Addendum to: On volumes of arithmetic quotients of SO(1, n)

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Abstract. There are errors in the proof of uniqueness of arithmetic subgroups of the smallest covolume. In this note we correct the proof, obtain certain results which were stated as a conjecture, and we give several remarks on further developments.

Mathematics Subject Classification (2000): 11E57 (primary); 22E40 (secondary).

1.1. Let us recall some notation and basic notions. Following [1] we will assume that \( n \) is even and \( n \geq 4 \). The group of orientation preserving isometries of hyperbolic \( n \)-space is isomorphic to \( \text{SO}(1, n)^o \), the connected component of the identity of the special orthogonal group of signature \( (1, n) \), which can be identified with \( \text{SO}_0(1, n) \), the subgroup of \( \text{SO}(1, n) \) preserving the upper half space. This group is not Zariski closed in \( \text{SL}_{n+1} \) thus in order to construct arithmetically defined subgroups of \( \text{SO}(1, n)^o \) we consider arithmetic subgroups of the orthogonal group \( \text{SO}(1, n) \) or, more precisely, of groups \( G = \text{SO}(f) \) where \( f \) is an admissible quadratic form defined over a totally real number field \( k \) (see [1, Section 2.1]).

We have an exact sequence of \( k \)-isogenies:

\[
1 \to C \to \widetilde{G} \xrightarrow{\phi} G \to 1, \tag{1.1}
\]

where \( \widetilde{G}(k) \simeq \text{Spin}(f) \) is the simply connected cover of \( G \) and \( C \simeq \mu_2 \) is the center of \( \widetilde{G} \). This induces an exact sequence in Galois cohomology (see [5, Section 2.2.3])

\[
\widetilde{G}(k) \xrightarrow{\phi} G(k) \xrightarrow{\delta} H^1(k, C) \to H^1(k, \widetilde{G}). \tag{1.2}
\]

The main idea of this note is that by using (1.2) certain questions about arithmetic subgroups of \( G \) can be reduced to questions about the Galois cohomology group \( H^1(k, C) \).

A coherent collection of parahoric subgroups \( P = (P_v)_{v \in V_f} \) of \( \widetilde{G}(V_f = V_f(k) \) denotes the set of finite places of the field \( k \) defines a principal arithmetic subgroup

Received October 16, 2006; accepted in revised form March 12, 2007.
\[ \Lambda = \widetilde{G}(k) \cap \prod_{v \in V_f} P_v \subset \widetilde{G}(k) \text{ (see [2]).} \] We fix an infinite place \( v \) of \( k \) for which \( \widetilde{G}(k_v) \simeq SO(1, n) \) and denote it by \( Id \). The image of \( \Lambda \) under the central \( k \)-isogeny \( \phi \) is an arithmetic subgroup of \( G \) and every maximal arithmetic subgroup of \( G(k_{Id}) \) can be obtained as a normalizer of some \( \phi(\Lambda) \) [2, Proposition 1.4]. We will also consider the local stabilizers of \( P \) in the adjoint group \( G(\overline{k}) \), defining \( \overline{P}_v \) to be the stabilizer of \( P_v \) in \( G(k_v) \) and \( \overline{P} = (\overline{P}_v)_{v \in V_f} \). Clearly, \( \overline{P}_v \supset \phi(P_v) \). In the notation of [1] the subgroups \( \phi(P_v) \) are called parahoric subgroups of \( G \), however this terminology is non-standard and we will avoid using it here.

ACKNOWLEDGEMENTS. This article was written in Summer 2006 while I have been visiting MPIM in Bonn. I would like to thank Gopal Prasad for his remarks on a preliminary version of this note.

1.2. Given a totally real number field \( k \) with the group of units \( U \), let
\[ k^*_\infty = \{ a \in k^* \mid a_v > 0 \text{ for } v \in V_\infty \setminus Id \}, \quad U_\infty = U \cap k^*_\infty. \]

Lemma 1.1. \( \text{Im}(\delta) \simeq k^*_\infty/(k^*)^2. \)

Proof. From (1.2) we have \( \text{Im}(\delta : G(k) \to H^1(k, \mu_2)) = \text{Ker}(H^1(k, \mu_2) \to H^1(k, \widetilde{G})) \). The Hasse principle for simply connected \( k \)-groups implies that \( H^1(k, \widetilde{G}) \) is isomorphic to \( \prod_{v \in V_\infty} H^1(k_v, \widetilde{G}) \) [5, Theorem 6.6], and hence
\[ \text{Im}(\delta) = \text{Ker}(H^1(k, \mu_2) \to \prod_{v \in V_\infty} H^1(k_v, \widetilde{G})). \]

The group \( H^1(k, \mu_2) \) is canonically isomorphic to \( k^*/(k^*)^2 \) [5, Lemma 2.6]. It is well known that for all \( v \in V_\infty \) such that the group \( G(k_v) \) is anisotropic, the map \( \phi \) in (1.2) is surjective and hence for all such \( v \), \( \text{Im}(\delta_v) = \text{Ker}(H^1(k_v, \mu_2) \to H^1(k_v, \widetilde{G})) \) is trivial. For the remaining one infinite place \( v(= Id) \in V_\infty \), \( \phi(G(k_v)) \) is a subgroup of index 2 in \( G(k_v) \) which consists of the orthogonal transformations with the trivial spinor norm. Collecting this information together we obtain the required isomorphism.

