

Sharp upper bounds for a singular perturbation problem related to micromagnetics

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Abstract. We construct an upper bound for the following family of functionals $\{E_\varepsilon\}_{\varepsilon>0}$, which arises in the study of micromagnetics:

$$E_\varepsilon(u) = \int_{\Omega} \varepsilon |\nabla u|^2 + \frac{1}{\varepsilon} \int_{\mathbb{R}^2} |H_u|^2.$$

Here Ω is a bounded domain in \mathbb{R}^2 , $u \in H^1(\Omega, S^1)$ (corresponding to the magnetization) and H_u , the demagnetizing field created by u , is given by

$$\begin{cases} \operatorname{div}(\tilde{u} + H_u) = 0 & \text{in } \mathbb{R}^2, \\ \operatorname{curl} H_u = 0 & \text{in } \mathbb{R}^2, \end{cases}$$

where \tilde{u} is the extension of u by 0 in $\mathbb{R}^2 \setminus \Omega$. Our upper bound coincides with the lower bound obtained by Rivière and Serfaty.

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1. Introduction

In this paper we study the following energy-functional, related to micromagnetics:

$$E_\varepsilon(u) := \int_{\Omega} \varepsilon |\nabla u|^2 + \frac{1}{\varepsilon} \int_{\mathbb{R}^2} |H_u|^2. \quad (1.1)$$

Here Ω is a bounded domain in \mathbb{R}^2 with Lipschitz boundary, u is a unit-valued vector-field (corresponding to the magnetization) in $H^1(\Omega, S^1)$ and H_u , the demagnetizing field created by u , is given by

$$\begin{cases} \operatorname{div}(\tilde{u} + H_u) = 0 & \text{in } \mathbb{R}^2 \\ \operatorname{curl} H_u = 0 & \text{in } \mathbb{R}^2, \end{cases} \quad (1.2)$$

where \tilde{u} is the extension of u by 0 in $\mathbb{R}^2 \setminus \Omega$. For the physical models related to E_ε , we refer to [18] and all the references therein.

We can rewrite (1.1) in the following form. Denoting by $\Delta^{-1}\tilde{u}$ the Newtonian potential of \tilde{u} , we observe that the vector-field $\bar{H}_u := -\nabla(\operatorname{div}(\Delta^{-1}\tilde{u}))$ belongs to $L^2(\mathbb{R}^2, \mathbb{R}^2)$. Moreover,

$$\begin{cases} \operatorname{div} \bar{H}_u = -\operatorname{div} \tilde{u} & \text{in } \mathbb{R}^2 \\ \operatorname{curl} \bar{H}_u = 0 & \text{in } \mathbb{R}^2. \end{cases}$$

So $H_u = \bar{H}_u$ and we obtain

$$E_\varepsilon(u) = \int_\Omega \varepsilon |\nabla u|^2 + \frac{1}{\varepsilon} \int_{\mathbb{R}^2} |\nabla(\operatorname{div}(\Delta^{-1}\tilde{u}))|^2. \tag{1.3}$$

In [19] T. Rivière and S. Serfaty proved the following theorem, giving compactness and a lower bound for the energies E_ε .

Theorem 1.1. *Let Ω be a bounded simply connected domain in \mathbb{R}^2 . Let $\varepsilon_n \rightarrow 0$ and $u_n \in H^1(\Omega, S^1)$ with a lifting $\varphi_n \in H^1(\Omega, \mathbb{R})$ i.e., $u_n = e^{i\varphi_n}$ a.e., and such that*

$$E_{\varepsilon_n}(u_n) \leq C \tag{1.4}$$

$$\|\varphi_n\|_{L^\infty(\Omega)} \leq N. \tag{1.5}$$

Then, up to extraction of a subsequence, there exists u and φ in $\cap_{q<\infty} L^q(\Omega)$ such that

$$\begin{aligned} \varphi_n &\rightarrow \varphi \text{ in } \cap_{q<\infty} L^q(\Omega) \\ u_n &\rightarrow u \text{ in } \cap_{q<\infty} L^q(\Omega). \end{aligned}$$

Moreover, if we consider

$$\begin{cases} T^t \varphi(x) := \inf(\varphi(x), t) \\ T^t u(x) := e^{i T^t \varphi(x)}, \end{cases}$$

then $\operatorname{div}_x T^t u$ is a bounded Radon measure on $\Omega \times \mathbb{R}$, with $t \mapsto \operatorname{div}_x T^t u$ continuous from \mathbb{R} to $\mathcal{D}'(\Omega)$. In addition

$$2 \int_{\mathbb{R}} \int_\Omega |\operatorname{div}_x T^t u| dx dt \leq \varliminf_{n \rightarrow \infty} \int_\Omega 2|\nabla \varphi_n \cdot H_{u_n}| \leq \varliminf_{n \rightarrow \infty} E_{\varepsilon_n}(u_n) < \infty.$$

The main contribution of this paper is to establish the upper bound for E_ε in the case where u and its lifting φ belong to BV . First of all we observe that if $\varepsilon_n \rightarrow 0$,

$E_{\varepsilon_n}(u_n) \leq C$, and $u_n \rightarrow u$ in L^q , where $|u_n| = 1$ and $u \in BV$ then clearly $\lim_{n \rightarrow \infty} \int_{\mathbb{R}^2} |H_{u_n}|^2 = 0$, which implies

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^2} \tilde{u}_n \cdot \nabla \delta = - \lim_{n \rightarrow \infty} \int_{\mathbb{R}^2} H_{u_n} \cdot \nabla \delta = 0 \quad \forall \delta \in C_c^\infty(\mathbb{R}^2, \mathbb{R}).$$

Therefore, $\operatorname{div} \tilde{u} = 0$ as a distribution, *i.e.*,

$$\begin{cases} |u| = 1 & \text{a.e. in } \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega, \\ u \cdot \mathbf{n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

The main result of this paper is the following theorem.

Theorem 1.2. *Let Ω be a bounded domain in \mathbb{R}^2 with Lipschitz boundary. Consider $u \in BV(\Omega, S^1)$, satisfying $\operatorname{div} u = 0$ in Ω and $u \cdot \mathbf{n} = 0$ on $\partial\Omega$ and assume there exist $\varphi \in BV(\Omega, \mathbb{R})$, such that $u = e^{i\varphi}$ a.e. in Ω . Then there exists a family of functions $\{v_\varepsilon\} \subset C^2(\mathbb{R}^N, \mathbb{R})$ satisfying*

$$\lim_{\varepsilon \rightarrow 0^+} v_\varepsilon(x) = \varphi(x) \quad \text{in } L^1(\Omega, \mathbb{R})$$

and

$$\lim_{\varepsilon \rightarrow 0} E_\varepsilon(e^{iv_\varepsilon}) = 2 \int_{\mathbb{R}} \int_{\Omega} |\operatorname{div}_x T^t u| \, dx \, dt.$$

Moreover, if $\varphi \in BV(\Omega, \mathbb{R}) \cap L^\infty$, then we have

$$\lim_{\varepsilon \rightarrow 0^+} v_\varepsilon(x) = \varphi(x) \quad \text{in } L^p(\Omega, \mathbb{R}) \quad \forall p \in [1, \infty).$$

In order to construct $\{v_\varepsilon\}$ we take the convolution of φ with a varying smoothing kernel, *i.e.*, we set $v_\varepsilon(x) := \varepsilon^{-2} \int_{\mathbb{R}^2} \eta(\frac{y-x}{\varepsilon}, x) \varphi(y) dy$ where $\eta \in C_c^2(\mathbb{R}^2 \times \mathbb{R}^2)$ satisfies $\int_{\mathbb{R}^2} \eta(z, x) dx = 1$ for every $x \in \Omega$, and we optimize the choice of the kernel η . A similar approach was used in [16] and [17], but a new ingredient is required here, since the non-local term $\int_{\mathbb{R}^2} |H_u|^2$ gives certain difficulties.

1.1. The basic idea

We shall follow essentially the strategy of [16] and [17]. The main new ingredient here is the calculation of

$$\lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \int_{\mathbb{R}^2} |\nabla(\operatorname{div}(\Delta^{-1}\{\chi_\Omega e^{iv_\varepsilon}\}))|^2. \tag{1.7}$$

We first calculate

$$L(l) := \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \int_{\mathbb{R}^2} |\nabla(\operatorname{div}(\Delta^{-1}\{\varphi_\varepsilon\}))|^2,$$

where $\varphi_\varepsilon(x) := \varepsilon^{-2} \int_{\mathbb{R}^2} l\left(\frac{y-x}{\varepsilon}, x\right)\varphi(y)dy$, with $l \in C_c^2(\mathbb{R}^2 \times \Omega, \mathbb{R}^2)$ satisfying $\int_{\mathbb{R}^2} l(z, x)dx = 0$ for every $x \in \Omega$. Since $\nabla(\operatorname{div}(\Delta^{-1}\{\phi_\varepsilon\}))$ has the same asymptotic behavior as $\varepsilon^{-2} \int_{\mathbb{R}^2} s\left(\frac{y-x}{\varepsilon}, x\right)\varphi(y)dy$, where $s(z, x) := \nabla_z(\operatorname{div}_z(\Delta_z^{-1}l(z, x)))$, we can calculate the limit $L(l)$ in a similar way to what was done in [16] and [17] (see Lemmas 3.1 and 3.2 for the details). Using the results of [17] (see Proposition 2.2 below) it is easy to calculate

$$D(l) := \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{\Omega} |e^{iv_\varepsilon} - \varphi_\varepsilon - u|^2 dx.$$

Now, given a fixed η and a small $\delta > 0$, we choose $l = l_\delta$ in such a way that $D(l_\delta) < \delta$. Then, using the estimate

$$\int_{\mathbb{R}^2} |\nabla(\operatorname{div}(\Delta^{-1}\{\chi_\Omega f\}))|^2 \leq C \int_{\Omega} |f|^2,$$

we deduce that

$$\begin{aligned} & \overline{\lim}_{\varepsilon \rightarrow 0} \left| \frac{1}{\varepsilon} \int_{\mathbb{R}^2} |\nabla(\operatorname{div}(\Delta^{-1}\{\chi_\Omega e^{iv_\varepsilon}\}))|^2 - \frac{1}{\varepsilon} \int_{\mathbb{R}^2} |\nabla(\operatorname{div}(\Delta^{-1}\{\chi_\Omega \varphi_\varepsilon\}))|^2 \right| \\ & \leq \overline{\lim}_{\varepsilon \rightarrow 0} C \left\{ \frac{1}{\varepsilon} \int_{\mathbb{R}^2} |\nabla(\operatorname{div}(\Delta^{-1}\{\chi_\Omega (e^{iv_\varepsilon} - \varphi_\varepsilon - u)\}))|^2 \right\}^{1/2} \leq C\delta^{1/2}. \end{aligned}$$

Finally, letting δ tend to 0, we conclude that the limit in (1.7) should be equal to $\lim_{\delta \rightarrow 0} L(l_\delta)$. We follow basically this strategy in the proof of Proposition 4.1.

