 0$ on $\partial\Omega^*$, $\nabla\varphi, \nabla\varphi^* \neq 0$ near $\partial\Omega, \partial\Omega^*$. Then if $u \in C^2(\overline{\Omega})$ is an elliptic solution of the second boundary value problem (1.4), (1.7), we must have

$$\varphi^*(Tu) = 0 \quad \text{on} \ \partial\Omega, \quad \varphi^*(Tu) < 0 \quad \text{near} \ \partial\Omega. \tag{2.3}$$

By tangential differentiation, we obtain

$$\varphi_i^*(D_j T^i u)\tau_j = 0 \tag{2.4}$$

for all unit tangent vectors τ, whence

$$\varphi_i^*(D_j T^i) = \chi\gamma_j \tag{2.5}$$

for some $\chi \geq 0$. Consequently

$$\varphi_i^* c^{i,k}(u_{jk} - c_{jk}) = \chi\gamma_j,$$

that is

$$\varphi_i^* c^{i,k} w_{jk} = \chi\gamma_j, \tag{2.6}$$

where

$$w_{ij} = u_{ij} - c_{ij}. \tag{2.7}$$

At this point we observe that $\chi > 0$ on $\partial\Omega$ since $|\nabla\varphi^*| \neq 0$ on $\partial\Omega$ and $\det DT \neq 0$. Using the ellipticity of (1.14) and letting $[w^{ij}]$ denote the inverse matrix of $[w_{ij}]$, we then have

$$\varphi_i^* c^{i,k} = \chi w^{jk}\gamma_j. \tag{2.8}$$

Now writing

$$G(x, p) = \varphi^*(Y(x, p)), \tag{2.9}$$

we have

$$\beta_k := G_{p_k}(\cdot, Du) = \chi w^{jk}\gamma_j, \tag{2.10}$$

whence

$$\beta \cdot \gamma = \chi w^{ij}\gamma_i\gamma_j > 0 \tag{2.11}$$

on $\partial\Omega$. We obtain a further formula for $\beta \cdot \gamma$, from (2.6), namely

$$\varphi_i^* c^{i,k} w_{jk} \varphi_l^* c^{l,j} = \chi \varphi_l^* c^{l,j}\gamma_j = \chi(\beta \cdot \gamma). \tag{2.12}$$

Eliminating χ from (2.11) and (2.12), we have

$$(\beta \cdot \gamma)^2 = (w^{ij}\gamma_i\gamma_j)(w_{kl} c^{i,k} c^{j,l} \varphi_i^* \varphi_j^*). \tag{2.13}$$

We call (2.13) a formula of Urbas type, as it was proved by Urbas [22] for the special case, $c(x, y) = x \cdot y$, $Y(\xi, p) = p$, of the Monge-Ampère equation. Note that to prove (2.13), we only used conditions (A1) and (A2) and moreover (2.13) continues to hold in the generality of (1.4).

Our main task now is to estimate $\beta \cdot \gamma$ from below for solutions of (1.14), (1.7). For this in addition to conditions (A1), (A2), we also need the uniform c and c^* convexity of Ω and Ω^* respectively. Our approach is similar to [22] for the special case of the Monge-Ampère equation and begins by invoking the key idea from [17] for estimating double normal derivatives of solution of the Dirichlet problem. Namely we fix a point x_0 on $\partial\Omega$ where $\beta \cdot \gamma$ is minimized, for an elliptic solution $u \in C^3(\overline{\Omega})$, and use a comparison argument to estimate $\gamma \cdot D(\beta \cdot \gamma)$ from above. Without some concavity condition in p the quantity $\beta \cdot \gamma$ does not satisfy a nice differential inequality so we will get around this by considering instead the function

$$v = \beta \cdot \gamma - \kappa\varphi^*(Tu) \tag{2.14}$$

for sufficiently large κ, where now the defining function φ^* is chosen so that

$$(D_{ij}\varphi^*(Tu) - c^{k,l}c_{l,ij}(\cdot, Tu)D_k\varphi^*(Tu)\xi_i\xi_j \geq \delta_0^*|\xi|^2 \tag{2.15}$$

near $\partial\Omega$, for all $\xi \in \mathbb{R}^n$ and some positive constant δ_0^*. Inequality (2.15) is possible by virtue of the uniform c^*-convexity of Ω^*, with the function φ^* given, for example by

$$\varphi^* = -ad^* + b(d^*)^2, \tag{2.16}$$

where a and b are positive constants and d^* denotes the distance function for Ω^*, [7].

By differentiation of equation (1.14), in the form (1.1), we obtain, for $r = 1, \cdots, n$,

$$w^{ij}\{D_{ij}u_r - D_{p_k}A_{ij}(x, Du)D_ku_r - D_{x_r}A_{ij}(x, Du)\} = D_r\log B. \tag{2.17}$$

Introducing the linearized operator L,

$$Lv = w^{ij}(D_{ij}v - D_{p_k}A_{ij}D_kv), \tag{2.18}$$

we need to compute Lv for v given by (2.14). Setting

$$F(x, p) = G_p(x, p) \cdot \gamma(x) - \kappa G(x, p), \tag{2.19}$$

where G is defined by (2.9), we see that

$$v(x) = F(x, Du(x)). \tag{2.20}$$

Writing $b_k^{ij} = -D_{p_k}A_{ij}$, we then have

$$\begin{aligned} Lv =& w^{ij}\{F_{p_r}D_{ij}u_r + F_{p_rp_s}D_{ir}uD_{js}u \\ &+ F_{x_ix_j} + 2F_{x_ip_r}D_{jr}u + b_k^{ij}(F_{x_k} + F_{p_r}D_ku_r)\} \end{aligned} \tag{2.21}$$

In the ensuing calculations, we will often employ the following formulae,

$$\begin{aligned} c_k^{i,j}(x,y) &= D_{x_k} c^{i,j}(x,y) = -c^{i,l}c^{r,j}c_{kl,r}(x,y), \\ c_{,k}^{i,j}(x,y) &= D_{y_k} c^{i,j}(x,y) = -c^{i,l}c^{r,j}c_{l,kr}(x,y), \end{aligned} \tag{2.22}$$

as well as (1.13). Indeed, using (1.13) and (2.22), we have

$$\begin{aligned} G_{p_i p_j} &= D_{p_j}(\varphi_k^* c^{k,i}) = \varphi_{kl}^* c^{k,i} c^{l,j} - \varphi_k^* c^{s,j} c_{,s}^{k,i} \\ &= c^{k,i} c^{l,j} \{\varphi_{kl}^* - \varphi_r^* c^{r,s} c_{s,kl}\} \end{aligned} \tag{2.23}$$

so that

$$G_{p_i p_j}(x, Du)\xi_i\xi_j \geq \delta_0^* \sum |c^{i,j}\xi_j|^2 \geq \kappa_0^* |\xi|^2 \tag{2.24}$$

for a further positive constant κ_0^*. By choosing κ sufficiently large, we can then ensure that

$$F_{p_i p_j}(x, Du)\xi_i\xi_j \leq -\frac{1}{2}\kappa|\xi|^2 \tag{2.25}$$

near $\partial\Omega$. Substituting into (2.20) and using (2.16), it follows that

$$Lv \leq -\frac{1}{4}\kappa w_{ii} + C(w^{ii} + 1) + D_{p_k} log B D_k v, \tag{2.26}$$

where C is a constant depending on c, ψ, Ω and Ω^*, as well as κ.

Next we observe that unless the defining function φ^* extends to all of Ω^* so that (2.15) is satisfied for all $Tu \in \Omega^*$, we have no control on the neighbourhood of $\partial\Omega$, where (2.26) holds. This is remedied by replacing G in (2.19) by a function satisfying (2.24) in all of Ω , agreeing with (2.9) near $\partial\Omega$, for example by taking

$$G(x, p) = m_h\{(\varphi^*(Y(x, p)), a_1(|p|^2 - K^2)\}, \tag{2.27}$$

where a_1 and K are positive constants, with a_1 sufficiently small and $K > \max|Du|$, and for h sufficiently small, m_h is the mollification of the max-function of two variables.