1.3 The proof of the uniqueness part in [1, Theorem 4.1] contains errors but the result is correct. We will now give another argument for it. In order to do so we first establish a more general fact and then apply it to the cases considered in [1].

Let \( P = (P_v)_{v \in V_f} \) and \( P' = (P'_v)_{v \in V_f} \) be two coherent collections of parahoric subgroups of \( \widetilde{G} \) such that for all \( v \in V_f \), \( P'_v \) is conjugate to \( P_v \) under an element of \( G(k_v) \). For all but finitely many \( v \), \( P_v = P'_v \) hence there is an element \( g \in G(\mathbb{A}_f) \) (\( \mathbb{A}_f \) denotes the ring of finite adèles of \( k \)) such that \( P' \) is the conjugate of \( P \) under \( g \). We have \( \overline{P} = \prod_{v \in V_f} \overline{P}_v \) is the stabilizer of \( P \) in \( G(\mathbb{A}_f) \). The number of distinct
G(k)-conjugacy classes of coherent collections $P'$ as above is the cardinality $c(P)$ of $\mathcal{C}(P) = G(k)\backslash G(\mathbb{A}_f)/\mathcal{P}$, which is called the class group of $G$ relative to $\mathcal{P}$. The class number $c(P)$ is known to be finite (see e.g. [2, Proposition 3.9]). The following result can be used for obtaining further information about its value.

**Proposition 1.2.** Let $G = \text{SO}(f)$, $\tilde{G} = \text{Spin}(f)$ for an admissible quadratic form $f$ defined over $k$ and let $P = (P_v)_{v \in V_f}$ a coherent collection of parahoric subgroups of $\tilde{G}$. The class number $c(P)$ divides the order $h_{\infty,2}$ of a restricted 2-class group of $k$ given by

$$h_{\infty,2} = \frac{2^{[k:Q] - 1} h_2}{[U : U_\infty]},$$

where $h_2$ is the order of the 2-class group of $k$.

**Proof.** Recall two isomorphisms (see [5, Proposition 8.8], a minor modification is needed in order to adjust the statement to our setting but the argument remains the same):

$$G(k)\backslash G(\mathbb{A}_f)/\mathcal{P} \simeq G(\mathbb{A}_f)/\mathcal{P}G(k);$$

$$G(\mathbb{A}_f)/\mathcal{P}G(k) \simeq \delta_{\mathbb{A}_f}(G(\mathbb{A}_f))/\delta_{\mathbb{A}_f}(\mathcal{P}G(k)),$$

where $\delta_{\mathbb{A}_f}$ is the restriction of the product map $\prod_v G(k_v) \to \prod_v \mathbb{H}^1(k_v, C)$ to $G(\mathbb{A}_f)$.

For every finite place $v$, $\mathbb{H}^1(k_v, \tilde{G})$ is trivial (see [5, Theorem 6.4]) which implies $\delta_v : G(k_v) \to \mathbb{H}^1(k_v, C)$ is surjective. Thus the image of $\delta_{\mathbb{A}_f}(G(\mathbb{A}_f))$ can be identified with the restricted direct product $\prod' H^1(k_v, C)$ with respect to the subgroups $\delta_v(\mathcal{P}v)$. Also $\delta_{\mathbb{A}_f}(G(k))$ naturally identifies with the image of $\delta(G(k))$ in $H^1(k, C)$ under the embedding $\psi : H^1(k, C) \to \prod' H^1(k_v, C)$. Hence we have an isomorphism

$$\delta_{\mathbb{A}_f}(G(\mathbb{A}_f))/\delta_{\mathbb{A}_f}(\mathcal{P}G(k)) \simeq \prod' H^1(k_v, C)/\left(\prod_v \delta_v(\mathcal{P}v) \cdot \psi(\text{Im } \delta(G(k)))\right).$$

The group $H^1(k_v, \mu_2)$ is canonically isomorphic to $k_v^*/(k_v^*)^2$, by Lemma 1.1 $\text{Im } \delta(G(k)) \simeq k_{\infty}^*/(k_{\infty}^*)^2$, so we obtain

$$\prod' H^1(k_v, C)/\prod_v \delta_v(\mathcal{P}v) \cdot \psi(\text{Im } \delta(G(k))) \simeq \prod' k_v^*/(k_v^*)^2 \simeq \frac{J_f}{J_f^2} \cdot \frac{k^*}{k_{\infty}^*},$$

where $J_f$ is the ring of finite idèles of $k$ and $\delta_P$ denotes $\prod_v \delta_v(\mathcal{P}v)$.