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2. Preliminaries

Throughout this section we assume that Ω is a bounded domain in \mathbb{R}^2 with Lipschitz boundary. We begin by introducing some notation. For every $\mathbf{v} \in S^1$ (the unit sphere in \mathbb{R}^2) and $R > 0$ we denote

$$B_R^+(x, \mathbf{v}) = \{y \in \mathbb{R}^2 : |y - x| < R, (y - x) \cdot \mathbf{v} > 0\}, \tag{2.1}$$

$$B_R^-(x, \mathbf{v}) = \{y \in \mathbb{R}^2 : |y - x| < R, (y - x) \cdot \mathbf{v} < 0\}, \tag{2.2}$$

$$H_+(x, \mathbf{v}) = \{y \in \mathbb{R}^2 : (y - x) \cdot \mathbf{v} > 0\}, \tag{2.3}$$

$$H_-(x, \mathbf{v}) = \{y \in \mathbb{R}^2 : (y - x) \cdot \mathbf{v} < 0\} \tag{2.4}$$

and

$$H_{\mathbf{v}}^0 = \{y \in \mathbb{R}^2 : y \cdot \mathbf{v} = 0\}. \tag{2.5}$$

Definition 2.1. Consider a function $f \in BV(\Omega, \mathbb{R}^m)$ and a point $x \in \Omega$.

i) We say that x is a point of *approximate continuity* of f if there exists $z \in \mathbb{R}^m$ such that

$$\lim_{\rho \rightarrow 0^+} \frac{\int_{B_\rho(x)} |f(y) - z| dy}{\mathcal{L}^2(B_\rho(x))} = 0.$$

In this case z is called an *approximate limit* of f at x and we denote z by $\tilde{f}(x)$. The set of points of approximate continuity of f is denoted by G_f .

ii) We say that x is an *approximate jump point* of f if there exist $a, b \in \mathbb{R}^m$ and $\mathbf{v} \in S^{N-1}$ such that $a \neq b$ and

$$\lim_{\rho \rightarrow 0^+} \frac{\int_{B_\rho^+(x, \mathbf{v})} |f(y) - a| dy}{\mathcal{L}^2(B_\rho(x))} = 0, \quad \lim_{\rho \rightarrow 0^+} \frac{\int_{B_\rho^-(x, \mathbf{v})} |f(y) - b| dy}{\mathcal{L}^2(B_\rho(x))} = 0. \tag{2.6}$$

The triple (a, b, \mathbf{v}) , uniquely determined by (2.6) up to a permutation of (a, b) and a change of sign of \mathbf{v} , is denoted by $(f^+(x), f^-(x), \mathbf{v}_f(x))$. We shall call $\mathbf{v}_f(x)$ the *approximate jump vector* and we shall sometimes write simply $\mathbf{v}(x)$ if the reference to the function f is clear. The set of approximate jump points is denoted by J_f . A choice of $\mathbf{v}(x)$ for every $x \in J_f$ (which is unique up to sign) determines an orientation of J_f . At a point of approximate continuity x , we shall use the convention $f^+(x) = f^-(x) = \tilde{f}(x)$.

We refer to [2] for the results on BV-functions that we shall use in the sequel.

Consider a function $\Phi = (\varphi_1, \varphi_2, \dots, \varphi_d) \in BV(\Omega, \mathbb{R}^d)$. By [2, Proposition 3.21] we may extend Φ to a function $\bar{\Phi} \in BV(\mathbb{R}^2, \mathbb{R}^d)$, such that $\bar{\Phi} = \Phi$ a.e. in Ω , $\text{supp } \bar{\Phi}$ is compact and $\|D\bar{\Phi}\|(\partial\Omega) = 0$. From the proof of Proposition 3.21 in [2] it follows that if $\Phi \in BV(\Omega, \mathbb{R}^d) \cap L^\infty$ then its extension $\bar{\Phi}$ is also in $BV(\mathbb{R}^2, \mathbb{R}^d) \cap L^\infty$. Consider also a matrix valued function $\Xi \in C_c^2(\mathbb{R}^2 \times \mathbb{R}^2, \mathbb{R}^{l \times d})$. For every $\varepsilon > 0$ define a function $\Psi_\varepsilon(x) : \mathbb{R}^2 \rightarrow \mathbb{R}^l$ by

$$\begin{aligned} \Psi_\varepsilon(x) &:= \frac{1}{\varepsilon^2} \int_{\mathbb{R}^2} \Xi\left(\frac{y-x}{\varepsilon}, x\right) \cdot \bar{\Phi}(y) dy \\ &= \int_{\mathbb{R}^2} \Xi(z, x) \cdot \bar{\Phi}(x + \varepsilon z) dz, \quad \forall x \in \mathbb{R}^2. \end{aligned} \tag{2.7}$$

Thanks to [17, Proposition 3.2], we have the following result. It generalizes Proposition 3.2 from [16] and provides the key tool for the calculation of the upper bound, both in [17] and in the current paper. In the proof of Lemma 3.2 we shall also follow the general strategy of its proof in [17].

Proposition 2.2. *Let $W \in C^1(\mathbb{R}^l \times \mathbb{R}^q, \mathbb{R})$ satisfying*

$$\nabla_a W(a, b) = 0 \text{ whenever } W(a, b) = 0. \tag{2.8}$$

Consider $\Phi \in BV(\Omega, \mathbb{R}^d) \cap L^\infty$ and $u \in BV(\Omega, \mathbb{R}^q) \cap L^\infty$ satisfying

$$W\left(\left\{\int_{\mathbb{R}^2} \Xi(z, x) dz\right\} \cdot \Phi(x), u(x)\right) = 0 \text{ for a.e. } x \in \Omega,$$

where $\Xi \in C_c^2(\mathbb{R}^2 \times \mathbb{R}^2, \mathbb{R}^{l \times d})$, as above. Let Ψ_ε be as in (2.7). Then,

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \frac{1}{\varepsilon} W(\Psi_\varepsilon(x), u(x)) dx \\ &= \int_{J_\Phi} \left\{ \int_{-\infty}^0 W(\Gamma(t, x), u^+(x)) dt + \int_0^{+\infty} W(\Gamma(t, x), u^-(x)) dt \right\} d\mathcal{H}^1(x), \end{aligned} \tag{2.9}$$

where

$$\Gamma(t, x) = \left(\int_{-\infty}^t P(s, x) ds \right) \cdot \Phi^-(x) + \left(\int_t^{+\infty} P(s, x) ds \right) \cdot \Phi^+(x), \tag{2.10}$$

with

$$P(t, x) = \int_{H_{v(x)}^0} \Xi(t v(x) + y, x) d\mathcal{H}^1(y), \tag{2.11}$$

$v(x)$ is the jump vector of Φ and it is assumed that the orientation of J_u coincides with the orientation of J_Φ \mathcal{H}^1 a.e. on $J_u \cap J_\Phi$.

Definition 2.3. Given $f \in L^\infty(\mathbb{R}^2, \mathbb{R}^k)$ with compact support, we define its Newtonian potential

$$(\Delta^{-1} f)(x) := \int_{\mathbb{R}^2} \frac{1}{2\pi} \ln|x - y| f(y) dy.$$

Then it is well known that

$$\int_{\mathbb{R}^2} |\nabla^2(\Delta^{-1} f)(x)|^2 dx = \int_{\mathbb{R}^2} |f(x)|^2 dx, \tag{2.12}$$

where given $v = (v_1, \dots, v_k) : \mathbb{R}^2 \rightarrow \mathbb{R}^k$ we denote by $\nabla^2 v \in \mathbb{R}^{k \times 2 \times 2}$ the tensor with lij -th component $\partial_i^2 v_l$.

Definition 2.4. Let \mathcal{V} be the class of all functions $\eta \in C_c^2(\mathbb{R}^2 \times \mathbb{R}^2, \mathbb{R})$ such that

$$\int_{\mathbb{R}^2} \eta(z, x) dz = 1 \quad \forall x \in \Omega. \tag{2.13}$$

Let \mathcal{U} be the class of all functions $l(z, x) \in C_c^2(\mathbb{R}^2 \times \Omega, \mathbb{R}^2)$ such that

$$\int_{\mathbb{R}^2} l(z, x) dz = 0 \quad \forall x \in \mathbb{R}^2. \tag{2.14}$$

In [17, Lemma 5.1], we proved the following statement. This statement generalize Claim 3 of Lemma 3.4 from [16] and was an essential tool in the optimizing the upper bound in [17].

Lemma 2.5. *Let μ be positive finite Borel measure on Ω and $\mathbf{v}_0(x) : \Omega \rightarrow \mathbb{R}^2$ a Borel measurable function with $|\mathbf{v}_0| = 1$. Let \mathcal{W}_1 denote the set of functions $p(t, x) : \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ satisfying the following conditions:*

- i) p is Borel measurable and bounded,
- ii) there exists $M > 0$ such that $p(t, x) = 0$ for $|t| > M$ and any $x \in \Omega$,
- iii) $\int_{\mathbb{R}} p(t, x) dt = 1, \forall x \in \Omega$.

Then for every $p(t, x) \in \mathcal{W}_1$, there exists a sequence of functions $\{\eta_n\} \subset \mathcal{V}$ (see Definition 2.4), such that the sequence of functions $\{p_n(t, x)\}$ defined on $\mathbb{R} \times \Omega$ by

$$p_n(t, x) = \int_{H_{\mathbf{v}_0(x)}^0} \eta_n(t\mathbf{v}_0(x) + y, x) d\mathcal{H}^1(y),$$

has the following properties:

- i) there exists C_0 such that $\|p_n\|_{L^\infty} \leq C_0$ for every n ,
- ii) there exist $M > 0$ such that $p_n(t, x) = 0$ for $|t| > M$ and every $x \in \Omega$, for all n ,
- iii) $\lim_{n \rightarrow \infty} \int_{\Omega} \int_{\mathbb{R}} |p_n(t, x) - p(t, x)| dt d\mu(x) = 0$.

With the same method it is not difficult to prove

Lemma 2.6. *Let μ be positive finite Borel measure on Ω and $\mathbf{v}_0(x) : \Omega \rightarrow \mathbb{R}^2$ a Borel measurable function with $|\mathbf{v}_0| = 1$. Let \mathcal{W}_0 denote the set of functions $q(t, x) : \mathbb{R} \times \Omega \rightarrow \mathbb{R}^2$ satisfying the following conditions:*

- i) q is Borel measurable and bounded,
- ii) there exists $M > 0$ such that $q(t, x) = 0$ for $|t| > M$ and every $x \in \Omega$.
- iii) $\int_{\mathbb{R}} q(t, x) dt = 0, \forall x \in \Omega$.

Then for every $q(t, x) \in \mathcal{W}_0$, there exists a sequence of functions $\{l_n\} \subset \mathcal{U}$ (see Definition 2.4), such that the sequence of functions $\{q_n(t, x)\}$ defined on $\mathbb{R} \times \Omega$ by

$$q_n(t, x) = \int_{H_{\mathbf{v}_0(x)}^0} l_n(t\mathbf{v}_0(x) + y, x) d\mathcal{H}^1(y),$$

has the following properties:

- i) there exists C_0 such that $\|q_n\|_{L^\infty} \leq C_0$ for every n ,
- ii) there exist $M > 0$ such that $q_n(t, x) = 0$ for $|t| > M$ and every $x \in \Omega$, for all n ,
- iii) $\lim_{n \rightarrow \infty} \int_{\Omega} \int_{\mathbb{R}} |q_n(t, x) - q(t, x)| dt d\mu(x) = 0$.

3. First estimates

Throughout this section we assume that Ω is a bounded domain in \mathbb{R}^2 with Lipschitz boundary.

Let $l \in \mathcal{U}$ (see Definition 2.4). Consider $r(z, x) := \Delta_z^{-1}l(z, x)$. Then $r \in C^2(\mathbb{R}^2 \times \mathbb{R}^2, \mathbb{R}^2)$ with $\text{supp } r \subset \mathbb{R}^2 \times K$, where $K \Subset \Omega$. Moreover, since $\int_{\mathbb{R}^2} l(z, x) dz = 0$, for every $k = 0, 1, 2, \dots$ we have the estimates

$$\begin{aligned} |\nabla_x^k r(z, x)| &\leq \frac{C_k}{|z| + 1}, \\ |\nabla_x^k (\nabla_z r(z, x))| &\leq \frac{C_k}{|z|^2 + 1}, \\ |\nabla_x^k (\nabla_z^2 r(z, x))| &\leq \frac{C_k}{|z|^3 + 1}, \end{aligned} \tag{3.1}$$

where $C_k > 0$ does not depend on z and x .

