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# Addendum to: On volumes of arithmetic quotients of SO $(1, n)$

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**Abstract.** There are errors in the proof of uniqueness of arithmetic subgroups of the smallest covolume. In this note we correct the proof, obtain certain results which were stated as a conjecture, and we give several remarks on further developments.

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**1.1.** Let us recall some notation and basic notions. Following [1] we will assume that $n$ *is even* and $n \geq 4$. The group of orientation preserving isometries of hyperbolic $n$-space is isomorphic to $\mathrm{SO}(1, n)^o$, the connected component of the identity of the special orthogonal group of signature $(1, n)$, which can be identified with $\mathrm{SO}_0(1, n)$, the subgroup of $\mathrm{SO}(1, n)$ preserving the upper half space. This group is not Zariski closed in $\mathrm{SL}_{n+1}$ thus in order to construct arithmetically defined subgroups of $\mathrm{SO}(1, n)^o$ we consider arithmetic subgroups of the orthogonal group $\mathrm{SO}(1, n)$ or, more precisely, of groups $\mathrm{G} = \mathrm{SO}(f)$ where $f$ is an admissible quadratic form defined over a totally real number field $k$ (see [1, Section 2.1]).

We have an exact sequence of $k$-isogenies:

$$1 \to \mathrm{C} \to \widetilde{\mathrm{G}} \xrightarrow{\phi} \mathrm{G} \to 1, \tag{1.1}$$

where $\widetilde{\mathrm{G}}(k) \simeq \mathrm{Spin}(f)$ is the simply connected cover of G and $\mathrm{C} \simeq \mu_2$ is the center of $\widetilde{\mathrm{G}}$. This induces an exact sequence in Galois cohomology (see [5, Section 2.2.3])

$$\widetilde{\mathrm{G}}(k) \xrightarrow{\phi} \mathrm{G}(k) \xrightarrow{\delta} \mathrm{H}^1(k, \mathrm{C}) \to \mathrm{H}^1(k, \widetilde{\mathrm{G}}). \tag{1.2}$$

The main idea of this note is that by using (1.2) certain questions about arithmetic subgroups of G can be reduced to questions about the Galois cohomology group $\mathrm{H}^1(k, \mathrm{C})$.

A coherent collection of parahoric subgroups $P = (P_v)_{v \in V_f}$ of $\widetilde{\mathrm{G}}$ ($V_f = V_f(k)$ denotes the set of finite places of the field $k$) defines a principal arithmetic subgroup

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$\Lambda = \widetilde{\mathrm{G}}(k) \cap \prod_{v\in V_f} P_v \subset \widetilde{\mathrm{G}}(k)$ (see [2]). We fix an infinite place $v$ of $k$ for which $\mathrm{G}(k_v) \simeq \mathrm{SO}(1,n)$ and denote it by $Id$. The image of $\Lambda$ under the central $k$-isogeny $\phi$ is an arithmetic subgroup of G and every maximal arithmetic subgroup of $\mathrm{G}(k_{Id})$ can be obtained as a normalizer of some $\phi(\Lambda)$ [2, Proposition 1.4]. We will also consider the local stabilizers of $P$ in the adjoint group $\mathrm{G}(= \overline{\mathrm{G}})$, defining $\overline{P}_v$ to be the stabilizer of $P_v$ in $\mathrm{G}(k_v)$ and $\overline{P} = (\overline{P}_v)_{v\in V_f}$. Clearly, $\overline{P}_v \supset \phi(P_v)$. In the notation of [1] the subgroups $\phi(P_v)$ are called parahoric subgroups of G, however this terminology is non-standard and we will avoid using it here.

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**1.2.** Given a totally real number field $k$ with the group of units $U$, let

$$k^*_\infty = \{a \in k^* \mid a_v > 0 \text{ for } v \in V_\infty \setminus Id\}, \quad U_\infty = U \cap k^*_\infty.$$

**Lemma 1.1.** $\mathrm{Im}(\delta) \simeq k^*_\infty/(k^*)^2$.

*Proof.* From (1.2) we have $\mathrm{Im}(\delta : \mathrm{G}(k) \to \mathrm{H}^1(k,\mu_2)) = \mathrm{Ker}(\mathrm{H}^1(k,\mu_2) \to \mathrm{H}^1(k,\widetilde{\mathrm{G}}))$. The Hasse principle for simply connected $k$-groups implies that $\mathrm{H}^1(k,\widetilde{\mathrm{G}})$ is isomorphic to $\prod_{v\in V_\infty} \mathrm{H}^1(k_v,\widetilde{\mathrm{G}})$ [5, Theorem 6.6], and hence

$$\mathrm{Im}(\delta) = \mathrm{Ker}(\mathrm{H}^1(k,\mu_2) \to \textstyle\prod_{v\in V_\infty} \mathrm{H}^1(k_v,\widetilde{\mathrm{G}})).$$