A suitable barrier is now provided by the uniform c-convexity of Ω which implies, analogously to the case of Ω^* above, that there exists a defining function φ for Ω satisfying

$$[D_{ij}\varphi - c^{l,k}c_{ij,l}(x, Tu)D_k\varphi]\xi_i\xi_j \geq \delta_0|\xi|^2, \tag{2.28}$$

in a fixed neighbourhood of $\partial\Omega$ (for a constant $\delta_0 > 0$). By appropriate choice, of say the constants a and b in (2.16), without the $*$ (or following the uniformly convex case in [7, Chapter 14]), we may obtain, by virtue of (2.21),

$$L\varphi \geq \delta_0 w^{ii} + K w^{ij} D_i\varphi D_j\varphi, \tag{2.29}$$

for a given constant K. Combining (2.26) and (2.29), and using the positivity of B, we then infer by the usual barrier argument (which entails fixing a small enough neighbourhood of $\partial\Omega$, [7]),

$$\gamma \cdot Dv(x_0) \le C, \tag{2.30}$$

where again C is a constant depending on c, Ω, Ω^* and ψ. From (2.28) and since x_0 is a minimum point of v on $\partial\Omega$, we can write

$$Dv(x_0) = \tau\gamma(x_0) \tag{2.31}$$

where $\tau \le C$. To use the information embodied in (2.31), we need to calculate

$$\begin{aligned} D_i(\beta\cdot\gamma) &= D_i\{\varphi^*_k c^{k,j}\gamma_j\} \\ &= \varphi^*_{kl} D_i(T^l u)c^{k,j}\gamma_j + \varphi^*_k c^{k,j} D_i\gamma_j + \varphi^*_k\gamma_j(c^{k,j}_i + c^{k,j}_{,l} D_i T^l u) \\ &= \varphi^*_k c^{k,j}(D_i\gamma_j - c^{s,r}c_{ij,s}\gamma_r) + (\varphi^*_{kl} - \varphi^*_r c^{r,s}c_{s,kl})c^{k,j}\gamma_j D_i T^l u\,. \end{aligned} \tag{2.32}$$

Multiplying by $\varphi^*_t c^{t,i}$ and summing over i, we obtain

$$\begin{aligned} \varphi^*_t c^{t,i} D_i(\beta\cdot\gamma) &= \varphi^*_k\varphi^*_t c^{k,j}c^{t,i}(D_i\gamma_j - c^{s,r}c_{ij,s}\gamma_r) \\ &\qquad + \varphi^*_t c^{t,i}c^{k,j}\gamma_j c^{l,m} w_{im}(\varphi^*_{kl} - \varphi^*_r c^{r,s}c_{s,kl}) \\ &\ge \delta_0 \sum |\varphi^*_i c^{i,j}|^2 \end{aligned} \tag{2.33}$$

by virtue of the uniform c-convexity of Ω, the c^*-convexity of Ω^* and (2.6). Consequently, from (2.19) and (2.31), we obtain at x_0,

$$-\kappa w_{kl}c^{i,k}c^{j,l}\varphi^*_i\varphi^*_j \le C(\beta\cdot\gamma) - \tau_0 \tag{2.34}$$

for positive constants, C and τ_0. Hence if $\beta\cdot\gamma \le \tau_0/2C$, we have the lower bound

$$w_{kl}c^{i,k}c^{j,l}\varphi^*_i\varphi^*_j \ge \frac{\tau_0}{2\kappa}. \tag{2.35}$$

To complete the estimation of $\beta\cdot\gamma$ we may invoke the dual problem to estimate $w^{ij}\gamma_i\gamma_j$ at x_0. Assuming for the moment that Tu is one to one, we let u^* denote the c-transform of u, defined for $y = Tu(x) \in \Omega^*$ by

$$u^*(y) = c(x, y) - u(x). \tag{2.36}$$

It follows that

$$Du^*(y) = c_y(x, y) = c_y(T^*u^*(y), y), \tag{2.37}$$

where

$$T^*u^*(y) = X(Du^*, y) = (Tu)^{-1}(y), \tag{2.38}$$

and the second boundary value problem (1.14), (1.7) is equivalent to

$$|\det D_y(T^*u^*)| = g(y)/f(T^*u^*) \ \text{ in } \ \Omega^*, \tag{2.39}$$

$$T^*\Omega^* = \Omega. \tag{2.40}$$

Noting that the defining functions φ and φ^* may be chosen so that $\nabla\varphi = \gamma$, $\nabla\varphi^* = \gamma^*$ on $\partial\Omega$, $\partial\Omega^*$ respectively, we clearly have for $x \in \partial\Omega$, $y \in Tu(x) \in \partial\Omega^*$,

$$\beta \cdot \gamma(x) = c^{k,i}(x, y)\varphi_i\varphi^*_k(y) = \beta^* \cdot \gamma^*(y), \tag{2.41}$$

where

$$\beta^*(y) = D_q\varphi(Y^*(D_yu^*, y)). \tag{2.42}$$

Hence the quantity $\beta^* \cdot \gamma^*$ is minimized on $\partial\Omega^*$ at the point $y_0 = Tu(x_0)$. Furthermore, for $y = Tu(x)$, $x \in \partial\Omega$,

$$w^{ij}\gamma_i\gamma_j(x) = w^*_{kl}(y)c^{k,i}c^{l,j}(x, y)\varphi_i\varphi_j(x), \tag{2.43}$$

where

$$w^*_{kl}(y) = u^*_{y_k y_l}(y) - c_{,kl}(x, y). \tag{2.44}$$

Applying now the estimate (2.35) to u^* at the point $y_0 \in \partial\Omega^*$, we finally conclude from (2.13) the desired obliqueness estimate

$$\beta \cdot \gamma \geq \delta \tag{2.45}$$

on $\partial\Omega$ for some positive constant δ depending only on Ω, Ω^*, c, and ψ.

The above argument clearly extends to arbitrary positive terms B in (1.15). Noting also that it suffices in the above argument that T need only be one-to-one from a neighbourhood of the point x_0 to a neighbourhood of y_0, we have the following theorem.

Theorem 2.1. *Let $c \in C^3(\mathbb{R}^n \times \mathbb{R}^n)$ be a cost function satisfying hypotheses* (A1), (A2)*, with respect to bounded C^3 domains Ω, $\Omega^* \subset \mathbb{R}^n$, which are respectively uniformly c-convex, c^*-convex with respect to each other. Let ψ be a positive function in $C^1(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$. Then any elliptic solution $u \in C^3(\overline{\Omega})$, of the second boundary value problem* (1.14), (1.7) *satisfies the obliqueness estimate* (2.45).

Note that Tu is automatically globally one-to-one under the hypotheses of Theorem 1.2 by virtue of the change of variables formula. We remark that Theorem 2.1 extends to the more general prescribed Jacobian equation (1.4), [19]. The main difference is that we cannot directly use the c-transform to get the complementary estimate to (2.35) Instead the quantities there may be transformed using the local diffeomorphism Tu. Indeed we could also have avoided the use of duality in the proof of Theorem 2.1, by direct transformation of (2.35); see also [26] for further simplification and other Hessian type equations.

3. Global second derivative bounds

In this section we show that the second derivatives of elliptic solutions of equation (1.14) may be estimated in terms of their boundary values. For this estimation and the boundary estimates in the next section, it suffices to consider the general form (1.1) under the assumption that the matrix valued function $A \in C^2(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ satisfies condition (A3w), that is

$$D_{p_k p_l} A_{ij}(x, z, p)\xi_i \xi_j \eta_k \eta_l \geq 0 \tag{3.1}$$

for all $(x, z, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$, $\xi, \eta \in \mathbb{R}^n$, $\xi \perp \eta$. We also assume A is symmetric,which is the case for the optimal transportation equation (1.14). When (3.1) is strengthened to the condition (A3) in [15], that is

$$D_{p_k p_l} A_{ij}(x, z, p)\xi_i \xi_j \eta_k \eta_l \geq c_0 |\xi|^2 |\eta|^2 \tag{3.2}$$

for some constant $\delta > 0$, for all $(x, z, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$, $\xi, \eta \in \mathbb{R}^n$, $\xi \perp \eta$, then the global second derivative estimate follows immediately from our derivation of interior estimates in [15]. In the general case the proof is much more complicated and we need to also assume some kind of barrier condition,(corresponding to c-boundedness in the optimal transportation case), namely that there exists a function $\widetilde{\varphi} \in C^2(\overline{\Omega})$ satisfying

$$[D_{ij}\widetilde{\varphi}(x) - D_{p_k} A_{ij}(x, z, p) D_k \widetilde{\varphi}(x)]\xi_i \xi_j \geq |\xi|^2 \tag{3.3}$$

for some positive for all $\xi \in \mathbb{R}^n$, $x, z, p \in$ some set $\mathcal{U} \subset \Omega \times \mathbb{R} \times \mathbb{R}^n$, whose projection on Ω is Ω. In general, condition (3.1) implies some restriction on the domain Ω, but for the case of equations arising in optimal transportation, it can be avoided by a duality argument.

Our reduction to the boundary estimation follows the approach in [7], originating with Pogorelov, with some modification analogous to that in [13]. Let $u \in C^4(\Omega)$ be an elliptic solution of equation (1.1), with $x, u(x), Du(x) \in U$ for $x \in \Omega$ and ξ a unit vector in $\mathbb{R}^n$. Let v be the auxiliary function given by

$$v = v(\cdot, \xi) = \log(w_{ij}\xi_i \xi_j) + \tau |Du|^2 + \kappa \widetilde{\varphi}, \tag{3.4}$$

where $w_{ij} = D_{ij}u - A_{ij}$. By differentiation of equation (1.1), we have

$$\begin{aligned} w^{ij}\big[D_{ij}u_\xi - D_\xi A_{ij} - (D_z A_{ij})u_\xi - (D_{p_k} A_{ij}) D_k u_\xi\big] \\ = D_\xi \widetilde{B} + (D_z \widetilde{B})u_\xi + (D_{p_k} \widetilde{B}) D_k u_\xi, \end{aligned} \tag{3.5}$$

where $\widetilde{B} = \log B$. A further differentiation yields

$$
\begin{aligned}
w^{ij}\big[&D_{ij}u_{\xi\xi} - D_{\xi\xi}A_{ij} - (D_{zz}A_{ij})(u_\xi)^2 - (D_{p_k p_l}A_{ij})D_k u_\xi D_l u_\xi \\
&- (D_z A_{ij})u_{\xi\xi} - (D_{p_k}A_{ij})D_k u_{\xi\xi} - 2(D_{\xi z}A_{ij})u_\xi \\
&- 2(D_{\xi p_k}A_{ij})D_k u_\xi - 2(D_{z p_k}A_{ij})(D_k u_\xi)u_\xi\big] \\
&- w^{ik}w^{jl}D_\xi w_{ij} D_\xi w_{kl} \\
= D_{\xi\xi}\widetilde{B} &+ (D_{zz}\widetilde{B})u_\xi^2 + (D_{p_k p_l}\widetilde{B})D_k u_\xi D_l u_\xi \\
&+ 2(D_{\xi z}\widetilde{B})u_\xi + 2(D_{\xi p_k}\widetilde{B})D_k u_\xi + 2(D_{z p_k}\widetilde{B})(D_k u_\xi)u_\xi \\
&+ (D_z\widetilde{B})u_{\xi\xi} + (D_{p_k}\widetilde{B})D_k u_{\xi\xi}.
\end{aligned}
\tag{3.6}
$$