Lemma 3.1. *Let $\varphi \in BV(\Omega, \mathbb{R}) \cap L^\infty$ and $l \in \mathcal{U}$ (see Definition 2.4). For every $\varepsilon > 0$ consider the function $\varphi_\varepsilon \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ by*

$$\varphi_\varepsilon(x) := \frac{1}{\varepsilon^2} \int_{\mathbb{R}^2} l\left(\frac{y-x}{\varepsilon}, x\right) \bar{\varphi}(y) dy = \int_{\mathbb{R}^2} l(z, x) \bar{\varphi}(x + \varepsilon z) dz, \tag{3.2}$$

where $\bar{\varphi}$ is some bounded BV extension of φ to \mathbb{R}^2 with compact support. Next consider $r(z, x) := \Delta_z^{-1}l(z, x)$ and set

$$\begin{aligned} \xi_\varepsilon(x) &:= \frac{1}{\varepsilon^2} \int_{\mathbb{R}^2} \nabla_1(\text{div}_1 r)\left(\frac{y-x}{\varepsilon}, x\right) \bar{\varphi}(y) dy \\ &= \int_{\mathbb{R}^2} \nabla_z(\text{div}_z r(z, x)) \bar{\varphi}(x + \varepsilon z) dz, \end{aligned} \tag{3.3}$$

where $\nabla_1(\text{div}_1 r)$ is the gradient of divergence of $r(z, x)$ in its first variable, namely z . Then,

$$\int_{\mathbb{R}^2} \frac{1}{\varepsilon} \left| \nabla(\text{div}(\Delta^{-1}\varphi_\varepsilon))(x) \right|^2 dx = o_\varepsilon(1) + \int_{\Omega} \frac{1}{\varepsilon} \varphi_\varepsilon(x) \cdot \xi_\varepsilon(x) dx. \tag{3.4}$$

Proof. Since $l(z, x) = 0$ if $x \notin K$, where K is some compact subset of Ω , we have, in particular, $\varphi_\varepsilon(x) = 0$ for every $x \in \mathbb{R}^2 \setminus \Omega$. Then, integrating by part two times, we conclude

$$\begin{aligned} &\int_{\mathbb{R}^2} \frac{1}{\varepsilon} \left| \nabla(\text{div}(\Delta^{-1}\varphi_\varepsilon))(x) \right|^2 dx \\ &= - \int_{\mathbb{R}^2} \frac{1}{\varepsilon} \Delta(\text{div}(\Delta^{-1}\varphi_\varepsilon))(x) \cdot (\text{div}(\Delta^{-1}\varphi_\varepsilon))(x) dx \\ &= - \int_{\mathbb{R}^2} \frac{1}{\varepsilon} \text{div } \varphi_\varepsilon(x) \cdot (\text{div}(\Delta^{-1}\varphi_\varepsilon))(x) dx \\ &= \int_{\Omega} \frac{1}{\varepsilon} \varphi_\varepsilon(x) \cdot \nabla(\text{div}(\Delta^{-1}\varphi_\varepsilon))(x) dx. \end{aligned} \tag{3.5}$$

Next consider the function $\zeta_\varepsilon \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ given by

$$\zeta_\varepsilon(x) := \frac{1}{\varepsilon^2} \int_{\mathbb{R}^2} r\left(\frac{y-x}{\varepsilon}, x\right) \bar{\varphi}(y) dy = \int_{\mathbb{R}^2} r(z, x) \bar{\varphi}(x + \varepsilon z) dz. \tag{3.6}$$

We will prove now that

$$\left| \varepsilon^2 \nabla^2 \zeta_\varepsilon(x) - \int_{\mathbb{R}^2} \nabla_z^2 r(z, x) \bar{\varphi}(x + \varepsilon z) dz \right| \leq C \varepsilon^{2/3} \quad \forall x \in \Omega. \tag{3.7}$$

We shall denote by $\nabla_1 l$ and $\nabla_2 l$ the gradient of $l(z, x)$ with respect to the variables z and x respectively. We have,

$$\begin{aligned} & \varepsilon^2 \nabla^2 \zeta_\varepsilon(x) - \int_{\mathbb{R}^2} \nabla_z^2 r(z, x) \bar{\varphi}(x + \varepsilon z) dz \\ &= \int_{\mathbb{R}^2} \nabla_x^2 r\left(\frac{y-x}{\varepsilon}, x\right) \bar{\varphi}(y) dy - \frac{1}{\varepsilon^2} \int_{\mathbb{R}^2} \nabla_1^2 r\left(\frac{y-x}{\varepsilon}, x\right) \bar{\varphi}(y) dy \\ &= -\frac{1}{\varepsilon} \int_{\mathbb{R}^2} \left\{ \nabla_1 \nabla_2 r\left(\frac{y-x}{\varepsilon}, x\right) + \nabla_2 \nabla_1 r\left(\frac{y-x}{\varepsilon}, x\right) \right\} \bar{\varphi}(y) dy \\ & \quad + \int_{\mathbb{R}^2} \nabla_2^2 r\left(\frac{y-x}{\varepsilon}, x\right) \bar{\varphi}(y) dy. \end{aligned} \tag{3.8}$$

Therefore, by the Hölder inequality and the estimates in (3.1), we obtain

$$\begin{aligned} & \left| \varepsilon^2 \nabla^2 \zeta_\varepsilon(x) - \int_{\mathbb{R}^2} \nabla_z^2 r(z, x) \bar{\varphi}(x + \varepsilon z) dz \right| \\ & \leq \varepsilon^{2/3} \left(\frac{1}{\varepsilon^2} \int_{\mathbb{R}^2} \left| \nabla_1 \nabla_2 r\left(\frac{y-x}{\varepsilon}, x\right) + \nabla_2 \nabla_1 r\left(\frac{y-x}{\varepsilon}, x\right) \right|^{6/5} dy \right)^{5/6} \left(\int_{\mathbb{R}^2} |\bar{\varphi}(y)|^6 dy \right)^{1/6} \\ & \quad + \varepsilon^{2/3} \left(\frac{1}{\varepsilon^2} \int_{\mathbb{R}^2} \left| \nabla_2^2 r\left(\frac{y-x}{\varepsilon}, x\right) \right|^3 dy \right)^{1/3} \left(\int_{\mathbb{R}^2} |\bar{\varphi}(y)|^{3/2} dy \right)^{2/3} \\ & = \varepsilon^{2/3} \left(\int_{\mathbb{R}^2} \left| \nabla_1 \nabla_2 r(z, x) + \nabla_2 \nabla_1 r(z, x) \right|^{6/5} dz \right)^{5/6} \left(\int_{\mathbb{R}^2} |\bar{\varphi}(y)|^6 dy \right)^{1/6} \\ & \quad + \varepsilon^{2/3} \left(\int_{\mathbb{R}^2} \left| \nabla_2^2 r(z, x) \right|^3 dz \right)^{1/3} \left(\int_{\mathbb{R}^2} |\bar{\varphi}(y)|^{3/2} dy \right)^{2/3} \leq C \varepsilon^{2/3} \end{aligned}$$

which gives (3.7). In particular,

$$\begin{aligned} \left| \varepsilon^2 \Delta \zeta_\varepsilon(x) - \varphi_\varepsilon(x) \right| &= \left| \varepsilon^2 \Delta \zeta_\varepsilon(x) - \int_{\mathbb{R}^2} \Delta_z r(z, x) \bar{\varphi}(x + \varepsilon z) dz \right| \\ & \leq C_0 \varepsilon^{2/3}. \end{aligned} \tag{3.9}$$

Next by (3.5),

$$\begin{aligned} \int_{\mathbb{R}^2} \frac{1}{\varepsilon} \left| \nabla(\operatorname{div}(\Delta^{-1}\varphi_\varepsilon))(x) \right|^2 dx &= \int_{\Omega} \frac{1}{\varepsilon} \varphi_\varepsilon(x) \cdot \nabla(\operatorname{div}(\Delta^{-1}\varphi_\varepsilon))(x) dx \\ &= \int_{\Omega} \frac{1}{\varepsilon} \varphi_\varepsilon(x) \cdot \nabla(\operatorname{div}(\varepsilon^2\zeta_\varepsilon(x))) dx \\ &\quad - \int_{\Omega} \frac{1}{\varepsilon} \varphi_\varepsilon(x) \cdot \nabla(\operatorname{div}(\Delta^{-1}(\varepsilon^2\Delta\zeta_\varepsilon - \varphi_\varepsilon)))(x) dx. \end{aligned} \tag{3.10}$$

The last integral can be estimated by

$$\begin{aligned} &\left| \int_{\Omega} \frac{1}{\varepsilon} \varphi_\varepsilon(x) \cdot \nabla(\operatorname{div}(\Delta^{-1}(\varepsilon^2\Delta\zeta_\varepsilon - \varphi_\varepsilon)))(x) dx \right| \\ &\leq \left(\int_{\Omega} \frac{1}{\varepsilon} |\varphi_\varepsilon(x)|^2 \right)^{1/2} \left(\frac{1}{\varepsilon} \int_{\mathbb{R}^2} \left| \nabla(\operatorname{div}(\Delta^{-1}(\varepsilon^2\Delta\zeta_\varepsilon - \varphi_\varepsilon)))(x) \right|^2 dx \right)^{1/2} \\ &\leq \left(\int_{\Omega} \frac{1}{\varepsilon} |\varphi_\varepsilon(x)|^2 \right)^{1/2} \left(\frac{2}{\varepsilon} \int_{\mathbb{R}^2} \left| \nabla^2(\Delta^{-1}(\varepsilon^2\Delta\zeta_\varepsilon - \varphi_\varepsilon))(x) \right|^2 dx \right)^{1/2} \\ &= \left(\int_{\Omega} \frac{1}{\varepsilon} |\varphi_\varepsilon(x)|^2 \right)^{1/2} \left(\int_{\Omega} \frac{2}{\varepsilon} |\varepsilon^2\Delta\zeta_\varepsilon(x) - \varphi_\varepsilon(x)|^2 dx \right)^{1/2}. \end{aligned}$$

Then, since

$$\begin{aligned} \int_{\Omega} \frac{1}{\varepsilon} |\varphi_\varepsilon(x)|^2 &\leq C \int_{\Omega} \frac{1}{\varepsilon} |\varphi_\varepsilon(x)| \\ &\leq \bar{C} \int_{\Omega} \frac{1}{\varepsilon} \left| \int_{B_R(0)} l(z, x) (\bar{\varphi}(x + \varepsilon z) - \varphi(x)) dz \right| dx \\ &\leq \bar{C} \int_{B_R(0)} \frac{1}{\varepsilon} |l(z, x)| \left(\int_{\Omega} |\bar{\varphi}(x + \varepsilon z) - \varphi(x)| dx \right) dz \\ &\leq \bar{C} \|D\bar{\varphi}\|(\mathbb{R}^2) \int_{B_R(0)} |l(z, x)| \cdot |z| dz = O(1), \end{aligned} \tag{3.11}$$

using (3.9), from (3.10) we infer

$$\int_{\mathbb{R}^2} \frac{1}{\varepsilon} \left| \nabla(\operatorname{div}(\Delta^{-1}\varphi_\varepsilon))(x) \right|^2 dx = o_\varepsilon(1) + \int_{\Omega} \frac{1}{\varepsilon} \varphi_\varepsilon(x) \cdot \nabla(\operatorname{div}(\varepsilon^2\zeta_\varepsilon(x))) dx. \tag{3.12}$$

Next we remind that ξ_ε is defined by (3.3). By (3.7), we have,

$$\left| \nabla(\operatorname{div}(\varepsilon^2\zeta_\varepsilon(x))) - \xi_\varepsilon(x) \right| \leq \bar{C}\varepsilon^{2/3} \quad \forall x \in \Omega. \tag{3.13}$$

Then as before,

$$\begin{aligned} & \left| \int_{\Omega} \frac{1}{\varepsilon} \varphi_{\varepsilon}(x) \cdot \left(\nabla \left(\operatorname{div}(\varepsilon^2 \zeta_{\varepsilon}(x)) \right) - \xi_{\varepsilon}(x) \right) dx \right| \\ & \leq \left(\int_{\Omega} \frac{1}{\varepsilon} |\varphi_{\varepsilon}(x)|^2 \right)^{1/2} \left(\int_{\Omega} \frac{1}{\varepsilon} \left| \nabla \left(\operatorname{div}(\varepsilon^2 \zeta_{\varepsilon}(x)) \right) - \xi_{\varepsilon}(x) \right|^2 dx \right)^{1/2} \\ & \leq C\varepsilon^{1/6}. \end{aligned}$$

Therefore, from (3.12) we infer (3.4). □

Lemma 3.2. *Let $\varphi \in BV(\Omega, \mathbb{R}) \cap L^{\infty}$ and $l \in \mathcal{U}$ (see Definition 2.4). For every $\varepsilon > 0$ and every $x \in \mathbb{R}^2$ consider the function $\varphi_{\varepsilon} \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ as in (3.2). Then,*

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^2} \frac{1}{\varepsilon} \left| \nabla \left(\operatorname{div}(\Delta^{-1} \varphi_{\varepsilon})(x) \right) \right|^2 dx \\ & = \int_{J_{\varphi}} \left\{ \int_{-\infty}^{+\infty} |\varphi^+(x) - \varphi^-(x)|^2 \cdot \left| \mathbf{v}(x) \cdot \int_t^{+\infty} q(s, x) ds \right|^2 dt \right\} d\mathcal{H}^1(x), \end{aligned} \tag{3.14}$$

where

$$q(t, x) = \int_{H_{\mathbf{v}(x)}^0} l(t\mathbf{v}(x) + y, x) d\mathcal{H}^1(y), \tag{3.15}$$

and $\mathbf{v}(x)$ is the jump vector of φ .