The group $\mathrm{H}^1(k,\mu_2)$ is canonically isomorphic to $k^*/(k^*)^2$ [5, Lemma 2.6]. It is well known that for all $v \in V_\infty$ such that the group $\mathrm{G}(k_v)$ is anisotropic, the map $\phi$ in (1.2) is surjective and hence for all such $v$, $\mathrm{Im}(\delta_v) = \mathrm{Ker}(\mathrm{H}^1(k_v,\mu_2) \to \mathrm{H}^1(k_v,\widetilde{\mathrm{G}}))$ is trivial. For the remaining one infinite place $v(= Id) \in V_\infty$, $\phi(\widetilde{\mathrm{G}}(k_v))$ is a subgroup of index 2 in $\mathrm{G}(k_v)$ which consists of the orthogonal transformations with the trivial spinor norm. Collecting this information together we obtain the required isomorphism. □

**1.3** The proof of the uniqueness part in [1, Theorem 4.1] contains errors but the result is correct. We will now give another argument for it. In order to do so we first establish a more general fact and then apply it to the cases considered in [1].

Let $P = (P_v)_{v\in V_f}$ and $P' = (P'_v)_{v\in V_f}$ be two coherent collections of parahoric subgroups of $\widetilde{\mathrm{G}}$ such that for all $v \in V_f$, $P'_v$ is conjugate to $P_v$ under an element of $\mathrm{G}(k_v)$. For all but finitely many $v$, $P_v = P'_v$ hence there is an element $g \in \mathrm{G}(\mathbb{A}_f)$ ($\mathbb{A}_f$ denotes the ring of finite adèles of $k$) such that $P'$ is the conjugate of $P$ under $g$. We have $\overline{P} = \prod_{v\in V_f} \overline{P}_v$ is the stabilizer of $P$ in $\mathrm{G}(\mathbb{A}_f)$. The number of distinct

G($k$)-conjugacy classes of coherent collections $P'$ as above is the cardinality $c(\overline{P})$ of $\mathcal{C}(\overline{P}) = \mathrm{G}(k)\backslash \mathrm{G}(\mathbb{A}_f)/\overline{P}$, which is called the class group of G relative to $\overline{P}$. The class number $c(\overline{P})$ is known to be finite (see *e.g.* [2, Proposition 3.9]). The following result can be used for obtaining further information about its value.

**Proposition 1.2.** *Let* $\mathrm{G} = \mathrm{SO}(f)$, $\widetilde{\mathrm{G}} = \mathrm{Spin}(f)$ *for an admissible quadratic form* $f$ *defined over* $k$ *and let* $P = (P_v)_{v \in V_f}$ *a coherent collection of parahoric subgroups of* $\widetilde{\mathrm{G}}$. *The class number* $c(P)$ *divides the order* $h_{\infty,2}$ *of a restricted* 2*-class group of* $k$ *given by*

$$h_{\infty,2} = \frac{2^{[k:\mathbb{Q}]-1} h_2}{[U : U_\infty]},$$

*where* $h_2$ *is the order of the* 2*-class group of* $k$.

*Proof.* Recall two isomorphisms (see [5, Proposition 8.8], a minor modification is needed in order to adjust the statement to our setting but the argument remains the same):

$$\mathrm{G}(k)\backslash \mathrm{G}(\mathbb{A}_f)/\overline{P} \simeq \mathrm{G}(\mathbb{A}_f)/\overline{P}\mathrm{G}(k);$$

$$\mathrm{G}(\mathbb{A}_f)/\overline{P}\mathrm{G}(k) \simeq \delta_{\mathbb{A}_f}(\mathrm{G}(\mathbb{A}_f))/\delta_{\mathbb{A}_f}(\overline{P}\mathrm{G}(k)),$$

where $\delta_{\mathbb{A}_f}$ is the restriction of the product map $\prod_v \mathrm{G}(k_v) \to \prod_v \mathrm{H}^1(k_v, \mathrm{C})$ to $\mathrm{G}(\mathbb{A}_f)$.