Furthermore differentiating (3.4) we have

$$
D_i v = \frac{D_i w_{\xi\xi}}{w_{\xi\xi}} + 2\tau D_k u D_{ki} u + \kappa D_i \widetilde{\varphi}, \tag{3.7}
$$

$$
\begin{aligned}
D_{ij} v = \frac{D_{ij} w_{\xi\xi}}{w_{\xi\xi}} &- \frac{D_i w_{\xi\xi} D_j w_{\xi\xi}}{w_{\xi\xi}^2} \\
&+ 2\tau(D_{ik}uD_{jk}u + D_k u D_{ijk}u) + \kappa D_{ij}\widetilde{\varphi},
\end{aligned}
\tag{3.8}
$$

where we have written $w_{\xi\xi} = D_{ij}w\xi_i\xi_j$. Using condition (A3w) in (3.6) and retaining all terms involving third derivatives, we estimate

$$
\begin{aligned}
Lu_{\xi\xi} &:= w^{ij}(D_{ij}u_{\xi\xi} + b_k^{ij}D_k u_{\xi\xi}) - (D_{p_k}\widetilde{B})D_k u_{\xi\xi} \\
&\geq w^{ik}w^{jl}D_\xi w_{ij}D_\xi w_{kl} - C\{(1+w_{ii})w^{ii} + (w_{ii})^2\}
\end{aligned}
\tag{3.9}
$$

where, as in the previous section, $b_k^{ij} = -D_{p_k}A_{ij}$ and C is a constant depending on the first and second derivatives of A and $\log B$ and $\sup_\Omega(|u| + |Du|)$. To apply (A3w), we fix a point $x \in \Omega$ and choose coordinate vectors as the eigenfunctions of the matrix $[w_{ij}]$ corresponding to eigenvalues $0 < \lambda_1 \leq \cdots \leq \lambda_n$. Writing $A_{ij,kl} = D_{p_k p_l}A_{ij}$, we then estimate

$$
\begin{aligned}
w^{ij}A_{ij,kl}u_{k\xi}u_{l\xi} &\geq w^{ij}A_{ij,kl}w_{k\xi}w_{l\xi} - Cw^{ii}(1+w_{ii}) \\
&\geq \sum_{k \text{ or } l = r} \frac{1}{\lambda_r} A_{rr,kl}(\lambda_k\xi_k)(\lambda_l\xi_l) - Cw^{ii}(1+w_{ii}) \\
&\geq -C\{w^{ii}(1+w_{ii}) + w_{ii}\}.
\end{aligned}
$$

From (3.9), we obtain also

$$
Lw_{\xi\xi} \geq w^{ik}w^{jl}D_\xi w_{ij}D_\xi w_{kl} - C\{(1+w_{ii})w^{ii} + w_{ii}^2\} \tag{3.10}
$$

for a further constant C. Here we use equation (3.5) to control the third derivative term arising from differentiating $A_{kl}\xi_k\xi_l$. From (3.8) and (3.10), we obtain, after some reduction,

$$\begin{aligned} Lv \geq & \frac{1}{w_{\xi\xi}} w^{ik} w^{jl} D_\xi w_{ij} D_\xi w_{kl} - \frac{1}{w_{\xi\xi}^2} w^{ij} D_i w_{\xi\xi} D_j w_{\xi\xi} \\ & + 2\tau w_{ii} + \kappa w^{ii} - C \left\{ \frac{1}{w_{\xi\xi}} [(1 + w_{ii}) w^{ii} + w_{ii}^2] + \tau + \kappa \right\} . \end{aligned} \tag{3.11}$$

Now suppose v takes its maximum at a point $x_0 \in \Omega$ and a vector ξ, which we take to be e_1. We need to control the first two terms on the right hand side of (3.11). To do this we choose remaining coordinates so that $[w_{ij}]$ is diagonal at x_0. Then we estimate

$$\begin{aligned} & \frac{1}{w_{\xi\xi}} w^{ik} w^{jl} D_\xi w_{ij} D_\xi w_{kl} - \frac{1}{w_{\xi\xi}^2} w^{ij} D_i w_{\xi\xi} D_j w_{\xi\xi} \\ & = \frac{1}{w_{11}} w^{ii} w^{jj} (D_1 w_{ij})^2 - \frac{1}{w_{11}^2} w^{ii} (D_i w_{11})^2 \\ & \geq \frac{1}{w_{11}^2} \sum_{i>1} [2 w^{ii} (D_1 w_{1i}^2 - w^{ii} (D_i w_{11})^2] \\ & = \frac{1}{w_{11}^2} \sum_{i>1} w^{ii} (D_i w_{11})^2 \\ & \quad + \frac{2}{w_{11}^2} \sum_{i>1} w^{ii} [D_1 w_{1i} - D_i w_{11}] [D_1 w_{1i} + D_i w_{11}] \\ & \geq \frac{1}{w_{11}^2} \sum_{i>1} w^{ii} (D_i w_{11})^2 \\ & \quad + \frac{2}{w_{11}^2} \sum_{i>1} w^{ii} [D_i A_{11} - D_1 A_{1i}] [2 D_i w_{11} + D_i A_{11} - D_1 A_{1i}] \\ & \geq -C w^{ii} . \end{aligned} \tag{3.12}$$

Combining (3.11) with (3.12), we obtain the estimate, at x_0,

$$Lv \geq \tau w_{ii} + \kappa w^{ii} - C\{\tau + \kappa\}, \tag{3.13}$$

for either τ or κ sufficiently large. Note that when we use (3.7) in the second last line of (3.12), we improve (3.13) by retention of the term $\sum_{i>1} w^{ii}(D_i w_{11})^2$ on the right hand side , which corresponds to the key term in the Pogorelov argument for interior estimates [7] ; (see [14] for an extension of this argument to the derivation of interior second order derivative estimates under (A3w)).

From (3.13), we finally obtain an estimate from above for $w_{ii}(x_0)$, which we formulate in the following theorem.

Theorem 3.1. *Let $u \in C^4(\Omega)$ be an elliptic solution of equation* (1.1) *in Ω , with $x, u(x), Du(x) \in \mathcal{U}$, for all $x \in \Omega$. Suppose the conditions* (A3w) *and* (3.3) *hold and B is a positive function in $C^2(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$. Then we have the estimate*

$$\sup_{\Omega} |D^2 u| \le C(1 + \sup_{\partial\Omega} |D^2 u|), \tag{3.14}$$

where the constant C depends on A, B, Ω, and $\sup_U(|z| + |p|)$.

Note that we only need the condition (A3w) to hold on the set $\mathcal{U}$. When only the non-degenerate condition (A3) holds then Theorem 3.1 trivially follows by taking $v = w_{ii}$, [15].

From the proof of Theorem 3.1 we obtain the corresponding estimate for equation (1.14), without the barrier condition (3.3).

Theorem 3.2. *Let $u \in C^4(\Omega)$ be an elliptic solution of equation* (1.14) *in Ω with $Tu(\Omega) \subset \Omega^*$. Suppose the cost function c satisfies hypotheses* (A1), (A2), (A3w)*, with A depending only on p and B is a positive function in $C^2(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$. Then we have the estimate* (3.14)*, with C depending only on C, B, Ω and Ω^*.*

To prove Theorem 3.2, we take $\kappa = 0$ in the proof of Theorem 3.1, to obtain an estimate for w_{ii} in terms of w^{ii}, that is

$$w_{ii} \le \varepsilon \sup_{\Omega} w^{ii} + C_\varepsilon\big(1 + \sup_{\partial\Omega} |D^2 u|\big), \tag{3.15}$$

for arbitrary $\varepsilon > 0$, with constant C_ε also depending on ε. If T is globally one-to-one, we then conclude (3.13), in the optimal transportation case, by using the dual problem (2.37), (2.38). More generally, we consider the dual function v^* in place of (3.4), given by

$$v^* = v(x, \xi) = \log\big(w^{ij} c_{i,k} c_{j,l} \xi_k \xi_l\big) + \tau |c_y(x, Tu(x))|^2 \tag{3.16}$$

and suppose it is maximized at a point x_0^* in Ω. Since T will now be one-to-one from a neighborhood $\mathcal{N}$ of x_0^* to a neighbourhood $\mathcal{N}^*$ of $y_0^* = Tu(x_0^*)$, we may then proceed as before, noting that in $\mathcal{N}^*$, v^* is given by (3.4) with u replaced by its c-transform u^*.

The estimate (3.14) arose from investigation of the classical Dirichlet problem (see [19]). We remark also that from (3.15), we see that (3.3) is also not needed when $n = 2$.

4. Boundary estimates for the second derivatives

This part of our argument is similar to the treatment of the oblique boundary value problems for Monge-Ampère equations in [13, 24]. The paper [13] concerned the Neumann problem, utilizing a delicate argument which did not extend to other linear oblique boundary conditions. For nonlinear oblique conditions of the form (2.1) where the function G is uniformly convex in the gradient, the twice tangential differentiation of (2.1) yields quadratic terms in second derivatives which compensate for the deviation of $\beta = G_p$ from the geometric normal and permit some technical simplification for general inhomogeneous terms ψ [24].