Proof. By Lemma 3.1 we have

$$\int_{\mathbb{R}^2} \frac{1}{\varepsilon} \left| \nabla \left(\operatorname{div}(\Delta^{-1} \varphi_{\varepsilon})(x) \right) \right|^2 dx = o_{\varepsilon}(1) + \int_{\Omega} \frac{1}{\varepsilon} \varphi_{\varepsilon}(x) \cdot \xi_{\varepsilon}(x) dx. \tag{3.16}$$

From this point the strategy of the proof is similar to that in [16] and [17] (see also Proposition 2.2). The only difference is that here ξ_{ε} is defined by a convolution of $\bar{\varphi}$ with a kernel whose support is not compact. However, it turns out that this difference is not crucial and we can use almost the same approach.

Step 1. We prove a useful expression,

$$\begin{aligned} & \int_{\Omega} \frac{1}{\varepsilon} \varphi_{\varepsilon}(x) \cdot \xi_{\varepsilon}(x) dx \\ & = \int_0^1 \left\{ \iint_{\mathbb{R}^2} \left(\frac{1}{t^2 \varepsilon^2} \int_{\Omega \cap B_{Rt\varepsilon}(y)} \left\{ \xi_{\varepsilon}(x) \cdot l\left(\frac{y-x}{t\varepsilon}, x \right) \right\} \frac{y-x}{t\varepsilon} dx \right) \cdot d[D\bar{\varphi}](y) \right\} dt, \end{aligned} \tag{3.17}$$

where ξ_ε is as in (3.3) and $R > 0$ is such that $l(z, x) = 0$ whenever $|z| \geq R$. As before, we shall denote by $\nabla_1 l$ and $\nabla_2 l$ the gradient of l with respect to the first and second variables respectively. Denote $(\varphi_{t\varepsilon,1}(x), \varphi_{t\varepsilon,2}(x)) := \varphi_{t\varepsilon}(x)$ and $(l_1(z, x), l_2(z, x)) := l(z, x)$. Then for every $t \in (0, 1]$, every $j \in \{1, 2\}$ and every $x \in \mathbb{R}^2$ we have

$$\begin{aligned} \frac{d(\varphi_{t\varepsilon,j}(x))}{dt} &= \frac{d}{dt} \left(\frac{1}{t^2\varepsilon^2} \int_{\mathbb{R}^2} l_j\left(\frac{y-x}{t\varepsilon}, x\right) \bar{\varphi}(y) dy \right) \\ &= -\frac{1}{t^3\varepsilon^2} \int_{\mathbb{R}^2} \left\{ \nabla_1 l_j\left(\frac{y-x}{t\varepsilon}, x\right) \cdot \frac{y-x}{t\varepsilon} + 2l_j\left(\frac{y-x}{t\varepsilon}, x\right) \right\} \bar{\varphi}(y) dy \\ &= -\frac{1}{t^2\varepsilon} \int_{\mathbb{R}^2} \operatorname{div}_y \left\{ l_j\left(\frac{y-x}{t\varepsilon}, x\right) \frac{y-x}{t\varepsilon} \right\} \bar{\varphi}(y) dy \\ &= \frac{1}{t^2\varepsilon} \int_{\mathbb{R}^2} l_j\left(\frac{y-x}{t\varepsilon}, x\right) \frac{y-x}{t\varepsilon} \cdot d[D\bar{\varphi}](y). \end{aligned} \tag{3.18}$$

Therefore, for any $\rho \in (0, 1)$ we have,

$$\begin{aligned} &\int_{\Omega} \frac{1}{\varepsilon} (\varphi_\varepsilon(x) - \varphi_{\rho\varepsilon}(x)) \cdot \xi_\varepsilon(x) dx \\ &= \int_{\Omega} \frac{1}{\varepsilon} \xi_\varepsilon(x) \cdot \left(\int_{\rho}^1 \frac{d(\varphi_{t\varepsilon}(x))}{dt} dt \right) dx \\ &= \int_{\Omega} \left\{ \int_{\rho}^1 \xi_\varepsilon(x) \cdot \left(\frac{1}{t^2\varepsilon^2} \int_{\mathbb{R}^2} l\left(\frac{y-x}{t\varepsilon}, x\right) \left\{ \frac{y-x}{t\varepsilon} \cdot d[D\bar{\varphi}](y) \right\} dt \right) dx \right\} dx \\ &= \int_{\rho}^1 \left\{ \int_{\Omega} \xi_\varepsilon(x) \cdot \left(\frac{1}{t^2\varepsilon^2} \int_{\mathbb{R}^2} l\left(\frac{y-x}{t\varepsilon}, x\right) \left\{ \frac{y-x}{t\varepsilon} \cdot d[D\bar{\varphi}](y) \right\} dx \right) dx \right\} dt \\ &= \int_{\rho}^1 \left\{ \int_{\mathbb{R}^2} \left(\frac{1}{t^2\varepsilon^2} \int_{\Omega \cap B_{Rt\varepsilon}(y)} \left\{ \xi_\varepsilon(x) \cdot l\left(\frac{y-x}{t\varepsilon}, x\right) \right\} \frac{y-x}{t\varepsilon} dx \right) \cdot d[D\bar{\varphi}](y) \right\} dt. \end{aligned} \tag{3.19}$$

From our assumptions on φ , by (3.1), it follows that there exists a constant $C > 0$, independent of ρ , such that $|\xi_\rho(x)| \leq C$ for every $\rho > 0$ and every $x \in \Omega$. Therefore, letting ρ tend to zero in (3.19), using the fact that $\lim_{\rho \rightarrow 0} \|\varphi_\rho(x)\|_{L^1(\Omega)} = 0$ (see (3.11)), we get (3.17).

Step 2. We prove the identity

$$\begin{aligned}
 & \int_{\mathbb{R}^2} \frac{1}{\varepsilon} \left| \nabla (\operatorname{div}(\Delta^{-1} \varphi_\varepsilon))(x) \right|^2 dx \\
 &= o_\varepsilon(1) + \int_0^1 \left(\int_{J_\varphi} \int_{B_R(0)} \{ l(z, x) \cdot \xi_\varepsilon(x - \varepsilon tz) \} z dz \right) \cdot d[D\varphi](x) dt \\
 & \quad + \int_0^1 \left(\int_{G_\varphi} \int_{B_R(0)} \{ l(z, x) \cdot \xi_\varepsilon(x - \varepsilon tz) \} z dz \right) \cdot d[D\varphi](x) dt,
 \end{aligned} \tag{3.20}$$

where G_φ is the set of approximate continuity of φ . By (3.16) and (3.17) we deduce

$$\begin{aligned}
 & \int_{\mathbb{R}^2} \frac{1}{\varepsilon} \left| \nabla (\operatorname{div}(\Delta^{-1} \varphi_\varepsilon))(x) \right|^2 dx \\
 &= o_\varepsilon(1) + \int_0^1 \left\{ \int_{\mathbb{R}^2} \left(\frac{1}{t^2 \varepsilon^2} \int_{\Omega \cap B_{Rt\varepsilon}(y)} \left\{ \xi_\varepsilon(x) \cdot l\left(\frac{y-x}{t\varepsilon}, x\right) \right\} \frac{y-x}{t\varepsilon} dx \right) \cdot d[D\bar{\varphi}](y) \right\} dt \\
 &= o_\varepsilon(1) + \int_0^1 \left\{ \int_{\mathbb{R}^2} \left(\frac{1}{t^2 \varepsilon^2} \int_{K \cap B_{Rt\varepsilon}(y)} \left\{ \xi_\varepsilon(x) \cdot l\left(\frac{y-x}{t\varepsilon}, x\right) \right\} \frac{y-x}{t\varepsilon} dx \right) \cdot d[D\bar{\varphi}](y) \right\} dt,
 \end{aligned} \tag{3.21}$$

where $K \Subset \Omega$ is a compact set (see Definition 2.4). But, for every $\varepsilon < \frac{1}{R} \operatorname{dist}(K, \partial\Omega)$ we have

$$\begin{aligned}
 & \int_{\mathbb{R}^2} \left(\frac{1}{t^2 \varepsilon^2} \int_{K \cap B_{Rt\varepsilon}(y)} \left\{ \xi_\varepsilon(x) \cdot l\left(\frac{y-x}{t\varepsilon}, x\right) \right\} \frac{y-x}{t\varepsilon} dx \right) \cdot d[D\bar{\varphi}](y) \\
 &= \int_{\Omega} \left(\frac{1}{t^2 \varepsilon^2} \int_{K \cap B_{Rt\varepsilon}(y)} \left\{ \xi_\varepsilon(x) \cdot l\left(\frac{y-x}{t\varepsilon}, x\right) \right\} \frac{y-x}{t\varepsilon} dx \right) \cdot d[D\bar{\varphi}](y) \\
 &= \int_{\Omega} \left(\frac{1}{t^2 \varepsilon^2} \int_{\mathbb{R}^2} \left\{ \xi_\varepsilon(x) \cdot l\left(\frac{y-x}{t\varepsilon}, x\right) \right\} \frac{y-x}{t\varepsilon} dx \right) \cdot d[D\bar{\varphi}](y).
 \end{aligned}$$

Therefore, by (3.21), we obtain

$$\begin{aligned}
 & \int_{\mathbb{R}^2} \frac{1}{\varepsilon} \left| \nabla (\operatorname{div}(\Delta^{-1} \varphi_\varepsilon))(x) \right|^2 dx \\
 &= o_\varepsilon(1) + \int_0^1 \int_{\Omega} \left(\frac{1}{t^2 \varepsilon^2} \int_{\mathbb{R}^2} \left\{ \xi_\varepsilon(x) \cdot l\left(\frac{y-x}{t\varepsilon}, x\right) \right\} \frac{y-x}{t\varepsilon} dx \right) \cdot d[D\bar{\varphi}](y) dt \\
 &= o_\varepsilon(1) + \int_0^1 \left(\int_{\Omega} \left\{ \int_{B_R(0)} \{l(z, y - \varepsilon tz) \cdot \xi_\varepsilon(y - \varepsilon tz)\} z dz \right\} \cdot d[D\varphi](y) \right) dt \tag{3.22} \\
 &= o_\varepsilon(1) + \int_0^1 \left(\int_{\Omega} \left\{ \int_{B_R(0)} \{l(z, x) \cdot \xi_\varepsilon(x - \varepsilon tz)\} z dz \right\} \cdot d[D\varphi](x) \right) dt,
 \end{aligned}$$

where in the last equality we used the estimate $|l(z, x - \varepsilon tz) - l(z, x)| \leq C\varepsilon t|z|$. Therefore we obtain (3.20).