For every finite place $v$, $\mathrm{H}^1(k_v, \widetilde{\mathrm{G}})$ is trivial (see [5, Theorem 6.4]) which implies $\delta_v : \mathrm{G}(k_v) \to \mathrm{H}^1(k_v, \mathrm{C})$ is surjective. Thus the image of $\delta_{\mathbb{A}_f}(\mathrm{G}(\mathbb{A}_f))$ can be identified with the restricted direct product $\prod' \mathrm{H}^1(k_v, \mathrm{C})$ with respect to the subgroups $\delta_v(\overline{P}_v)$. Also $\delta_{\mathbb{A}_f}(\mathrm{G}(k))$ naturally identifies with the image of $\delta(\mathrm{G}(k))$ in $\mathrm{H}^1(k, \mathrm{C})$ under the embedding $\psi : \mathrm{H}^1(k, \mathrm{C}) \to \prod' \mathrm{H}^1(k_v, \mathrm{C})$. Hence we have an isomorphism

$$\delta_{\mathbb{A}_f}(\mathrm{G}(\mathbb{A}_f))/\delta_{\mathbb{A}_f}(\overline{P}\mathrm{G}(k)) \simeq {\prod}' \mathrm{H}^1(k_v, \mathrm{C})/\left(\prod_v \delta_v(\overline{P}_v) \cdot \psi(\mathrm{Im}\, \delta(\mathrm{G}(k)))\right).$$

The group $\mathrm{H}^1(k_v, \mu_2)$ is canonically isomorphic to $k_v^* / (k_v^*)^2$, by Lemma 1.1 $\mathrm{Im}\, \delta(\mathrm{G}(k)) \simeq k_\infty^*/(k^*)^2$, so we obtain

$$\frac{\prod' \mathrm{H}^1(k_v, \mathrm{C})}{\prod_v \delta_v(\overline{P}_v) \cdot \psi(\mathrm{Im}\, \delta(\mathrm{G}(k)))} \simeq \frac{\prod' k_v^*/(k_v^*)^2}{\delta_P \cdot k_\infty^*/(k^*)^2} \simeq \frac{J_f}{\delta_P \cdot J_f^2 k^*} \cdot \frac{k^*}{k_\infty^*},$$

where $J_f$ is the ring of finite idèles of $k$ and $\delta_P$ denotes $\prod_v \delta_v(\overline{P}_v)$.

Now, $\#(J_f/J_f^2 k^*) = h_2$, the group $k^*/k_\infty^*$ splits as a product of local factors and $\#(k^*/k_\infty^*) = 2^{[k:\mathbb{Q}]-1}/[U : U_\infty]$ (see [4, Chapter 6]). This implies the proposition. □

In order to give a precise formula for the class number $c(P)$ one has to analyze the image of $\prod_v \delta_v(\overline{P}_v)$ in $\prod' \mathrm{H}^1(k_v, \mathrm{C})$. Still in many practical cases this appears to be unnecessary. Thus in order to prove the uniqueness of the minimal hyperbolic orbifolds we need to consider $k = \mathbb{Q}[\sqrt{5}]$ (in the compact case) and $k = \mathbb{Q}$ (for the non-compact orbifolds). In both cases $h_2 = h = 1$. For $k = \mathbb{Q}[\sqrt{5}]$, $U_\infty = \{1, \frac{1-\sqrt{5}}{2}\}$ and thus $[U : U_\infty] = 2$ which implies $h_{\infty,2} = 1$. For $k = \mathbb{Q}$, clearly, $h_{\infty,2} = 1$ as well. So in all the cases $c(P) = 1$ which implies that the corresponding arithmetic subgroups are defined uniquely up to a conjugation by $g \in \mathrm{SO}(1,n)$. It is clear that we can always chose $g \in \mathrm{SO}_0(1,n)$ and therefore the smallest orbifolds constructed in [1] are unique up to an (orientation preserving) isometry.

**1.4.** We now turn to Conjecture 4.1 and its analogue for the non-cocompact orbifolds in [1, Section 4.4]. Recall that in [1] the numbers $N(r)$, $N'(r)$ were defined for every $r \geq 2$ and estimated from above. These numbers are related to the index of the principal arithmetic subgroups in their normalizers. We now prove

**Proposition 1.3.** *For every $r \geq 2$, $N(r) = N'(r) = 1$.*

*Proof.* Let $\Lambda$ be a principal arithmetic subgroup of $\widetilde{\mathrm{G}}$ which corresponds to a compact or non-compact hyperbolic $n$-orbifold of the minimal volume, $\Lambda' = \phi(\Lambda)$ and $\Gamma = N_{\mathrm{G}}(\Lambda')$.