First we deal with the non-tangential second derivatives. Letting $F \in C^2(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$ and $v = F(\cdot, u, Du)$, where $u \in C^3(\overline{\Omega})$ is an elliptic solution of equation (1.1), we have from our calculation in Section 2,

$$|Lv| \leq C(w^{ii} + w_{ii} + 1), \tag{4.1}$$

where L is given by (2.18) and C is a constant depending on A, B, G, Ω and $|u|_{1;\Omega}$. Now using the equation (1.1) itself, we may estimate

$$w_{ii}^{\frac{1}{n-1}} \leq C w^{ii}, \tag{4.2}$$

so that, writing $M = \sup_\Omega w_{ii}$, we have from (4.1)

$$|Lv| \leq C(1 + M)^{\frac{n-2}{n-1}} w^{ii}. \tag{4.3}$$

Hence, if there exists a C^2 defining function φ satisfying (3.3) near $\partial\Omega$, together with $\varphi = 0$ on $\partial\Omega$, we obtain by the usual barrier argument, taking $F = G$,

$$|D(\beta \cdot Du)| \leq C(1 + M)^{\frac{n-2}{n-1}} \tag{4.4}$$

on $\partial\Omega$, so that in particular

$$w_{\beta\beta} \leq C(1 + M)^{\frac{n-2}{n-1}} \tag{4.5}$$

on $\partial\Omega$. Now for any vector $\xi \in \mathbb{R}^n$, we have

$$w_{\xi\xi} = w_{\tau\tau} + b(w_{\tau\beta} + w_{\beta\tau}) + b^2 w_{\beta\beta}, \tag{4.6}$$

where

$$b = \frac{\xi \cdot \gamma}{\beta \cdot \gamma}, \quad \tau = \xi - b\beta. \tag{4.7}$$

Suppose $w_{\xi\xi}$ takes its maximum over $\partial\Omega$ and tangential ξ, $|\xi| = 1$ at $x_0 \in \partial\Omega$ and $\xi = e_1$. Then from (4.5) and (4.6) and tangential differentiation of the boundary condition (2.1) we have on $\partial\Omega$,

$$w_{11} \leq |e_1 - b\beta|^2 w_{11}(0) + bF(\cdot, u, Du) + Cb^2(1 + M)^{\frac{n-2}{n-1}}, \tag{4.8}$$

for a given function $F \in C^2(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$. Combining (2.26), (3.10), (4.1) and (4.2) and utilizing a similar barrier argument to that is Section 2, we thus obtain the third derivative estimate

$$-D_\beta w_{11}(x_0) \leq C(1 + M)^{\frac{2n-3}{n-1}}. \tag{4.9}$$

Differentiating (2.1) twice in a tangential direction τ, with $\tau(x_0) = e_1$, we obtain at x_0,

$$(D_{p_k p_l} G) u_{1k} u_{1l} + (D_{p_k} G) u_{11k} \leq C(1 + M), \tag{4.10}$$

whence we conclude from (4.9)

$$\max_{\partial\Omega} |D^2 u| \leq C(1 + \sup_{\Omega} |D^2 u|)^{\frac{2n-3}{2(n-1)}} \tag{4.11}$$

by virtue of the uniform convexity of G with respect to p. Taking account of the global estimate (3.14), we complete the proof of Theorem 1.1. □

Once the second derivatives are bounded, the equation (1.1) is effectively uniformly elliptic so that from the obliqueness estimate (2.45), we obtain global $C^{2,\alpha}$ estimates from the theory of oblique boundary value problems for uniformly elliptic equations in [12]. By the theory of linear elliptic equations with oblique boundary conditions [7], we then infer estimates in $C^{3,\alpha}(\overline{\Omega})$ for any $\alpha < 1$ from the assumed smoothness of our data. We may also have assumed that our solution $u \in C^2(\overline{\Omega})$.

As in the previous section, the technicalities are simpler when condition (A3w) is strengthened to condition (A3) and we also obtain local boundary estimates for the second derivatives. To see this we estimate the tangential second derivatives first by differentiating the equation (1.1) and boundary condition (2.1) twice with respect to a tangential vector field τ near a point $y \in \partial\Omega$. We then obtain an estimate for $\eta D_{\tau\tau} u$, for an appropriately chosen cut-off function η. The mixed tangential-normal second derivatives $D_{\tau n} u$ are estimated as above by a single tangential differentiation of (2.1) so that the double normal derivative may be obtained either from (4.5) or from the equation (1.1) itself and the estimates in Section 2 for $w^{ij}\gamma_i\gamma_j$ from below, similarly to the Dirichlet problem, see [18].

5. Method of continuity

To complete the proof of Theorem 1.2, we adapt the method of continuity for nonlinear oblique boundary value problems, presented in [7] and already used in the special case (1.16), (1.17), [22]. The situation here is more complicated unless we know in advance that there exists a smooth function u_0, satisfying the ellipticity condition (1.2) together with the boundary condition (1.7). Later in Section 7, we shall prove the existence of such a function (at least for approximating domains). Otherwise we need to consider families of subdomains. To commence the procedure, we fix a point $x_0 \in \Omega$. Then for sufficiently small radius $r > 0$, the ball $\Omega_0 = B_r(x_0) \subset \Omega$ will be uniformly c-convex with respect to Ω^* and the function u_0, given by

$$u_0(x) = \frac{\kappa}{2}|x - x_0|^2 + p_0 \cdot (x - x_0), \tag{5.1}$$

will satisfy the ellipticity condition (1.2). Moreover the image $\Omega_0^* = Tu_0(\Omega_0)$ will be uniformly c^*-convex with respect to Ω with Tu_0 a diffeomorphism from Ω_0 to

Ω_0^*. To see this we observe that

$$c_x(x_0, \Omega_0^*) = B_{\kappa r}(p_0), \tag{5.2}$$

so that by taking κr small enough, we can fulfill condition (1.25) on $\partial\Omega_0^*$, with respect to $x_0 \in \Omega$, for constant $\delta_0 = \frac{1}{\kappa r}$ as large as we wish. Suppose now we can foliate $\Omega - \Omega_0$ and $\Omega^* - \Omega_0^*$ by boundaries of c-convex and c^*- convex domains, respectively. That is there exist increasing families of domains $\{\Omega_t\}$, $\{\Omega_t^*\}$, $0 \le t \le 1$, continuously depending on the parameter t, such that

(i) $\Omega_t \subset \Omega$, $\Omega_t^* \subset \Omega^*$,
(ii) $\Omega_1 = \Omega$, $\Omega_1^* = \Omega^*$,
(iii) $\partial\Omega_t, \partial\Omega_t^* \in C^4$, uniformly with respect to t,
(iv) Ω_t, Ω_t^* are uniformly c-convex, c^*-convex with respect to Ω^*, Ω, respectively.

The construction of such a family is discussed below.

Given our families of domains Ω_t, Ω_t^*, $0 < t \le 1$, we need to define corresponding equations. Let B be a positive function in $C^2(\overline{\Omega} \times \mathbb{R}^n)$ and f a positive function in $C^2(\overline{\Omega})$ such that

$$f = -\sigma u_0 + \log[\det\{D^2 u_0 - c_{xx}(\cdot, Y(\cdot, Du_0))\}/B(\cdot, Du_0)] \tag{5.3}$$

in Ω_0, for some fixed constant $\sigma > 0$. We then consider the family of boundary value problems:

$$\begin{aligned} F[u] : &= \det\{D^2 u - c_{xx}(\cdot, Y(\cdot, Du))\} = e^{\sigma u + (1-t) f} B(\cdot, Du), \\ Tu(\Omega_t) &= Y(\cdot, Du)(\Omega_t) = \Omega_t^*. \end{aligned} \tag{5.4}$$

From our construction and the obliqueness, we see that u_0 is the unique elliptic solution of (5.4) at $t = 0$.

From Section 2, we also see that the boundary condition in (5.4) is equivalent to the oblique condition

$$G_t(\cdot, Du) := \varphi_t^*(Y(\cdot, Du)) = 0 \quad \text{on } \partial\Omega_t. \tag{5.5}$$

To adapt the method of continuity from [7], we fix $\alpha \in (0, 1)$ and let Σ denote the subset of $[0, 1]$ for which the problem (5.4) is solvable for an elliptic solution $u = u_t \in C^{2,\alpha}(\overline{\Omega}_t)$, with Tu invertible. We then need to show that Σ is both closed and open in $[0, 1]$. First we note that the boundary condition (5.4) implies a uniform bound for Du_t. Integrating the equation (5.4), we then obtain uniform bounds for the quantities

$$\int_{\Omega_t} e^{\sigma u_t},$$

so that the solutions u_t will be uniformly bounded for $\sigma > 0$. Uniform estimates in $C^{2,1}(\overline{\Omega})$ now follow from our a priori estimates in Section 4, which are also clearly independent of $t \in [0, 1]$. By compactness, we then infer that Σ is closed. To show Σ is open, we use the implicit function theorem and the linear theory of oblique boundary value problems, as in [7]. The varying domains $\{\Omega_t\}$ may be handled by means of diffeomorphisms approximating the identity, which transfer the problem (5.4) for t close to some $t_0 \in \Sigma$ to a problem in Ω_{t_0}. We then conclude the solvability of (5.4) for all $t \in [0, 1]$, which implies there exists a unique elliptic solution $u = u_\sigma \in C^3(\overline{\Omega})$ of the boundary value problem

$$\begin{aligned} F[u] &= e^{\sigma u} B(\cdot, Du), \\ Tu(\Omega) &= \Omega^* \end{aligned} \tag{5.6}$$

for arbitrary $\sigma > 0$, with Tu one-to-one. To complete the proof of Theorem 1.2,(at least when the above foliations exist), we assume that B satisfies (1.6), (1.8) and (1.9). As above we see that the integrals

$$\int_\Omega e^{\sigma u_\sigma}$$

are uniformly bounded, with $D(\sigma u_\sigma) \to 0$ as $\sigma \to 0$. Consequently $\sigma u_\sigma \to$ constant$= 0$ by (1.9) and modulo additive constants, $u_\sigma \to u$ as $\sigma \to 0$, where u is the solution of (1.14), (1.7), as required.