Step 3. We will prove that the second integral in the r.h.s of (3.20) vanishes as $\varepsilon \rightarrow 0$. For every x in G_φ we have,

$$\lim_{\rho \rightarrow 0^+} \frac{1}{\rho^2} \int_{B_\rho(x)} |\bar{\varphi}(y) - \tilde{\varphi}(x)| dy = 0.$$

Taking $\rho = L\varepsilon$, for every $L > 0$, gives

$$\lim_{\varepsilon \rightarrow 0^+} \int_{B_L(0)} |\bar{\varphi}(x + \varepsilon z) - \tilde{\varphi}(x)| dz = 0, \text{ for } x \text{ in } G_\varphi. \tag{3.23}$$

Using (3.1), since $\int_{\mathbb{R}^2} \nabla_z (\operatorname{div}_z r(z, x - \varepsilon ty)) dz = 0$, for every x in G_φ , $y \in B_R(0)$, $t \in [0, 1]$ and $L > 0$ we have,

$$\begin{aligned}
 |\xi_\varepsilon(x - \varepsilon ty)| &= \left| \int_{\mathbb{R}^2} \nabla_z (\operatorname{div}_z r(z, x - \varepsilon ty)) \bar{\varphi}(x + \varepsilon z - \varepsilon ty) dz \right| \\
 &= \left| \int_{\mathbb{R}^2} \nabla_z (\operatorname{div}_z r(z, x - \varepsilon ty)) (\bar{\varphi}(x + \varepsilon z - \varepsilon ty) - \tilde{\varphi}(x)) dz \right| \\
 &\leq \int_{B_L(0)} \left| \nabla_z (\operatorname{div}_z r(z, x - \varepsilon ty)) \right| \cdot |\bar{\varphi}(x + \varepsilon z - \varepsilon ty) - \tilde{\varphi}(x)| dz \\
 &\quad + \int_{\mathbb{R}^2 \setminus B_L(0)} \left| \nabla_z (\operatorname{div}_z r(z, x - \varepsilon ty)) \right| \cdot |\bar{\varphi}(x + \varepsilon z - \varepsilon ty) - \tilde{\varphi}(x)| dz \tag{3.24} \\
 &\leq A_L \int_{B_L(0)} |\bar{\varphi}(x + \varepsilon z - \varepsilon ty) - \tilde{\varphi}(x)| dz + B \int_{\mathbb{R}^2 \setminus B_L(0)} \frac{1}{|z|^3 + 1} dz \\
 &\leq A_L \int_{B_{(L+R)}(0)} |\bar{\varphi}(x + \varepsilon z) - \tilde{\varphi}(x)| dz + B \int_{\mathbb{R}^2 \setminus B_L(0)} \frac{1}{|z|^3 + 1} dz,
 \end{aligned}$$

where $B > 0$ is a constant and $A_L > 0$ depends only on L . Given $\delta > 0$ we can take $L > 0$ such that

$$B \int_{\mathbb{R}^2 \setminus B_L(0)} \frac{1}{|z|^3 + 1} dz < \delta,$$

Then, using (3.24) and (3.23), we infer $\overline{\lim}_{\varepsilon \rightarrow 0^+} |\xi_\varepsilon(x - \varepsilon ty)| < \delta$ and since δ was arbitrary,

$$\lim_{\varepsilon \rightarrow 0^+} \xi_\varepsilon(x - \varepsilon ty) = 0 \quad \forall x \in G_\varphi, y \in B_R(0), t \in [0, 1]. \tag{3.25}$$

Using (3.1), we also have $|\xi_\varepsilon(x - \varepsilon ty)| \leq C$, and therefore, plugging (3.25) into (3.20), we obtain

$$\begin{aligned} & \int_{\mathbb{R}^2} \frac{1}{\varepsilon} \left| \nabla(\operatorname{div}(\Delta^{-1} \varphi_\varepsilon))(x) \right|^2 dx \\ &= o_\varepsilon(1) + \int_0^1 \left(\int_{J_\varphi} \left\{ \int_{B_R(0)} \{l(z, x) \cdot \xi_\varepsilon(x - \varepsilon tz)\} z dz \right\} \cdot d[D\varphi](x) \right) dt. \end{aligned} \tag{3.26}$$

Step 4. Consider $\bar{l}(z, x) := \nabla_z(\operatorname{div}_z r(z, x))$. For every $\varepsilon, t \in (0, 1)$, $x \in J_\varphi$ and $z \in B_R(0)$, we have

$$\begin{aligned} \xi_\varepsilon(x - \varepsilon tz) &= \int_{\mathbb{R}^2} \bar{l}(y, x - \varepsilon tz) \bar{\varphi}(x + \varepsilon(y - tz)) dy \\ &= \int_{\mathbb{R}^2} \bar{l}(y + tz, x - \varepsilon tz) \bar{\varphi}(x + \varepsilon y) dy \\ &= \int_{H_+(0, \mathbf{v}(x))} \bar{l}(y + tz, x - \varepsilon tz) \bar{\varphi}(x + \varepsilon y) dy \\ &\quad + \int_{H_-(0, \mathbf{v}(x))} \bar{l}(y + tz, x - \varepsilon tz) \bar{\varphi}(x + \varepsilon y) dy \\ &= \int_{H_+(0, \mathbf{v}(x))} \bar{l}(y + tz, x - \varepsilon tz) \varphi^+(x) dy \\ &\quad + \int_{H_-(0, \mathbf{v}(x))} \bar{l}(y + tz, x - \varepsilon tz) \varphi^-(x) dy \\ &\quad + \int_{H_+(0, \mathbf{v}(x))} \bar{l}(y + tz, x - \varepsilon tz) (\bar{\varphi}(x + \varepsilon y) - \varphi^+(x)) dy \\ &\quad + \int_{H_-(0, \mathbf{v}(x))} \bar{l}(y + tz, x - \varepsilon tz) (\bar{\varphi}(x + \varepsilon y) - \varphi^-(x)) dy. \end{aligned} \tag{3.27}$$

By the definition of J_φ , for every $L > 0$ we obtain,

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} \int_{B_L^+(0, \nu(x))} |\varphi(x + \varepsilon z) - \varphi^+(x)| dz &= 0, \\ \lim_{\varepsilon \rightarrow 0^+} \int_{B_L^-(0, \nu(x))} |\varphi(x + \varepsilon z) - \varphi^-(x)| dz &= 0. \end{aligned} \quad \text{for } x \in J_\varphi. \tag{3.28}$$

Then, by (3.1), for every $L > R$ we have,

$$\begin{aligned} & \left| \int_{H_+(0, \nu(x))} \bar{l}(y + tz, x - \varepsilon tz) (\bar{\varphi}(x + \varepsilon y) - \varphi^+(x)) dy \right| \\ & \leq \int_{B_L^+(0, \nu(x))} |\bar{l}(y + tz, x - \varepsilon tz)| \cdot |\bar{\varphi}(x + \varepsilon y) - \varphi^+(x)| dy \\ & \quad + \int_{H_+(0, \nu(x)) \setminus B_L^+(0, \nu(x))} |\bar{l}(y + tz, x - \varepsilon tz)| \cdot |\bar{\varphi}(x + \varepsilon y) - \varphi^+(x)| dy \\ & \leq A_L \int_{B_L^+(0, \nu(x))} |\bar{\varphi}(x + \varepsilon y) - \varphi^+(x)| dy + B \int_{\mathbb{R}^2 \setminus B_L(0)} \frac{1}{(|y| - R)^3 + 1} dy, \end{aligned} \tag{3.29}$$

where $B > 0$ is a constant and $A_L > 0$ depends only on L . Given $\delta > 0$ we can take $L > 0$ such that

$$B \int_{\mathbb{R}^2 \setminus B_L(0)} \frac{1}{(|y| - R)^3 + 1} dy < \delta,$$

Then, using (3.29) and (3.28), we infer

$$\overline{\lim}_{\varepsilon \rightarrow 0^+} \left| \int_{H_+(0, \nu(x))} \bar{l}(y + tz, x - \varepsilon tz) (\bar{\varphi}(x + \varepsilon y) - \varphi^+(x)) dy \right| < \delta,$$

and since δ was arbitrary,

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} \int_{H_+(0, \nu(x))} \bar{l}(y + tz, x - \varepsilon tz) (\bar{\varphi}(x + \varepsilon y) - \varphi^+(x)) dy &= 0 \\ \forall x \in J_\varphi, z \in B_R(0), t \in [0, 1]. \end{aligned} \tag{3.30}$$

By the same method,

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} \int_{H_-(0, \nu(x))} \bar{l}(y + tz, x - \varepsilon tz) (\bar{\varphi}(x + \varepsilon y) - \varphi^-(x)) dy &= 0 \\ \forall x \in J_\varphi, z \in B_R(0), t \in [0, 1]. \end{aligned} \tag{3.31}$$

Therefore, by (3.27) for every $\varepsilon, t \in (0, 1), x \in J_\varphi$ and $z \in B_R(0)$, we have

$$\begin{aligned} \xi_\varepsilon(x - \varepsilon tz) &= o_\varepsilon(1) + \varphi^+(x) \\ &\quad \times \int_{H_+(0, \mathbf{v}(x))} \bar{l}(y + tz, x - \varepsilon tz) dy + \varphi^-(x) \int_{H_-(0, \mathbf{v}(x))} \bar{l}(y + tz, x - \varepsilon tz) dy \\ &= o_\varepsilon(1) + (\varphi^+(x) - \varphi^-(x)) \int_{H_+(0, \mathbf{v}(x))} \bar{l}(y + tz, x - \varepsilon tz) dy, \end{aligned} \tag{3.32}$$

where we used the equality $\int_{\mathbb{R}^2} \bar{l}(y + tz, x - \varepsilon tz) dy = 0$. Using (3.1), gives

$$\lim_{\varepsilon \rightarrow 0^+} \int_{H_+(0, \mathbf{v}(x))} \bar{l}(y + tz, x - \varepsilon tz) dy = \int_{H_+(0, \mathbf{v}(x))} \bar{l}(y + tz, x) dy.$$

Therefore, by (3.32), for every $x \in J_\varphi$, every $t \in (0, 1)$ and every $z \in B_R(0)$, we obtain,

$$\lim_{\varepsilon \rightarrow 0^+} \xi_\varepsilon(x - \varepsilon tz) = (\varphi^+(x) - \varphi^-(x)) \int_{H_+(0, \mathbf{v}(x))} \bar{l}(y + tz, x) dy. \tag{3.33}$$

Note that

$$\begin{aligned} \int_{H_+(0, \mathbf{v}(x))} \bar{l}(y + tz, x) dy &= \int_{H_+(tz, \mathbf{v}(x))} \bar{l}(y, x) dy \\ &= \int_{t\mathbf{v}(x)-z}^{+\infty} \left(\int_{H_{\mathbf{v}(x)}^0} \bar{l}(t\mathbf{v}(x) + y, x) d\mathcal{H}^1(y) \right) dt \tag{3.34} \\ &= \int_{t\mathbf{v}(x)-z}^{+\infty} \bar{q}(\tau, x) d\tau, \end{aligned}$$

where

$$\bar{q}(t, x) = \int_{H_{\mathbf{v}(x)}^0} \bar{l}(t\mathbf{v}(x) + y, x) d\mathcal{H}^1(y). \tag{3.35}$$

Combining (3.33) and (3.34), for every $x \in J_\varphi$, every $t \in (0, 1)$ and every $z \in B_R(0)$ we obtain,

$$\lim_{\varepsilon \rightarrow 0^+} \xi_\varepsilon(x - \varepsilon tz) = (\varphi^+(x) - \varphi^-(x)) \int_{t\mathbf{v}(x)-z}^{+\infty} \bar{q}(\tau, x) d\tau. \tag{3.36}$$

Using (3.36) in (3.26), we obtain,

$$\begin{aligned}
 & \int_{\mathbb{R}^2} \frac{1}{\varepsilon} \left| \nabla (\operatorname{div}(\Delta^{-1} \varphi_\varepsilon))(x) \right|^2 dx = o_\varepsilon(1) \\
 & + \int_0^1 \left(\int_{J_\varphi} (\varphi^+(x) - \varphi^-(x)) \left\{ \int_{B_R(0)} l(z, x) \cdot \int_{t\mathbf{v}(x)z}^{+\infty} \bar{q}(\tau, x) d\tau \right\} z dz \right) \cdot d[D\varphi](x) dt \\
 & = o_\varepsilon(1) \\
 & + \int_{J_\varphi} |\varphi^+(x) - \varphi^-(x)|^2 \left\{ \int_{B_R(0)} l(z, x) \cdot \int_0^1 \int_{t\mathbf{v}(x)z}^{+\infty} \bar{q}(\tau, x) d\tau dt \right\} (\mathbf{v}(x) \cdot z) dz \Big\} d\mathcal{H}^1(x).
 \end{aligned} \tag{3.37}$$