From [2, Proposition 2.9], which in turn follows from the work of J. Rohlfs, using the fact that the center of our group G is trivial, we obtain:

$$[\Gamma : \Lambda'] = \#(\mathrm{H}^1(k, \mu_2)_\Theta \cap \delta(\mathrm{G}(k))) = \#\mathrm{Im}(\delta : \mathrm{G}(k) \to \mathrm{H}^1(k, \mu_2)).$$

By Lemma 1.1 we can identify the image of $\delta$. The cases we are interested in are

$$k = \mathbb{Q} : \ \mathrm{Im}(\delta) = \left\{k^{*2}, (-1)k^{*2}\right\};$$
$$k = \mathbb{Q}[\sqrt{5}] : \ \mathrm{Im}(\delta) = \left\{k^{*2}, \frac{1-\sqrt{5}}{2}k^{*2}\right\}.$$

In both cases $[\Gamma : \Lambda'] = \#\mathrm{Im}(\delta) = 2$. Now it is easy to see that $\Lambda' = \phi(\Lambda) \subset \mathrm{SO}_0(1,n)$. From the other side there always exists $g \in \mathrm{SO}(1,n) \setminus \mathrm{SO}_0(1,n)$ which normalizes $\phi(\Lambda)$. For example take $g = \mathrm{diag}(-1,-1,1,\ldots,1)$. As in all the cases under consideration the quadratic form associated to $\Lambda$ is diagonal [1, Sections 4.3, 4.4], $g$ stabilizes $\Lambda$ and clearly $g \in \mathrm{SO}(1,n) \setminus \mathrm{SO}_0(1,n)$. From these facts it follows that $\Lambda'$ is a maximal arithmetic subgroup in $\mathrm{SO}_0(1,n)$ and thus $N(r)$ (or $N'(r)$)$=1$. □

This proposition makes *precise* the statements of Theorem 4.1 and 4.4 of [1]. It also implies that Table 2 of *loc. cit.* gives the precise values of the covolumes of the smallest $n$-dimensional hyperbolic orbifolds in even dimensions up to 18.

One other corollary is that cocompact and non-cocompact arithmetic subgroups of $\mathrm{SO}(1, 2r)^o$ of the smallest covolumes can be obtained as the stabilizers of certain lattices described in [1, Section 4.3]. We remark that since the fields of definition of the groups have class number 1, the lattices in both cases are free as $\mathcal{O}_k$-modules.

**1.5.** Correction: on p. 765, l. 9 one should read "grow super-exponentially" instead of "grow exponentially". (It follows from [1] that the Euler characteristic is bounded from below by const $\cdot \, (\prod_{i=1}^{r} \frac{(2i-1)!}{(2\pi)^{2i}})^{[k:\mathbb{Q}]}$ which for large enough $r$ is $\geq$ const $\cdot \, (2r-1)!$)

We conclude this addendum with a few remarks on related results which appeared after the paper was published.

**1.6.** In [1, Section 4.5] we observed that for $r > 2$ the minimal covolume among the arithmetic lattices in $\mathrm{SO}(1, 2r)$ is attained on a non-uniform lattice. This interesting phenomenon was first discovered by A. Lubotzky for $\mathrm{SL}_2$ over local fields of positive characteristic. Recently, in [6] A. Salehi Golsefidy proved that lattices of minimal covolume in classical Chevalley groups over local fields of characteristic $p > 7$ are all non-uniform. This result gives further support to a **conjecture** that *generically* (*i.e. for groups of high enough rank or fields of large enough positive characteristic*) *the minimal covolume is always attained on a non-uniform lattice.*

**1.7.** In [3] M. Conder and C. Maclachlan constructed a compact orientable hyperbolic 4-manifold which has Euler characteristic 16. The previously known smallest example which was used in order to formulate the main result in [1, Section 5] had $\chi = 26$. The construction of [3] agrees with our Theorem 5.5 and it also allows us to give a more precise formulation of the theorem:

**Theorem 5.5′.** *If there exists a compact orientable arithmetic hyperbolic* 4*-manifold* $M$ *with* $\chi(M) \leq 16$*, then* $M$ *is defined over* $\mathbb{Q}[\sqrt{5}]$ *and has the form* $\Gamma_M \backslash \mathcal{H}^4$ *with* $\Gamma_M$ *being a torsion-free subgroup of index* $7200\chi(M)$ *of the group* $\Gamma_1$ *of the smallest arithmetic hyperbolic* 4*-orbifold.*

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