We may construct the family of domains $\{\Omega_t\}$ used above, if we are given a C^4 defining function φ, satisfying

$$[D_{ij}\varphi(x) - c^{l,k} c_{ij,l}(x, y) D_k\varphi(x)]\xi_i\xi_j \geq \delta_0|\xi|^2 \tag{5.7}$$

for all $x \in \Omega$, $y \in \Omega^*$, $\xi \in \mathbb{R}^n$, which takes its minimum at x_0. Note that the uniform c-convexity of Ω implies the existence of a defining function satisfying (5.7) in a neighbourhood of $\partial\Omega$, as in (2.28) and if Ω is also c-bounded, we can extend to all of Ω.

There are various ways of constructing suitable families from a global defining function, φ. In particular taking $\varphi(x_0) = -1$, we may choose

$$\Omega_t = \{x \in \Omega \mid \varphi_t(x) < 0\}$$

where φ_t is defined by

$$\varphi_t = t(\varphi - a) + (t_0 - t)\varphi_0, \tag{5.8}$$

for $\varphi_0(x) = |x - x_0|^2 - r^2$, $t \leq t_0$, for some $0 < t_0 < 1$ and a close to -1 to first deform to a small sub-level set of φ, followed by taking $\varphi_t = (1 - t)a/(1 - t_0)$ for $t \geq t_0$. Alternatively, we could have chosen $\Omega_0 = \varphi < a$ at the outset and only used the second deformation.The domains Ω_t^* may be similarly constructed. If the curvatures of $\partial\Omega^*$ are sufficiently large, for example if Ω^* is a small ball, then the existence of a defining function satisfying (5.7) follows by pulling back from a single image $c_y(\cdot, y_0)(\Omega)$. As a byproduct, we see that if Ω is uniformly

c-convex with respect to a single point $y_0 \in \Omega^*$, there at least exists a smooth function $u_0 \in C^3(\overline{\Omega})$ satisfying the ellipticity condition (1.2). Moreover Tu_0 is a diffeomorphism from Ω to a small ball $B_r(y_0) \subset \Omega^*$.

From the above considerations, we see that the proof of Theorem 1.2 is completed in the cases where either Ω or Ω^* is a small ball. The general case will then follow by further use of the method of continuity if there exists a defining function satisfying (5.7) for either domain. However we will take up a different approach from this point and use the function u_0 constructed above to construct a further function u_1 approximately satisfying our given boundary conditions to which the method of continuity can be applied without domain variation. Specifically we will prove:

Lemma 5.1. *Let the domains Ω and Ω^* and cost function c satisfy the hypotheses of Theorem* 1.1. *Then for any $\epsilon > 0$, there exists a uniformly c^*-convex approximating domain, Ω^*_ϵ, lying within distance ϵ of Ω^*, and satisfying the corresponding condition* (1.20) *for fixed δ_0, together with a function $u_1 \in C^4(\overline{\Omega})$ satisfying the ellipticity condition* (1.2) *and the boundary condition* (1.7) *for Ω^*_ϵ.*

From Lemma 5.1 we complete the proof of Theorem 1.2. We defer the proof of Lemma 5.1 to Section 7 as the proof will use some of the same ingredients as in our discussion of optimal transportation in the next section. In Section 7, we will also indicate an alternative and direct construction of the function u_1, which avoids domain variation altogether in the method of continuity. The full procedure in this section is still needed though for extensions to the general prescribed Jacobian equation, (1.4), [19]. Moreover if we remove the invertibility of the mapping c_y from hypothesis (A1), we still conclude Theorems 1.1, 1.2 and 1.3 under the global uniform c-convexity hypothesis that there exists global defining functions for Ω and Ω^* satisfying (5.7), [19].

6. Optimal transportation

The interior regularity of solutions to the optimal transportation problem is considered in [15], under conditions (A1), (A2), (A3) and the c^*- convexity of the target domain Ω^*. Our approach is to first show that the Kantorovich potentials are generalized solutions of the boundary value problem (1.14), (1.7) in the sense of Aleksandrov and Bakel′man. The c^*-convexity of Ω^* is used to show the image of the generalized normal mapping lies in $\overline{\Omega}^*$ and condition (A3) is employed to obtain a priori second derivative estimates from which the desired regularity follows. The potential functions u and v solve the dual problem of minimizing the functional

$$I(u, v) = \int_\Omega fu + \int_{\Omega^*} gv \tag{6.1}$$

over the set K given by

$$K = \Big\{(u, v) \;\Big|\; u, v \in C^0(\Omega), C^0(\Omega^*) \text{ resp. } u(x) + v(y) \geq c(x, y) \text{ for all } x \in \Omega, y \in \Omega^*\Big\}. \tag{6.2}$$

The potential functions (u, v) satisfy the relations

$$\begin{aligned} u(x) &= \sup_{y\in\Omega} \big\{c(x, y) - v(y)\big\}, \\ v(y) &= \sup_{x\in\Omega} \big\{c(x, y) - u(y)\big\}, \end{aligned} \tag{6.3}$$

that is they are the c^* and c transforms of each other. Since $c \in C^{1,1}$, they will be semi-convex. The optimal mapping T is then given almost everywhere by (1.3) and the equation (1.14) will be satisfied with elliptic solution u almost everywhere in Ω. The functions u and v are respectively c and c^*- convex. A function $u \in C^0(\Omega)$ is called c-convex in Ω if for each $x_0 \in \Omega$, there exists $y_0 \in \mathbb{R}^n$ such that

$$u(x) \geq u(x_0) + c(x, y_0) - c(x_0, y_0) \tag{6.4}$$

for all $x \in \Omega$. If u is a c-convex function, for which the mapping T given by (1.3) is measure preserving, then it follows that u is a potential and again T is the unique optimal mapping. These results hold under the hypotheses (A1) and (A2) and it suffices to assume the densities $f, g \geq 0, \in L^1(\Omega), L^1(\Omega^*)$, respectively, whence the mapping T is only determined almost everywhere on the set where f is positive. The reader is referred to [3, 5, 6, 15, 23, 25] for further details.

From the above discussion we see that the solution of the boundary value problem (1.14), (1.7) will automatically furnish a potential for the optimal transportation problem if it is c-convex. Note that ellipticity only implies that the solution is locally c-convex and we need a further argument to conclude the global property, unlike the case of quadratic cost functions and convex solutions. First we recall the concept of generalized solution introduced in [15]. Let u be a c-convex function on the domain Ω. The c-normal mapping, χ_u, is defined by

$$\chi_u(x_0) = \Big\{y_0 \in \mathbb{R}^n \;\Big|\; u(x) \geq u(x_0) + c(x, y) - c(x_0, y_0), \text{ for all } x \in \Omega\Big\}. \tag{6.5}$$

Clearly, $\chi_u(x_0) \subset Y(x_0, \partial u(x_0))$ where ∂ denotes the subgradient of u. For $g \geq 0, \in L^1_{\ell oc}(\mathbb{R}^n)$, the generalized *Monge-Ampère measure* $\mu[u, g]$ is then defined by

$$\mu[u, g](e) = \int_{\chi_u(e)} g \tag{6.6}$$

for any Borel set $e \in \Omega$, so that u satisfies equation (1.14) in the generalized sense if

$$\mu[u, g] = f\,\mathrm{d}x\,. \tag{6.7}$$

The boundary condition (1.7) is satisfied in the generalized sense if

$$\Omega^* \subset \chi_u(\overline{\Omega}), \qquad \left|\left\{ x \in \Omega \;\middle|\; f(x) > 0 \text{ and } \chi_u(x) - \overline{\Omega}^* \neq \emptyset \right\}\right| = 0. \tag{6.8}$$

The theory of generalized solutions replicates that for the convex case, $c(x, y) = x \cdot y$, [13]. If f and g are positive, bounded measurable functions on Ω, Ω^* respectively satisfying the mass balance condition (1.9), and c satisfies (A1), (A2), then there exists a unique (up to constants) generalized solution of (6.7), (6.8) (with $g = 0$ outside Ω^*), which together with its c transform v, given by (6.3), uniquely solves the dual problem (6.1), (6.2), [15].