Step 5. We prove that

$$\bar{q}(t, x) = (q(t, x) \cdot \mathbf{v}(x)) \mathbf{v}(x). \tag{3.38}$$

Consider $(r_1(z, x), r_2(z, x)) := r(z, x)$. Then, by (3.1), for every $k = 1, 2$, we obtain,

$$\begin{aligned}
 & \int_{H_{\mathbf{v}(x)}^0} \nabla_z^2 r_k(t\mathbf{v}(x) + y, x) d\mathcal{H}^1(y) = \int_{H_{\mathbf{v}(x)}^0} \mathbf{v}(x) \otimes \mathbf{v}(x) \frac{\partial^2 r_k}{\partial (\mathbf{v}(x))^2} (t\mathbf{v}(x) + y, x) d\mathcal{H}^1(y) \\
 & + \int_{H_{\mathbf{v}(x)}^0} (\mathbf{v}(x) \otimes \mathbf{v}^\perp(x) + \mathbf{v}^\perp(x) \otimes \mathbf{v}(x)) \frac{\partial^2 r_k}{\partial (\mathbf{v}^\perp(x)) \partial (\mathbf{v}(x))} (t\mathbf{v}(x) + y, x) d\mathcal{H}^1(y) \\
 & + \int_{H_{\mathbf{v}(x)}^0} \mathbf{v}^\perp(x) \otimes \mathbf{v}^\perp(x) \frac{\partial^2 r_k}{\partial (\mathbf{v}^\perp(x))^2} (t\mathbf{v}(x) + y, x) d\mathcal{H}^1(y) \\
 & = \int_{H_{\mathbf{v}(x)}^0} \mathbf{v}(x) \otimes \mathbf{v}(x) \frac{\partial^2 r_k}{\partial (\mathbf{v}(x))^2} (t\mathbf{v}(x) + y, x) d\mathcal{H}^1(y),
 \end{aligned} \tag{3.39}$$

where $\mathbf{v}^\perp(x)$ is the vector orthogonal to $\mathbf{v}(x)$ in \mathbb{R}^2 and all derivatives are taken in the first argument- z of $r(z, x)$. In particular

$$\begin{aligned}
 q(t, x) &= \int_{H_{\mathbf{v}(x)}^0} l(t\mathbf{v}(x) + y, x) d\mathcal{H}^1(y) = \int_{H_{\mathbf{v}(x)}^0} \Delta r(t\mathbf{v}(x) + y, x) d\mathcal{H}^1(y) \\
 &= \int_{H_{\mathbf{v}(x)}^0} \frac{\partial^2 r}{\partial (\mathbf{v}(x))^2} (t\mathbf{v}(x) + y, x) d\mathcal{H}^1(y),
 \end{aligned}$$

and

$$\begin{aligned} \bar{q}(t, x) &= \int_{H_{\mathbf{v}(x)}^0} \bar{l}(t\mathbf{v}(x) + y, x) d\mathcal{H}^1(y) = \int_{H_{\mathbf{v}(x)}^0} \nabla_z(\operatorname{div}_z r)(t\mathbf{v}(x) + y, x) d\mathcal{H}^1(y) \\ &= \left(\mathbf{v}(x) \cdot \int_{H_{\mathbf{v}(x)}^0} \frac{\partial^2 r}{\partial(\mathbf{v}(x))^2}(t\mathbf{v}(x) + y, x) d\mathcal{H}^1(y) \right) \mathbf{v}(x). \end{aligned}$$

So, we obtain (3.38).

Step 6. Completing the proof. Plugging (3.38) into (3.37) gives

$$\begin{aligned} \int_{\mathbb{R}^2} \frac{1}{\varepsilon} \left| \nabla(\operatorname{div}(\Delta^{-1} \varphi_\varepsilon))(x) \right|^2 dx &= o_\varepsilon(1) + \int_{J_\varphi} |\varphi^+(x) - \varphi^-(x)|^2 \\ &\times \left\{ \int_{B_R(0)} \left((l(z, x) \cdot \mathbf{v}(x)) \int_0^1 \int_{t\mathbf{v}(x) \cdot z}^{+\infty} (q(\tau, x) \cdot \mathbf{v}(x)) d\tau dt \right) (\mathbf{v}(x) \cdot z) dz \right\} d\mathcal{H}^1(x). \end{aligned} \tag{3.40}$$

Next we have

$$\begin{aligned} &\int_{B_R(0)} \left((l(z, x) \cdot \mathbf{v}(x)) \int_0^1 \int_{t\mathbf{v}(x) \cdot z}^{+\infty} (q(\tau, x) \cdot \mathbf{v}(x)) d\tau dt \right) (\mathbf{v}(x) \cdot z) dz \\ &= \int_{\mathbb{R}^2} \left((l(z, x) \cdot \mathbf{v}(x)) \int_0^1 \int_{t\mathbf{v}(x) \cdot z}^{+\infty} (q(\tau, x) \cdot \mathbf{v}(x)) d\tau dt \right) (\mathbf{v}(x) \cdot z) dz \\ &= \int_{-\infty}^{+\infty} s \left(\int_0^1 \int_{ts}^{+\infty} (q(\tau, x) \cdot \mathbf{v}(x)) d\tau dt \right) \left(\int_{H_{\mathbf{v}(x)}^0} (l(s\mathbf{v}(x) + y, x) \cdot \mathbf{v}(x)) d\mathcal{H}^1(y) \right) ds \\ &= \int_{-\infty}^{+\infty} s (q(s, x) \cdot \mathbf{v}(x)) \left(\int_0^1 \int_{ts}^{+\infty} (q(\tau, x) \cdot \mathbf{v}(x)) d\tau dt \right) ds \\ &= \int_{-\infty}^{+\infty} (q(s, x) \cdot \mathbf{v}(x)) \left(\int_0^s \int_t^{+\infty} (q(\tau, x) \cdot \mathbf{v}(x)) d\tau dt \right) ds. \end{aligned} \tag{3.41}$$

Using the fact that $\int_{\mathbb{R}} q(\tau, x) d\tau = 0$ and integrating by path, we obtain,

$$\begin{aligned} & \int_{-\infty}^{+\infty} (q(s, x) \cdot \mathbf{v}(x)) \left(\int_0^s \int_t^{+\infty} (q(\tau, x) \cdot \mathbf{v}(x)) d\tau dt \right) ds \\ &= \int_{-\infty}^{+\infty} \left(\int_s^{+\infty} (q(\tau, x) \cdot \mathbf{v}(x)) d\tau \right)^2 ds. \end{aligned} \tag{3.42}$$

Therefore, returning to (3.41) we infer

$$\begin{aligned} & \int_{B_R(0)} \left((l(z, x) \cdot \mathbf{v}(x)) \int_0^1 \int_{t\mathbf{v}(x)\cdot z}^{+\infty} (q(\tau, x) \cdot \mathbf{v}(x)) d\tau dt \right) (\mathbf{v}(x) \cdot z) dz \\ &= \int_{-\infty}^{+\infty} \left(\int_t^{+\infty} (q(s, x) \cdot \mathbf{v}(x)) ds \right)^2 dt. \end{aligned} \tag{3.43}$$

Plugging (3.43) in (3.40) gives the desired result (3.14). □

4. Proof of the main result

As before, throughout this section we assume that Ω is a bounded domain in \mathbb{R}^2 with Lipschitz boundary. Next consider $u \in BV(\Omega, S^1)$, satisfying $\operatorname{div} u = 0$ in Ω and $u \cdot \mathbf{n} = 0$ on $\partial\Omega$ (\mathbf{n} is the unit normal to $\partial\Omega$). Let $\varphi \in BV(\Omega, \mathbb{R}) \cap L^\infty(\Omega, \mathbb{R})$, satisfying $u = e^{i\varphi}$ a.e. in Ω . By [2, Proposition 3.21] we may extend φ to a function $\bar{\varphi} \in BV(\mathbb{R}^2, \mathbb{R}) \cap L^\infty(\mathbb{R}^2, \mathbb{R})$ satisfying $\bar{\varphi} = \varphi$ a.e. in Ω , $\operatorname{supp} \bar{\varphi}$ is compact and $\|D\bar{\varphi}\|(\partial\Omega) = 0$ (from the proof of Proposition 3.21 in [2] it follows that if φ is bounded then its extension is also bounded). We also denote by $\bar{u} := e^{i\bar{\varphi}}$. Then $\bar{u} \in BV(\Omega'', \mathbb{R}^2) \cap L^\infty(\Omega'', \mathbb{R}^2)$ for some $\Omega'' \ni \Omega$, satisfying $\bar{u} = u$ a.e. in Ω and, by Volpert’s chain rule, $\|D\bar{u}\|(\partial\Omega) = 0$. Consider $\eta \in \mathcal{V}$. For any $\varepsilon > 0$ define a function $\psi_\varepsilon(x) : \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$\psi_\varepsilon(x) := \frac{1}{\varepsilon^2} \int_{\mathbb{R}^2} \eta\left(\frac{y-x}{\varepsilon}, x\right) \bar{\varphi}(y) dy = \int_{\mathbb{R}^2} \eta(z, x) \bar{\varphi}(x + \varepsilon z) dz, \quad \forall x \in \mathbb{R}^2. \tag{4.1}$$

Proposition 4.1. *Let $u, \varphi, \bar{u}, \bar{\varphi}$ and η be as above. Consider $\psi_\varepsilon(x)$ defined by (4.1). Then,*

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^2} \left| \nabla \operatorname{div} \Delta^{-1} (\chi_\Omega(x) e^{i\psi_\varepsilon(x)}) \right|^2 dx \\ &= \int_{J_\varphi} \left\{ \int_{-\infty}^{+\infty} \left| \mathbf{v}(x) \cdot \left(e^{i\gamma(t,x)} - e^{i\varphi^-(x)} \right) \right|^2 dt \right\} d\mathcal{H}^1(x), \end{aligned} \tag{4.2}$$

where

$$\gamma(t, x) = \varphi^-(x) \int_{-\infty}^t p(s, x) ds + \varphi^+(x) \int_t^{+\infty} p(s, x) ds, \tag{4.3}$$

with

$$p(t, x) = \int_{H_{\mathbf{v}(x)}^0} \eta(t\mathbf{v}(x) + y, x) d\mathcal{H}^1(y), \tag{4.4}$$

and χ_Ω is the characteristic function of Ω .

Proof. We follow basically the strategy that was described in Subsection 1.1.

Since $(u^+ - u^-) \cdot \mathbf{v} = 0$, the right hand side in (4.2) does not depend on the orientation of J_φ , we may assume that $\mathbf{v}(x)$ is Borel measurable. Together with $\eta \in \mathcal{V}$ we consider a second kernel $\bar{\eta} \in \mathcal{V}$. Let

$$\bar{p}(t, x) = \int_{H_{\mathbf{v}(x)}^0} \bar{\eta}(t\mathbf{v}(x) + y, x) d\mathcal{H}^1(y). \tag{4.5}$$

For any $\varepsilon > 0$ define a function $u_\varepsilon(x) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by

$$u_\varepsilon(x) := \frac{1}{\varepsilon^2} \int_{\mathbb{R}^2} \bar{\eta}\left(\frac{y-x}{\varepsilon}, x\right) \bar{u}(y) dy = \int_{\mathbb{R}^2} \bar{\eta}(z, x) e^{i\bar{\varphi}(x+\varepsilon z)} dz, \quad \forall x \in \mathbb{R}^2. \tag{4.6}$$

Define $Q : \mathbb{R} \times J_\varphi \rightarrow \mathbb{R}^2$ by

$$Q(t, x) := e^{i\gamma(t,x)} - \left(\left\{ \int_{-\infty}^t \bar{p}(s, x) ds \right\} e^{i\varphi^-(x)} + \left\{ \int_t^{+\infty} \bar{p}(s, x) ds \right\} e^{i\varphi^+(x)} \right), \tag{4.7}$$

where $\gamma(t, x)$ is defined by (4.3). Then define $q : \mathbb{R} \times \Omega \rightarrow \mathbb{R}^2$ by

$$q(t, x) = \begin{cases} -\frac{1}{(\varphi^+(x) - \varphi^-(x))} \frac{dQ(t, x)}{dt} & x \in J_\varphi, \\ 0 & x \in \Omega \setminus J_\varphi. \end{cases} \tag{4.8}$$

Then $q(t, x)$ is Borel measurable, q is bounded on $\mathbb{R} \times \Omega$, there exists $M > 0$ such that $\text{supp } q \subset [-M, M] \times \Omega$ and $\int_{\mathbb{R}} q(t, x) dt = 0 \forall x \in \Omega$. Moreover

$$(\varphi^+(x) - \varphi^-(x)) \int_t^{+\infty} q(s, x) ds = Q(t, x). \tag{4.9}$$

Then by Lemma 2.6, there exists a sequence of functions $l_n \in \mathcal{U}$ (see Definition 2.4), such that the sequence of functions $\{q_n\}$ defined on $\mathbb{R} \times \Omega$ by

$$q_n(t, x) = \int_{H_{\mathbf{v}_0(x)}^0} l_n(t\mathbf{v}_0(x) + y, x) d\mathcal{H}^1(y),$$

has the following properties:

$$\text{there exists } C_0 \text{ such that } \|q_n\|_{L^\infty} \leq C_0, \tag{4.10}$$

$$\text{there exists } M > 0 \text{ such that } q_n(t, x) = 0 \text{ for } |t| > M, \text{ and every } x \in \Omega, \tag{4.11}$$

$$\lim_{n \rightarrow \infty} \int_{\Omega} \int_{\mathbb{R}} |q_n(t, x) - q(t, x)| dt d\|D\varphi\|(x) = 0. \tag{4.12}$$