Now, $#(J_f/J_f^2 k^*) = h_2$, the group $k^*/k_{\infty}^*$ splits as a product of local factors and $#(k^*/k_{\infty}^*) = 2^{[k:Q] - 1}/[U : U_\infty]$ (see [4, Chapter 6]). This implies the proposition. \(\square\)
In order to give a precise formula for the class number $c(P)$ one has to analyze the image of $\prod_v \delta_v(\mathcal{P}_v)$ in $\prod_v H^1(k_v, \mathbb{C})$. Still in many practical cases this appears to be unnecessary. Thus in order to prove the uniqueness of the minimal hyperbolic orbifolds we need to consider $k = \mathbb{Q}[\sqrt{5}]$ (in the compact case) and $k = \mathbb{Q}$ (for the non-compact orbifolds). In both cases $h_2 = h = 1$. For $k = \mathbb{Q}[\sqrt{5}]$, $U_\infty = \{1, \frac{1-\sqrt{5}}{2}\}$ and thus $[U : U] = 2$ which implies $h_{\infty,2} = 1$. For $k = \mathbb{Q}$, clearly, $h_{\infty,2} = 1$ as well. So in all the cases $c(P) = 1$ which implies that the corresponding arithmetic subgroups are defined uniquely up to a conjugation by $g \in SO(1, n)$. It is clear that we can always chose $g \in SO_0(1, n)$ and therefore the smallest orbifolds constructed in [1] are unique up to an (orientation preserving) isometry.

1.4. We now turn to Conjecture 4.1 and its analogue for the non-cocompact orbifolds in [1, Section 4.4]. Recall that in [1] the numbers $N(r)$, $N'(r)$ were defined for every $r \geq 2$ and estimated from above. These numbers are related to the index of the principal arithmetic subgroups in their normalizers. We now prove

**Proposition 1.3.** For every $r \geq 2$, $N(r) = N'(r) = 1$.

**Proof.** Let $\Lambda$ be a principal arithmetic subgroup of $\tilde{G}$ which corresponds to a compact or non-compact hyperbolic $n$-orbifold of the minimal volume, $\Lambda' = \phi(\Lambda)$ and $\Gamma = N_G(\Lambda')$.

From [2, Proposition 2.9], which in turn follows from the work of J. Rohlfs, using the fact that the center of our group $G$ is trivial, we obtain:

$$[\Gamma : \Lambda'] = \#(H^1(k, \mu_2)_\Theta \cap \delta(G(k))) = \#\text{Im}(\delta : G(k) \rightarrow H^1(k, \mu_2)).$$

By Lemma 1.1 we can identify the image of $\delta$. The cases we are interested in are

$$k = \mathbb{Q} : \text{Im}(\delta) = \left\{k^{*2}, (-1)k^{*2}\right\};$$

$$k = \mathbb{Q}[\sqrt{5}] : \text{Im}(\delta) = \left\{k^{*2}, \frac{1-\sqrt{5}}{2}k^{*2}\right\}.$$
One other corollary is that cocompact and non-cocompact arithmetic subgroups of $\text{SO}(1, 2r)^o$ of the smallest covolumes can be obtained as the stabilizers of certain lattices described in [1, Section 4.3]. We remark that since the fields of definition of the groups have class number 1, the lattices in both cases are free as $O_k$-modules.

1.5. Correction: on p. 765, l. 9 one should read “grow super-exponentially” instead of “grow exponentially”. (It follows from [1] that the Euler characteristic is bounded from below by $\text{const} \cdot (\prod_{i=1}^{r} \frac{(2i-1)!}{(2\pi)^{2i}})^{[k:Q]}$ which for large enough $r$ is $\geq \text{const} \cdot (2r - 1)!$)

We conclude this addendum with a few remarks on related results which appeared after the paper was published.

1.6. In [1, Section 4.5] we observed that for $r > 2$ the minimal covolume among the arithmetic lattices in $\text{SO}(1, 2r)$ is attained on a non-uniform lattice. This interesting phenomenon was first discovered by A. Lubotzky for $\text{SL}_2$ over local fields of positive characteristic. Recently, in [6] A. Salehi Golsefidy proved that lattices of minimal covolume in classical Chevalley groups over local fields of characteristic $p > 7$ are all non-uniform. This result gives further support to a conjecture that generically (i.e., for groups of high enough rank or fields of large enough positive characteristic) the minimal covolume is always attained on a non-uniform lattice.

1.7. In [3] M. Conder and C. Maclachlan constructed a compact orientable hyperbolic 4-manifold which has Euler characteristic 16. The previously known smallest example which was used in order to formulate the main result in [1, Section 5] had $\chi = 26$. The construction of [3] agrees with our Theorem 5.5 and it also allows us to give a more precise formulation of the theorem:

**Theorem 5.5’.** If there exists a compact orientable arithmetic hyperbolic 4-manifold $M$ with $\chi(M) \leq 16$, then $M$ is defined over $\mathbb{Q}[\sqrt{5}]$ and has the form $\Gamma \backslash \mathcal{H}^4$ with $\Gamma$ being a torsion-free subgroup of index $7200\chi(M)$ of the group $\Gamma_1$ of the smallest arithmetic hyperbolic 4-orbifold.

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