Now let $u \in C^2(\overline{\Omega})$ be an elliptic solution of the boundary value problem (1.7), (1.14) and v a c-convex solution of the corresponding generalized problem. By adding constants, we may assume $\inf_\Omega(u - v) = 0$. We need to prove $u = v$ in Ω, that is the strong comparison principle holds. Let Ω' denote the subset of Ω where $u > v$ and first suppose that $\partial\Omega' \cap \Omega \neq \emptyset$. Note that if $v \in C^2(\Omega)$, this situation is immediately ruled out by the classical strong maximum principle [7]. Otherwise we may follow the proof of the strong maximum principle as there will exist a point $x_0 \in \partial\Omega' \cap \Omega$, where Ω' satisfies an interior sphere condition, that is there exists a ball $B \subset \Omega - \Omega'$ such that $x_0 \in \partial\Omega' \cap \partial B$, $u(x_0) = v(x_0)$ and $u > v$ in B. Since v is semi-convex, v will be twice differentiable at x_0, with $Dv(x_0) = Du(x_0)$. Moreover by passing to a smaller ball if necessary we may assume both u and v are c-convex in B. Since u is a smooth elliptic solution of (1.14), there will exist a strict supersolution $w \in C^2(\overline{B} - B_\rho)$, for some concentric ball B_ρ of radius $\rho < R$, satisfying $w(x_0) = u(x_0)$, $w \geq v$ on $\partial B \cup \partial B_\rho$, $Dw(x_0) \neq Du(x_0)$. By the comparison principle, [15], Lemma 5.2, we have $w \geq v$ in $B - B_\rho$, and hence $Dw(x_0) = Dv(x_0)$, which is a contradiction. Thus we may assume $\partial\Omega' \cap \Omega = \emptyset$, that is $u > v$ in Ω with $u(x_0) = v(x_0)$ for some point $x_0 \in \partial\Omega$. From our argument above, we obtain a function $w \in C^2(\overline{B} - B_\rho)$ satisfying $w(x_0)u(x_0) = v(x_0)$, $v \leq w \leq u$ in $\overline{B} - B_\rho$, together with

$$u(x) - w(x) \geq \epsilon|x - x_0| \tag{6.9}$$

for all $x \in B_R - B_\rho$. Since $v \leq w$ in $B_R - B_\rho$, this contradicts the obliqueness condition (2.45) if Ω^* is c^*-convex.

Alternatively we may proceed directly (and more simply) as follows to show that the solution u is c-convex, using the property that Tu is one-to-one. In fact, as mentioned previously in Section 2, this would follow automatically from the change of variables formula by virtue of the mass balance condition (1.9). Let $x_0 \in \Omega$ and $y_0 = Tu(x_0)$. Suppose there exists a point $x_1 \in \Omega$, where

$$u(x_1) < c(x_1, y_0) - c(x_0, y_0). \tag{6.10}$$

By downwards vertical translation, there exists a point $x_2 \in \partial\Omega$, satisfying

$$u(x) > u(x_2) + c(x, y_0) - c(x_2, y_0). \tag{6.11}$$

for all $x \in \Omega$. Putting $y_2 = Tu(x_2)$, we must also have

$$c_x(x_2, y_2).\gamma(x_2) < c_x(x_2, y_0).\gamma(x_2), \tag{6.12}$$

which again contradicts the c^*-convexity of Ω^*.

Remark 6.1. In the first proof above, we employed a comparison result that if u is a classical elliptic supersolution of (1.14) dominating a generalized subsolution v on the boundary of a subdomain Ω', then $u \geq v$ in Ω'. In our local uniqueness argument in [15], we also used implicitly the complementing result that if u is an elliptic subsolution dominated by a generalized supersolution v on $\partial\Omega'$, then $u \leq v$ in Ω'. However,in this case,we cannot apply Lemma 5.2 in [15] directly as local c-convexity of v may not imply global c-convexity in Ω, unless $v \in C^1(\Omega)$. This situation is rectified in [20], under the (A3) hypothesis; (see also [10,21]). However if Ω and Ω^* lie respectively in domains Ω_0 and Ω_0^* satisfying the hypotheses of Theorem1.2, with Ω^* also c^*-convex with respect to Ω_0, and f and g are positive in $L^1(\Omega)$ and $L^1(\Omega^*)$ respectively, then it follows directly by approximation from Theorem 1.3 that the local c-convexity of the potential u solving the Kantorovich dual problem implies its global c-convexity. Other results, such as the c-convexity of the contact set under condition (A3w), also follow from Theorem 1.3 by approximation. The reader is referred to Loeper [11] for a full treatment of this approach, including the sharpness of condition (A3w) for regularity.

We also note that we may alternatively conclude the global c-convexity of elliptic solutions of the boundary value problem, (1.14), (1.7), under condition (A3w) and the c-convexity of Ω, from [20, Theorem 2.1] (see also [21]).

7. Completion of the proof of Theorem 1.2

In this section, we provide the proof of Lemma 5.1, thereby completing that of Theorem 1.2. For this purpose, we need to draw on a geometric property of c-convex domains introduced in [20]. Namely, suppose that Ω is uniformly c-convex, with respect to Ω^*, and that the cost function c satisfies conditions (A1), (A2) and (A3w). Denoting as before the unit outer normal to $\partial\Omega$ by γ, we see that the level set $\mathcal{E}$ of the function e, given by

$$e(x) = e_y(x) = c(x, y) - c(x, y_0), \tag{7.1}$$

passing through x_0, is tangential to $\partial\Omega$ at x_0 if

$$y = Y(x_0, p_0 + t\gamma_0), \tag{7.2}$$

for $t > 0$, $p_0 = c_x(x_0, y_0)$, $\gamma_0 = \gamma(x_0)$ that is, y lies on the c^*-segment which is the image under $Y(x_0, \cdot)$ of the line from p_0 with slope γ_0. Then it follows from [20] that Ω lies strictly on one side of $\mathcal{E}$, whence

$$c(x, y) - c(x_0, y) < c(x, y_0) - c(x_0, y_0), \quad x \in \overline{\Omega} - \{x_0\}. \tag{7.3}$$

To prove (7.3) directly from (1.20), we take $x_0 = 0$, set $x' = (x_1, ...x_{n-1})$ and choose coordinates so that $\gamma_0 = (0, ..., -1)$. By Taylor's formula,

$$e(x) - e(x_0) \le -tx_n + [A_{ij}(0, p_0 + t\gamma_0) - A_{ij}(0, p_0)]x'_i x'_j + tO(|x||x_n| + |x|^3).$$

Using (1.13), (1.20), condition (A3w) and again Taylor's formula, we have

$$\begin{aligned}\big[A_{ij}(0, p_0 + t\gamma_0) - A_{ij}(0, p_0)\big] x'_i x'_j &\le -tc^{l,n} c_{ij,l}(x_0, y_0) x'_i x'_j \\ &\le tD_i\gamma_j(x_0)x'_i x'_j - t\delta_0|x'|^2,\end{aligned}$$

so that,

$$e(x) - e(x_0) \le -tx_n + tD_i\gamma_j(x_0)x'_i x'_j - t\delta_0|x'|^2 + tO(|x||x_n| + |x|^3) < 0,$$

for $x \in \overline{\Omega} - x_0$, sufficiently small. Consequently Ω lies locally, strictly on one side of $\mathcal{E}$. We can then verify the global inequality (7.3) by contradiction, as in [20]. For if (7.3) is violated, the set

$$U_a = \{x \in \partial\Omega; e(x) > -a|\}$$

contains two disjoint components, for sufficiently small $a > 0$. Increasing a, we see that these components will meet first at a point $x^* \in \partial\Omega$ at which the level set of the function e is tangential, contradicting the local inequality at x^*. For another approach the reader is referred to [21].

Now, to commence the proof of Lemma 5.1, we take u_0 to be a function as constructed in Section 6, that is u_0 is a smooth uniformly c-convex function on Ω, whose c-normal mapping $Tu_0 = Y(\cdot, Du_0)$ has image ω^*, which is a c^*-convex subdomain of Ω^*. Here we call a c-convex function uniformly c-convex if it also satisfies the ellipticity condition (1.2). We remark that the c-convexity of u_0 could also have been proved from (7.3), using the c-convexity of Ω and condition (A3w), instead of the c^*-convexity of Ω^* which we used in Section 6. Also by approximation, we may assume $u_0 \in C^\infty(\overline{\Omega})$. A function h is called a c-function if it has the form

$$h(\cdot) = c(\cdot, y_0) + a$$

for $y_0 \in \mathbb{R}^n$ and some constant a. When $c(x, y) = x \cdot y$, a c-function is a linear function. Obviously the c-normal mapping of h is the constant map, $Th(x) = y_0$ for all $x \in \mathbb{R}^n$. Let

$$u_1(x) = \sup\{u_0, h(x)\}, \quad x \in \Omega^\delta, \tag{7.4}$$

where the sup is taken over the set $\mathcal{S}$ of c-functions h with $h \le u_0$ in Ω, $T_h(\Omega) \subset \Omega^*$ and $\Omega^\delta = \{x \in \mathbb{R}^n : \ \operatorname{dist}(x, \Omega) < \delta\}$, for some $\delta > 0$, is a neighbourhood of Ω. The following lemma describes the properties of u_1.