In particular,

$$\lim_{n \rightarrow \infty} \int_{J_\varphi} \int_{\mathbb{R}} |\varphi^+(x) - \varphi^-(x)| \cdot |q_n(t, x) - q(t, x)| dt d\mathcal{H}^1(x) = 0. \quad (4.13)$$

For every positive integer n and for every $\varepsilon > 0$ consider the function $\varphi_{n,\varepsilon} \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ given by

$$\varphi_{n,\varepsilon}(x) := \frac{1}{\varepsilon^2} \int_{\mathbb{R}^2} l_n\left(\frac{y-x}{\varepsilon}, x\right) \bar{\varphi}(y) dy = \int_{\mathbb{R}^2} l_n(z, x) \bar{\varphi}(x + \varepsilon z) dz, \quad (4.14)$$

Next, we will use the following inequality, valid $\forall f(x), g(x), \lambda(x) \in L^2(\mathbb{R}^2, \mathbb{R}^2)$,

$$\begin{aligned} & \left| \int_{\mathbb{R}^2} |f(x)|^2 dx - \int_{\mathbb{R}^2} |g(x)|^2 dx \right| \\ & \leq \left(\|f - g - \lambda\|_{L^2} + \|\lambda\|_{L^2} \right) \sqrt{2 \left(\int_{\mathbb{R}^2} |f(x)|^2 dx + \int_{\mathbb{R}^2} |g(x)|^2 dx \right)}. \end{aligned} \quad (4.15)$$

Therefore, since $\varphi_{n,\varepsilon}(x) = 0$ for $x \notin \Omega$ and since $\operatorname{div}(\chi_\Omega \bar{u}) = 0$ as a distribution, we obtain,

$$\begin{aligned} & \left| \int_{\mathbb{R}^2} \left| \nabla \operatorname{div} \Delta^{-1}(\chi_\Omega(x) e^{i\psi_\varepsilon(x)}) \right|^2 dx - \int_{\mathbb{R}^2} \left| \nabla \operatorname{div} \Delta^{-1}(\varphi_{n,\varepsilon}(x)) \right|^2 dx \right| \\ & \leq 2 \left(\left\| \nabla \operatorname{div} \Delta^{-1}(\chi_\Omega(e^{i\psi_\varepsilon} - \varphi_{n,\varepsilon} - u_\varepsilon)) \right\|_{L^2} + \left\| \nabla \operatorname{div} \Delta^{-1}(\chi_\Omega u_\varepsilon) \right\|_{L^2} \right) \\ & \quad \times \sqrt{\int_{\mathbb{R}^2} \left| \nabla \operatorname{div} \Delta^{-1}(\chi_\Omega e^{i\psi_\varepsilon}) \right|^2 dx + \int_{\mathbb{R}^2} \left| \nabla \operatorname{div} \Delta^{-1}(\varphi_{n,\varepsilon}) \right|^2 dx} \\ & = 2 \left(\left\| \nabla \operatorname{div} \Delta^{-1}(\chi_\Omega(e^{i\psi_\varepsilon} - \varphi_{n,\varepsilon} - u_\varepsilon)) \right\|_{L^2} + \left\| \nabla \operatorname{div} \Delta^{-1}(\chi_\Omega(u_\varepsilon - \bar{u})) \right\|_{L^2} \right) \\ & \quad \times \sqrt{\int_{\mathbb{R}^2} \left| \nabla \operatorname{div} \Delta^{-1}(\chi_\Omega(e^{i\psi_\varepsilon} - e^{i\bar{\varphi}})) \right|^2 dx + \int_{\mathbb{R}^2} \left| \nabla \operatorname{div} \Delta^{-1}(\varphi_{n,\varepsilon}) \right|^2 dx}. \end{aligned} \quad (4.16)$$

But since for every $f \in L^\infty(\mathbb{R}^2, \mathbb{R}^2)$ with compact support we have

$$\int_{\mathbb{R}^2} |\nabla \operatorname{div} \Delta^{-1} f|^2 dx \leq 2 \int_{\mathbb{R}^2} |\nabla^2 \Delta^{-1} f|^2 dx = 2 \int_{\mathbb{R}^2} |f|^2 dx,$$

by (4.16), we obtain

$$\begin{aligned} & \left| \frac{1}{\varepsilon} \int_{\mathbb{R}^2} \left| \nabla \operatorname{div} \Delta^{-1}(\chi_{\Omega}(x)e^{i\psi_{\varepsilon}(x)}) \right|^2 dx - \frac{1}{\varepsilon} \int_{\mathbb{R}^2} \left| \nabla \operatorname{div} \Delta^{-1}(\varphi_{n,\varepsilon}(x)) \right|^2 dx \right| \\ & \leq 4 \left(\sqrt{\frac{1}{\varepsilon} \int_{\Omega} |e^{i\psi_{\varepsilon}} - \varphi_{n,\varepsilon} - u_{\varepsilon}|^2 dx} + \sqrt{\frac{1}{\varepsilon} \int_{\Omega} |u_{\varepsilon} - u|^2 dx} \right) \\ & \quad \times \sqrt{\frac{1}{\varepsilon} \int_{\Omega} |e^{i\psi_{\varepsilon}} - e^{i\varphi}|^2 dx + \frac{1}{\varepsilon} \int_{\mathbb{R}^2} |\nabla \operatorname{div} \Delta^{-1}(\varphi_{n,\varepsilon})|^2 dx}. \end{aligned} \tag{4.17}$$

Therefore, setting

$$L_0 := \int_{J_{\varphi}} \left\{ \int_{-\infty}^{+\infty} \left| \mathbf{v}(x) \cdot \left(e^{i\gamma(t,x)} - e^{i\varphi^{-}(x)} \right) \right|^2 dt \right\} d\mathcal{H}^1(x),$$

we have

$$\begin{aligned} & \left| \frac{1}{\varepsilon} \int_{\mathbb{R}^2} \left| \nabla \operatorname{div} \Delta^{-1}(\chi_{\Omega}e^{i\psi_{\varepsilon}}) \right|^2 dx - L_0 \right| \leq \left| L_0 - \frac{1}{\varepsilon} \int_{\mathbb{R}^2} \left| \nabla \operatorname{div} \Delta^{-1}(\varphi_{n,\varepsilon}) \right|^2 dx \right| \\ & \quad + 4 \left(\sqrt{\frac{1}{\varepsilon} \int_{\Omega} |e^{i\psi_{\varepsilon}} - \varphi_{n,\varepsilon} - u_{\varepsilon}|^2 dx} + \sqrt{\frac{1}{\varepsilon} \int_{\Omega} |u_{\varepsilon} - u|^2 dx} \right) \\ & \quad \times \sqrt{\frac{1}{\varepsilon} \int_{\Omega} |e^{i\psi_{\varepsilon}} - e^{i\varphi}|^2 dx + \frac{1}{\varepsilon} \int_{\mathbb{R}^2} |\nabla \operatorname{div} \Delta^{-1}(\varphi_{n,\varepsilon})|^2 dx}. \end{aligned} \tag{4.18}$$

Using Proposition 2.2 with $W(a, b) = W((a_1, a_2, a_3), b) : (\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2) \times \mathbb{R} \rightarrow \mathbb{R}$, defined by $W(a, b) := |e^{ia_1} - a_2 - a_3|^2$, we obtain,

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{\Omega} |e^{i\psi_{\varepsilon}} - \varphi_{n,\varepsilon} - u_{\varepsilon}|^2 dx = D_n \\ & := \int_{J_{\varphi}} \left\{ \int_{-\infty}^{+\infty} \left| e^{i\gamma(t,x)} - (\varphi^{+}(x) - \varphi^{-}(x)) \int_t^{+\infty} q_n(s, x) ds - \Gamma(t, x) \right|^2 dt \right\} d\mathcal{H}^1(x), \end{aligned} \tag{4.19}$$

where $\gamma(t, x)$ is defined by (4.3), and

$$\Gamma(t, x) := \left\{ \int_{-\infty}^t \bar{p}(s, x) ds \right\} e^{i\varphi^{-}(x)} + \left\{ \int_t^{+\infty} \bar{p}(s, x) ds \right\} e^{i\varphi^{+}(x)}.$$

Using Proposition 2.2 with $W(a, b) = W : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$, defined by $W(a, b) := |a - b|^2$, we also infer that

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{\Omega} |u_{\varepsilon} - u|^2 dx = T(\bar{\eta}) \\ & := \int_{J_{\varphi}} \left\{ \int_{-\infty}^0 |\Gamma(t, x) - u^+(x)|^2 dt + \int_0^{+\infty} |\Gamma(t, x) - u^-(x)|^2 dt \right\} d\mathcal{H}^1(x) \\ & = \int_{J_{\varphi}} \left\{ \int_{-\infty}^0 \left| (u^+ - u^-) \int_{-\infty}^t \bar{p}(s, \cdot) ds \right|^2 dt \right. \\ & \quad \left. + \int_0^{+\infty} \left| (u^+ - u^-) \int_t^{+\infty} \bar{p}(s, \cdot) ds \right|^2 dt \right\} d\mathcal{H}^1, \end{aligned} \tag{4.20}$$

and

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{\Omega} |e^{i\psi_{\varepsilon}} - e^{i\varphi}|^2 dx \\ & = M := \int_{J_{\varphi}} \left\{ \int_{-\infty}^0 |e^{i\gamma(t,x)} - e^{i\varphi^+(x)}|^2 dt + \int_0^{+\infty} |e^{i\gamma(t,x)} - e^{i\varphi^-(x)}|^2 dt \right\} d\mathcal{H}^1(x). \end{aligned} \tag{4.21}$$

By Lemma 3.2 we obtain

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^2} \frac{1}{\varepsilon} \left| \nabla(\operatorname{div}(\Delta^{-1}\varphi_{n,\varepsilon}))(x) \right|^2 dx \\ & = L_n := \int_{J_{\varphi}} \left\{ \int_{-\infty}^{+\infty} |\varphi^+(x) - \varphi^-(x)|^2 \cdot \left| \mathbf{v}(x) \cdot \int_t^{+\infty} q_n(s, x) ds \right|^2 dt \right\} d\mathcal{H}^1(x). \end{aligned} \tag{4.22}$$

Therefore, letting ε tend to 0 in (4.18), we get,

$$\begin{aligned} & \overline{\lim}_{\varepsilon \rightarrow 0^+} \left| \frac{1}{\varepsilon} \int_{\mathbb{R}^2} \left| \nabla \operatorname{div} \Delta^{-1}(\chi_{\Omega} e^{i\psi_{\varepsilon}}) \right|^2 dx - L_0 \right| \\ & \leq |L_0 - L_n| + 4\left(\sqrt{D_n} + \sqrt{T(\bar{\eta})}\right)\sqrt{M + L_n}. \end{aligned} \tag{4.23}$$

Using (4.7), (4.9), (4.13), (4.10) and (4.11) we obtain

$$\lim_{n \rightarrow \infty} D_n = 0, \tag{4.24}$$

and since $(u^+(x) - u^-(x)) \perp \mathbf{v}(x)$ (by $\operatorname{div} u = 0$), we also infer

$$\lim_{n \rightarrow \infty} L_n = L_0 := \int_{J_\varphi} \left\{ \int_{-\infty}^{+\infty} \left| \mathbf{v}(x) \cdot \left(e^{i\gamma(t,x)} - e^{i\varphi^-(x)} \right) \right|^2 dt \right\} d\mathcal{H}^1(x). \tag{4.25}$$

Therefore, letting n tend to $+\infty$ in (4.23), we obtain,

$$\begin{aligned} & \overline{\lim}_{\varepsilon \rightarrow 0^+} \left| \frac{1}{\varepsilon} \int_{\mathbb{R}^2} \left| \nabla \operatorname{div} \Delta^{-1} (\chi_\Omega e^{i\psi_\varepsilon}) \right|^2 dx - \int_{J_\varphi} \left\{ \int_{-\infty}^{+\infty} \left| \mathbf{v} \cdot \left(e^{i\gamma(t,\cdot)} - e^{i\varphi^-} \right) \right|^2 dt \right\} d\mathcal{H}^1 \right| \\ & \leq 4\sqrt{T(\bar{\eta})} \sqrt{M + L_0}. \end{aligned} \tag{4.26}$$