Lemma 7.1. *Assume that the cost function c satisfies* (A1), (A2), (A3w) *and the domains Ω and Ω^* are uniformly c-convex with respect to each other. Then, for* *sufficiently small δ, the function u_1 is a c-convex extension of u_0 to Ω^δ, whose c-normal image under u_1 is $\overline{\Omega}^*$. Moreover for any point $x \in \Omega^\delta - \overline{\Omega}$, there exist unique points, $x_b \in \partial\Omega$, $y_b \in \partial\Omega^*$, such that $\chi_{u_1} = y_b$ on the c-segment, ℓ_{x_b}, joining x to x_b, with respect to y_b,(except at the endpoint x_b) with the resultant mappings being C^2 diffeomorphisms from $\partial\Omega^r$ to $\partial\Omega, \partial\Omega^*$ respectively, for any $r < \delta$.*

Lemma 5.1 will follow from Lemma 7.1 by modification of u_1 outside Ω and mollification. To prove Lemma 7.1, we first take any c-function in the set $\mathcal{S}$, with c-normal image $y \in \Omega^* - \omega^*$ and increase it until its graph meets that of u_0 on $\overline{\Omega}$ at a point x_b, which will lie in $\partial\Omega$, since u_0 is uniformly c-convex. Accordingly we obtain a c-function $h \in \mathcal{S}$, given by

$$h(x) = h_{x_b,y}(x) = c(x, y) - c(x_b, y) + u_0(x_b). \tag{7.5}$$

Since $h \le u_0$ in Ω and $h(x_b) = u_0(x_b)$, we see that the point y must lie on the c^*-segment $\ell^*_{x_b}$,with respect to x_b, starting at $y_{0,b} = Tu_0(x_b)$ and given by

$$c_x(x_b, \ell^*_{x_b}) = \{Du_0(x_b) + t\gamma(x_b) : \ t \ge 0\}. \tag{7.6}$$

Conversely, for any $x_b \in \partial\Omega$, $y \in \ell^*_{x_b}$, we have

$$h \le u_0 \quad \text{in} \ \ \Omega, \tag{7.7}$$

by virtue of (7.3) (taking $x_0 = x_b$, $y_0 = y_{0,b}$). This proves that u_1 is indeed a c-convex extension of u_0 to Ω^δ.

To proceed further, we let y_b be the unique point in $\partial\Omega^*$, where $\ell^*_{x_b}$ intersects $\partial\Omega^*$. Since ω^* is also uniformly c^*-convex, $\ell^*_{x_b}$ only intersects $\partial\omega^*$ at the initial point $y_{0,b}$. Actually, only the uniform c-convexity of u_0 is needed to justify this. Henceforth we restrict $\ell^*_{x_b}$ to the segment joining $y_{0,b}$ to y_b. From our argument above, the mapping from x_b to y_b is onto $\partial\Omega^*$. From (7.3), it is also one-to-one as the c-function $h = h_{x_b,y_b}$ cannot meet $\partial\Omega$ at another point x'. It follows then that the mapping from x_b to y_b is a C^3 diffeomorphism from $\partial\Omega$ to $\partial\Omega^*$. Next, if B_r is a sufficiently small tangent ball of Ω at x_b, it will also be uniformly c-convex so again by (7.3), we obtain

$$h_{x_b,y_b}(x) > h_{x_b,y}(x) \quad \forall\, x \in B_r, \ y \in \ell^*_{x_b}, \tag{7.8}$$

and thus we have

$$u_1 = \max_{x_b \in \partial\Omega} \{u_0, h_{x_b,y_b}\}. \tag{7.9}$$

To complete the proof of Lemma 7.1, we need to show that for each $x \in \Omega^\delta - \Omega$, there exists a unique $x_b \in \partial\Omega$ where the maximum in (7.9) is attained. For this purpose, we invoke the c-transform of u_1,

$$v_0(y) = \sup\{c(x, y) - u_1(x) : \ x \in \Omega^\delta\}, \quad y \in \Omega^*, \tag{7.10}$$

which extends the c-transform of u_0 in ω^*. Moreover, by (7.7), we see that for $y \in \ell^*_{x_b}$, the sup is attained at x_b. Hence

$$v_0(y) = c(x_b, y) - u_0(x_b) \quad \forall\, y \in \ell^*_{x_b}. \tag{7.11}$$

One easily verifies that v_0 is smooth in $\overline{\Omega}^* - \partial\omega^*$. Using (7.11) and arguing as before, we infer that for any point $x \in \Omega^\delta - \overline{\Omega}$, there exists a unique point $y_b \in \partial\Omega^*$ such that

$$h^*_{x,y_b}(y) = c(x, y) - c(x, y_b) + v_0(y_b) \le v_0(y) \quad \forall\, y \in \overline{\Omega}^*. \tag{7.12}$$

Moreover x lies on the c-segment, ℓ_{y_b} given by

$$c_y(\ell_{y_b}, y_b) = \{Dv_0(y_b) + t\gamma^*(y_b) : \ t \in [0, \overline{\delta}]\}, \tag{7.13}$$

where γ^* denotes the unit outer normal to $\partial\Omega^*$ and $\overline{\delta}$ is a small constant. Note that $Dv_0(y_b) = c_y(x_b, y_b)$. From (7.12), we see that the maximum in (7.9) is attained at x_b, y_b so

$$u_1(x) = c(x, y_b) - c(x_b, y_b) + u_0(x_b) \quad x \in \ell_{y_b}, \tag{7.14}$$

with $\chi_{u_1}(\ell_{y_b} - \{x_b\}) = y_b$, $\chi_{u_1}(x_b) = \ell^*_{x_b}$. From the obliqueness of ℓ_{y_b} on $\partial\Omega$, we have that the mapping from $x \in \Omega^r$ to x_b is one-to-one for sufficiently small r. This completes the proof of Lemma 7.1.

From Lemma 7.1, we see that the function u_1 is smooth in $\Omega^\delta - \partial\Omega$. Furthermore, with δ sufficiently small, u_1 will be tangentially uniformly convex on $\partial\Omega^r$, that is

$$[D_{ij}u_1 - c_{ij}(\cdot, Tu_1)]\tau_i\tau_j \ge \lambda_0, \tag{7.15}$$

where τ is the unit tangent vector on $\partial\Omega^r$ and λ_0 a positive constant. To take care of the normal direction, we modify u_1 in $\Omega^\delta - \Omega$, by setting

$$u = u_1 + bd^2, \tag{7.16}$$

where b is a positive constant and d denotes distance from Ω. Again for δ sufficiently small, we infer that u satisfies

$$[D_{ij}u - c_{ij}(\cdot, Tu)]\xi_i\xi_j \ge \lambda_0,$$

in $\Omega^\delta - \partial\Omega$ for a further positive constant λ_0, and any unit vector ξ.

We complete the proof of Lemma 5.1 by mollification. Let $\rho \in C_0^\infty(B_1(0))$ be a mollifier, namely ρ is a smooth, nonnegative, and radially symmetric function supported in the unit ball $B_1(0)$ such that the integral $\int_{B_1(0)} \rho = 1$. We show that a mollification of u, given by

$$u_\varepsilon(x) = \rho * u = \int_{\mathbb{R}^n} \varepsilon^{-n}\rho\left(\frac{x-y}{\varepsilon}\right) u(y)\,dy = \int_{\mathbb{R}^n} \rho(y)u(x - \varepsilon y)\,dy \tag{7.17}$$

is uniformly c-convex in $\Omega^{\delta/2}$, provided $\varepsilon < \frac{1}{2}\delta$ is sufficiently small and $x \in \Omega^{\delta/2}$ (so that the value of u outside Ω^{δ} is irrelevant). Note that the image of the c-normal mapping of u_ε in $\Omega^{\delta/2}$ is a smooth perturbation of Ω^*, and so is also uniformly c-convex provided $\varepsilon > 0$ is sufficiently small.

It is easy to verify that

$$Du_\varepsilon(x) = \int_{\mathbb{R}^n} \rho(y) Du(x - \varepsilon y) dy, \tag{7.18}$$

$$\begin{aligned} D^2 u_\varepsilon(x) & \int_{\mathbb{R}^n} \rho(y) D^2 u(x - \varepsilon y) dy \\ & + \int_{\partial\Omega} \frac{1}{\varepsilon^n} \rho\left(\frac{x-y}{\varepsilon}\right) \gamma \cdot (D^+ u - D^- u)(y) \\ & \geq \int_{\mathbb{R}^n} \rho(y) D^2 u(x - \varepsilon y) dy, \end{aligned} \tag{7.19}$$

where

$$D^+ u(y) = \lim_{y' \notin \overline{\Omega}, y' \to y} Du(y'),$$
$$D^- u(y) = \lim_{y' \in \Omega, y' \to y} Du(y').$$

Since, $\omega^* \Subset \Omega^*$, we have

$$D^+_\gamma u - D^-_\gamma u \geq C_0 > 0 \text{ on } \partial\Omega \tag{7.20}$$

for some positive constant C_0. We divide $\Omega^{\delta/2}$ into three parts: $\Omega^{\delta/2} = U_1 \cup U_2 \cup U_3$, where

$$U_1 = \{x \in \Omega^{\delta/2} : \text{ dist}(x, \partial\Omega) \geq \varepsilon\},$$
$$U_2 = \{x \in \Omega^{\delta/2} : \text{ dist}(x, \partial\Omega) \in ((1-\sigma)\varepsilon, \varepsilon)\},$$
$$U_3 = \{x \in \Omega^{\delta/2} : \text{ dist}(x, \partial\Omega) \leq \varepsilon'\},$$

where $\sigma \in (\frac{1}{2}, 1)$ is a constant close to 1. Since u is smooth, uniformly c-convex away from $\partial\Omega$, u^ε is obviously smooth, uniformly c-convex in U_1 provided ε is sufficiently small. By taking $\sigma > 0$ sufficiently close to 1, for any point $x_0 \in U_2$, $Du^\varepsilon(x_0)$ is a small perturbation of $Du(x_0)$. By (7.19), we also see that the matrix

$$\{D^2 u^\varepsilon(x_0) - A(x_0, Du^\varepsilon(x_0))\} > 0, \tag{7.21}$$

namely u^ε is smooth, uniformly c-convex in U_2.

Finally we verify (7.21) in U_3. For any point $x_0 \in U_3$, we choose a coordinate system such that $x_0 = (0, \cdots, 0, x_{0,n})$, the origin $0 \in \partial\Omega$ and $\partial\Omega$ is tangent to $\{x_n = 0\}$. To verify (25), we first consider a tangential direction τ, namely τ is a unit vector tangential to $\partial\Omega$ at 0. Without loss of generality we assume that $\tau = (1, 0, \cdots, 0)$. Then we need to prove that

$$D_{11} u^\varepsilon(x_0) - A_{11}(x_0, Du^\varepsilon(x_0)) > 0. \tag{7.22}$$

By our choice of coordinates, $D_1u^\varepsilon(x_0)$ is a small perturbation of $D_1u(x_0)$. Hence it suffices to verify that

$$D_{11}u^\varepsilon(x_0) - A_{11}(x_0, D_1u(x_0), D'u^\varepsilon(x_0)) > 0, \tag{7.23}$$

where $D'u^\varepsilon = (D_2u^\varepsilon, \cdots, D_nu^\varepsilon)$. By (A3w), A_{11} is convex in $D'u^\varepsilon$, whence it follows readily that

$$A_{11}(x_0, D_1u(x_0), D'u^\varepsilon(x_0)) \le \int_{\mathbb{R}^n} \frac{1}{\varepsilon^n}\rho\left(\frac{x_0 - y}{\varepsilon}\right) A_{11}(x_0, D_1u(x_0), D'u(y))dy.$$

Inequality (7.22) now follows from (7.19) and (7.23). Note that the argument also applies to any direction η provided $\eta \cdot \gamma$ is sufficiently small. Next we observe from the second integral in (7.19) that (7.21) holds in the normal direction $\gamma = e_n$. Furthermore,

$$D_{nn}u^\varepsilon(x_0) - A_{nn}(x_0, Du^\varepsilon(x_0)) \ge K \tag{7.24}$$

for some K as large as we want, provided ε is sufficiently small. Now suppose the least eigenvalue of the matrix (7.21) is achieved in direction ξ. We can decompose $\xi = c_1\tau + c_2e_n$. If $c_2 \ge c_0$ for some constant $c_0 > 0$, then the matrix (7.21) in direction ξ is positive by (7.24). Otherwise the proof of (7.22) applies and we also see that the matrix (7.21) in direction ξ is positive. By appropriate adjustment of Ω, we complete the proof of Lemma 5.1 and consequently also Theorems 1.2 and 1.3.

To conclude this section we show that Lemma 5.1 may be proved independently of the arguments in Section 5 by direct construction of a uniformly c-convex function, u_0. To do this we let y_0 be a point in Ω^* and u_0 be the c^*- transform of the function

$$\psi(y) = -(r^2 - |y - y_0|^2)^{1/2}, \tag{7.25}$$

given by

$$u_0(x) = \sup\{c(x, y) - \psi(y), \quad y \in B_r(y_0)\},$$

for sufficiently small $r > 0$. Then u_0 is a locally uniformly c-convex function defined in some ball $B_R(0)$, with $R \to \infty$ as $r \to 0$, and the image of its c-normal mapping,

$$\omega^* := Tu_0(\Omega) \subset B_r(y_0),$$

where Tu_0 is a diffeomorphism between Ω and ω^*. As Tu_0 is defined on the ball $B_R(0) \Supset \Omega$, ω^* is a smooth domain. Locally u_0 is a smooth perturbation of the c-function

$$h_0(\cdot) = c(\cdot, y_0) + a_0, \tag{7.26}$$

for some constant a_0.

8. Examples

We repeat and somewhat expand the examples in [15], taking into account that our cost functions are the negatives of those there.

Example 1.

$$c(x, y) = -\sqrt{1 + |x - y|^2}\,. \tag{8.1}$$

Here the vector field Y and matrix A are given by

$$\begin{aligned} Y(x, p) &= x + \frac{p}{\sqrt{1 - |p|^2}}, \\ A(x, p) &= A(p) = -\big(1 - |p|^2\big)^{1/2}\big(I - p \otimes p\big). \end{aligned} \tag{8.2}$$

The cost function satisfies condition (A3). We remark that condition (A1) is only satisfied for $|p| < 1$ but this does not prohibit application of our results as the boundedness of the target domain Ω^* ensures that $|Du| < 1$ for solutions of (1.7), (1.14). More generally the conditions $p, q \in \mathbb{R}^n$ in (A1) may be replaced by $p, q \in$ some convex sub-domain.

Example 2.

$$c(x, y) = -\sqrt{1 - |x - y|^2}\,. \tag{8.3}$$

Here c is only defined for $|x - y| \leq 1$. The vector field Y and matrix A are given by

$$\begin{aligned} Y(x, p) &= -x + \frac{p}{\sqrt{1 + |p|^2}}, \\ A(x, p) &= A(p) = \big(1 + |p|^2\big)^{1/2}\big(I + p \otimes p\big). \end{aligned} \tag{8.4}$$

The cost function satisfies condition (A3). In order to directly apply our results we need to assume Ω and Ω^* are strictly contained in a ball of radius 1.

Example 3.
Let $f, g \in C^2(\Omega), C^2(\Omega^*)$ respectively and

$$c(x, y) = x \cdot y + f(x)g(y)\,. \tag{8.5}$$

If $|\nabla f.\nabla g| < 1$, then c satisfies (A1), (A2). If f, g are convex, then c satisfies (A3w), while if f, g are uniformly convex, then c satisfies (A3). As indicated in [15], the function (8.5) is equivalent to the square of the distance between points on the graphs of f and g. We also note that sublevel sets of f and g will be uniformly c-convex and c^*-convex respectively if f and g are uniformly c-convex, while the same is true for sublevel sets of f_ϵ, g_ϵ if f and g are only convex, for

$$f_\epsilon = f + \epsilon\sqrt{(1 + |x|^2)}$$

and positive ϵ sufficiently small. Also Ω and Ω^* are automatically c-bounded and c^*-bounded (take $\varphi = f$ or f_ϵ).

Example 4. Power costs

$$c(x, y) = \pm\frac{1}{m}|x - y|^m, \quad m \neq 0, \quad \log|x - y| \ , \ m = 0). \tag{8.6}$$

For $m \neq 1$ and $x \neq y$, when $m < 1$, the vector fields Y and matrices A are given by

$$\begin{aligned} Y(x, p) &= x \pm |p|^{\frac{2-m}{m-1}} p, \\ A(x, p) = A(p) &= \pm\big\{|p|^{\frac{m-2}{m-1}} I + (m-2)|p|^{-\frac{m}{m-1}} p \otimes p\big\}. \end{aligned} \tag{8.7}$$

The only cases for which condition (A3w) is satisfied are $m = 2(\pm)$ and $-2 \leq m < 1$ (+ only). For the latter, condition (A3) holds for $-2 < m < 1$. To apply our results directly in the latter cases, we need to assume Ω and Ω^* are disjoint.

In [15] we also considered the cost function

$$c(x, y) = -(1 + |x - y|^2)^{p/2} \tag{8.8}$$

for $1 \leq p \leq 2$, extending Example 1 to $p > 1$. We point out here that these functions only satisfy (A3) under the restriction $|x - y|^2 < \frac{1}{p-1}$. This condition was omitted in [15].

Example 5. Reflector antenna problem

Corresponding results and examples may be obtained on other manifolds such as the sphere S^n. Both the cost function conditions (A1), (A2), (A3w) (or (A3)) and the domain conditions (1.20) are invariant under local coordinate transformations. Indeed the considerations in [15] stemmed from the treatment of the reflector antenna problem by Wang in [27], which may be represented as an optimal transportation problem on the sphere S^n with cost function

$$c(x, y) = \log(1 - x \cdot y), \tag{8.9}$$

which is simply the spherical analogue of the case $m = 0$ in Example 4 above. The corresponding vector field Y is now given by

$$Y(x, p) = x - \frac{2}{1 + |p|^2}(x + p), \tag{8.10}$$

where now p belongs to the tangent space of S^n at x, while the matrix A is given by

$$A = \frac{1}{2}(|p|^2 - 1)g_0 - p \times p, \tag{8.11}$$

where g_0 denotes the metric on S^n. See [8, 27, 28] for more details. When the domains Ω and Ω^* have disjoint closures, and satisfy the appropriate analogues of uniform c-convexity, we obtain the global regularity of optimal mappings and potentials.

We will defer further examination and extensions to intersecting domains and other cost functions in a future work. We also point out here that Example 4 provides regularity for quadratic cost functions on spheres, that is $c(x, y) = x \cdot y$ when the points $x$and y are sufficiently close, in particular when domains Ω and Ω^* lie in the same quadrant and are uniformly spherically convex. Further examples of functions on spheres satisfying (A1),(A2) and (A3) are given by the intrinsic quadratic cost,

$$c(x, y) = -[\text{argcos}(x \cdot y)]^2,$$

for $x \cdot y > -1$, found by Loeper [11], and a generalized reflector antenna,

$$c(x, y) = \log(1 - kx \cdot y),$$

for $x \cdot y < k \leq 1$, corresponding to a refraction problem treated in [9]. Note that in all these cases the cost functions are functions of $x \cdot y$ but when we take the same functions in $\mathbb{R}^n$, they lose the strong (A3) property.

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