This equation is valid for any $\bar{\eta} \in \mathcal{V}$, and the constants M and L_0 do not depend on $\bar{\eta}$. For every $\delta > 0$ we always can choose $\bar{\eta}_\delta \in C^2(\mathbb{R}^2 \times \mathbb{R}^2, \mathbb{R})$, satisfying $\bar{\eta}_\delta \geq 0$, $\operatorname{supp} \bar{\eta}_\delta \subseteq B_\delta(0) \times \mathbb{R}^2$ and $\int_{\mathbb{R}^2} \bar{\eta}_\delta(z, x) dz = 1$ for any $x \in \Omega$. Then, as before, define $\bar{p}_\delta(t, x) : \mathbb{R} \times J_\varphi \rightarrow \mathbb{R}$ by

$$\bar{p}_\delta(t, x) = \int_{H_{\mathbf{v}(x)}^0} \bar{\eta}_\delta(t\mathbf{v}(x) + y, x) d\mathcal{H}^1(y).$$

Since $\bar{p}_\delta \geq 0$ and $\operatorname{supp} \bar{p}_\delta(t, x) \subset [-\delta, \delta] \times J_\varphi$ and $\int_{-\infty}^{+\infty} \bar{p}_\delta(t, x) dt = 1$, by (4.20) we infer

$$\begin{aligned} & T(\bar{\eta}_\delta) \\ & \leq \int_{J_\varphi} \left\{ \int_{-\delta}^0 \left| (u^+ - u^-) \int_{-\infty}^t \bar{p}_\delta(s, \cdot) ds \right|^2 dt + \int_0^\delta \left| (u^+ - u^-) \int_t^{+\infty} \bar{p}_\delta(s, \cdot) ds \right|^2 dt \right\} d\mathcal{H}^1 \\ & \leq 2\delta \int_{J_\varphi} |u^+ - u^-|^2 d\mathcal{H}^1 \leq 4\delta \int_{J_\varphi} |u^+ - u^-| d\mathcal{H}^1 \leq 4\delta \|Du\|(\Omega). \end{aligned}$$

Therefore, by (4.26) we obtain

$$\begin{aligned} & \overline{\lim}_{\varepsilon \rightarrow 0^+} \left| \frac{1}{\varepsilon} \int_{\mathbb{R}^2} \left| \nabla \operatorname{div} \Delta^{-1} (\chi_\Omega e^{i\psi_\varepsilon}) \right|^2 dx - \int_{J_\varphi} \left\{ \int_{-\infty}^{+\infty} \left| \mathbf{v} \cdot \left(e^{i\gamma(t,\cdot)} - e^{i\varphi^-} \right) \right|^2 dt \right\} d\mathcal{H}^1 \right| \\ & \leq 8\sqrt{\delta} \sqrt{\|Du\|(\Omega)} \sqrt{M + L_0}. \end{aligned} \tag{4.27}$$

For $\delta \rightarrow 0$, (4.27) gives (4.2). □

Let $\varphi, \bar{\varphi}$ and η be as in Proposition 4.1 and ψ_ε be defined by (4.1). Then using [16, Proposition 3.1], we obtain,

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} \varepsilon |\nabla \psi_\varepsilon(x)|^2 dx = \int_{J_\varphi} |\varphi^+(x) - \varphi^-(x)|^2 \cdot \left(\int_{\mathbb{R}} p^2(t, x) dt \right) d\mathcal{H}^1(x), \tag{4.28}$$

where $p(t, x)$ is defined by (4.4). As in [16] and [17] we also easily deduce that

$$\lim_{\varepsilon \rightarrow 0^+} \psi_\varepsilon(x) = \varphi(x) \quad \text{in } L^p(\Omega, \mathbb{R}) \quad \forall p \in [1, \infty).$$

Combining these facts with the result of Proposition 4.1, we infer the following.

Corollary 4.2. *Let $u \in BV(\Omega, S^1)$, satisfying $\operatorname{div} u = 0$ in Ω and $u \cdot \mathbf{n} = 0$ on $\partial\Omega$ (\mathbf{n} is the unit normal to $\partial\Omega$). Let $\varphi \in BV(\Omega, \mathbb{R}) \cap L^\infty$ such that $u = e^{i\varphi}$ a.e. in Ω . Consider a function $\bar{\varphi} \in BV(\mathbb{R}^2, \mathbb{R}) \cap L^\infty$ such that $\bar{\varphi} = \varphi$ a.e. in Ω , $\operatorname{supp} \bar{\varphi}$ is compact and $\|D\bar{\varphi}\|(\partial\Omega) = 0$. Given $\eta \in \mathcal{V}$, for every $\varepsilon > 0$ let ψ_ε be defined by (4.1). Then,*

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \left(\int_{\Omega} \varepsilon |\nabla e^{i\psi_\varepsilon(x)}|^2 dx + \int_{\mathbb{R}^2} \left| \nabla \operatorname{div} \Delta^{-1}(\chi_\Omega(x) e^{i\psi_\varepsilon(x)}) \right|^2 dx \right) \\ &= Y_\varphi(\eta) := \int_{J_\varphi} |\varphi^+(x) - \varphi^-(x)|^2 \cdot \left(\int_{-\infty}^{+\infty} p^2(t, x) dt \right) d\mathcal{H}^1(x) \tag{4.29} \\ &+ \int_{J_\varphi} \left\{ \int_{-\infty}^{+\infty} \left| \mathbf{v}(x) \cdot \left(e^{i\gamma(t,x)} - e^{i\varphi^-(x)} \right) \right|^2 dt \right\} d\mathcal{H}^1(x), \end{aligned}$$

where γ and p defined by (4.3) and (4.4) respectively. Moreover,

$$\lim_{\varepsilon \rightarrow 0^+} \psi_\varepsilon(x) = \varphi(x) \quad \text{in } L^p(\Omega, \mathbb{R}) \quad \forall p \in [1, \infty).$$

Next we turn to the minimization problem of the term on the right hand side of (4.29), over all kernels $\eta \in \mathcal{V}$, analogously to that was done in [16] and [17]. By the same method, as there, we can obtain the following.

Lemma 4.3. *Let $Y_\varphi(\eta) : \mathcal{V} \rightarrow \mathbb{R}$ be defined as the right hand side of (4.29). Then,*

$$\begin{aligned} & \inf_{\eta \in \mathcal{V}} Y_\varphi(\eta) = \mathcal{J}_0(\varphi) \\ &:= \int_{J_\varphi} 2 |\varphi^+(x) - \varphi^-(x)| \left\{ \int_0^1 \left| \mathbf{v}(x) \cdot \left(e^{i(s\varphi^-(x) + (1-s)\varphi^+(x))} - e^{i\varphi^-(x)} \right) \right| ds \right\} d\mathcal{H}^1(x) \\ &= \int_{J_\varphi} 2 \left| \int_{\varphi^-(x)}^{\varphi^+(x)} \left| \mathbf{v}(x) \cdot \left(e^{it} - e^{i\varphi^-(x)} \right) \right| dt \right| d\mathcal{H}^1(x). \end{aligned} \tag{4.30}$$

By [19, (II.36)] we infer that

$$\begin{aligned} \mathcal{J}_0(\varphi) &= \int_{J_\varphi} 2 \left| \int_{\varphi^-(x)}^{\varphi^+(x)} |\mathbf{v}(x) \cdot (e^{it} - e^{i\varphi^-(x)})| dt \right| d\mathcal{H}^1(x) \\ &= 2 \int_{\mathbb{R}} \int_{\Omega} |\operatorname{div}_x T^t u| dx dt, \end{aligned} \tag{4.31}$$

where we (as in [19]), consider $T^t \varphi := \inf(\varphi, t)$ and $T^t u := e^{i T^t \varphi}$.

Proof of Theorem 1.2. The case of $\varphi \in BV(\Omega, \mathbb{R}) \cap L^\infty$ follows easily from Corollary 4.2 and Lemma 4.3 by using a standard diagonal argument as in the proofs of [17, Theorem 1.1 and Theorem 1.2].

It remains to consider the case of an unbounded $\varphi \in BV(\Omega, \mathbb{R})$, such that $e^{i\varphi(x)} = u(x)$ a.e. in Ω . First recall that by [6] there exists $\varphi_0 \in BV(\Omega, \mathbb{R}) \cap L^\infty(\Omega, \mathbb{R})$ satisfying $e^{i\varphi_0(x)} = u(x)$ a.e. in Ω . Then $\varphi(x) = \varphi_0(x) + 2\pi l(x)$ where $l \in BV(\Omega, \mathbb{Z})$. For each integer $n \geq 1$ define,

$$l_n(x) := \begin{cases} l(x) & x \in \Omega, |l(x)| \leq n, \\ n & x \in \Omega, l(x) > n, \\ -n & x \in \Omega, l(x) < -n, \end{cases} \quad \varphi_n(x) := \varphi_0(x) + 2\pi l_n(x).$$

Clearly $\varphi_n \in BV(\Omega) \cap L^\infty(\Omega)$ and $e^{i\varphi_n(x)} = u(x)$ a.e. in Ω . From the case of a bounded φ , considered above, it follows that for each n there exists a family $\{v_{n,\varepsilon}\}_{\varepsilon>0} \subset C^2(\Omega, \mathbb{R})$ satisfying $\lim_{\varepsilon \rightarrow 0} v_{n,\varepsilon}(x) = \varphi_n(x)$ in $L^1(\Omega, \mathbb{R})$ and

$$\begin{aligned} &\lim_{\varepsilon \rightarrow 0} \left(\int_{\Omega} \varepsilon |\nabla e^{i v_{n,\varepsilon}(x)}|^2 dx + \int_{\mathbb{R}^2} \left| \nabla \operatorname{div} \Delta^{-1}(\chi_\Omega(x) e^{i v_{n,\varepsilon}(x)}) \right|^2 dx \right) \\ &= \mathcal{J}_0(\varphi_n) = \int_{J_{\varphi_n}} 2 \left| \int_{\varphi_n^-(x)}^{\varphi_n^+(x)} |\mathbf{v}_n(x) \cdot (e^{it} - e^{i\varphi_n^-(x)})| dt \right| d\mathcal{H}^1(x). \end{aligned}$$

Since for any $x \in \Omega$ we have $|\varphi_n(x)| \leq |\varphi_0(x)| + 2\pi |l(x)|$ while $\varphi_n(x) = \varphi(x)$ for n sufficiently large, we deduce by dominated convergence that

$$\lim_{n \rightarrow \infty} \varphi_n(x) = \varphi(x) \quad \text{in } L^1(\Omega, \mathbb{R}).$$

Put $\lambda_n(x) := |\varphi_n^+(x) - \varphi_n^-(x)|$. For \mathcal{H}^{N-1} -almost every $x \in J_{\varphi_0} \cup J_l$ we have $\lambda_n(x) \leq |\varphi_0^+(x) - \varphi_0^-(x)| + 2\pi |l^+(x) - l^-(x)|$, while $\lambda_n(x) = |\varphi^+(x) - \varphi^-(x)|$ for sufficiently large n . Moreover, $\mathcal{H}^{N-1}(J_{\varphi_n} \setminus (J_{\varphi_0} \cup J_l)) = 0$ and $\mathbf{v}_n(x) = \mathbf{v}(x)$

for \mathcal{H}^{N-1} -a.e. $x \in J_{\varphi_n} \cap J_\varphi$, for each n . Therefore, by dominated convergence,

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int_{J_{\varphi_n}} 2 \left| \int_{\varphi_n^-(x)}^{\varphi_n^+(x)} \mathbf{v}_n(x) \cdot (e^{it} - e^{i\varphi_n^-(x)}) dt \right| d\mathcal{H}^1(x) \\ &= \int_{J_\varphi} 2 \left| \int_{\varphi^-(x)}^{\varphi^+(x)} \mathbf{v}(x) \cdot (e^{it} - e^{i\varphi^-(x)}) dt \right| d\mathcal{H}^1(x). \end{aligned}$$

To complete the proof, we apply to $\{v_{n,\varepsilon}\}$ a standard diagonal argument. \square

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