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On surfaces with $p_g = q = 1$ and non-ruled bicanonical involution

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Abstract. This paper classifies surfaces S of general type with $p_g = q = 1$ having an involution i such that S/i has non-negative Kodaira dimension and that the bicanonical map of S factors through the double cover induced by i.

It is shown that S/i is regular and either: a) the Albanese fibration of S is of genus 2 or b) S has no genus 2 fibration and S/i is birational to a $K3$ surface. For case a) a list of possibilities and examples are given. An example for case b) with $K^2 = 6$ is also constructed.

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1. Introduction

Let S be a smooth irreducible projective surface of general type. The *pluricanonical map* ϕ_n of S is the map given by the linear system $|nK_S|$, where K_S is the canonical divisor of S. For minimal surfaces S, ϕ_n is a birational morphism if $n \geq 5$ (*cf.* [4, Chapter VII, Theorem (5.2)]). The *bicanonical map*

$$\phi_2 : S \longrightarrow \mathbb{P}^{K_S^2 + \chi(S) - 1}$$

is a morphism if $p_g(S) \geq 1$ (this result is due to various authors, see [7] for more details). This paper focuses on the study of surfaces S with $p_g(S) = q(S) = 1$ having an involution i such that the Kodaira dimension of S/i is non-negative and ϕ_2 is composed with i, *i.e.* it factors through the double cover $p : S \to S/i$.

There is an instance where the bicanonical map is necessarily composed with an involution: suppose that S has a fibration of genus 2, *i.e.* it has a morphism f from S to a curve such that a general fibre F of f is irreducible of genus 2. The

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system $|2K_S|$ cuts out on F a subseries of the bicanonical series of F, which is composed with the hyperelliptic involution of F, and then ϕ_2 is composed with an involution. This is the so called *standard case* of non-birationality of the bicanonical map.

By the results of Bombieri, [2], improved later by Reider, [22], a minimal surface S satisfying $K^2 > 9$ and ϕ_2 non-birational necessarily presents the standard case of non-birationality of the bicanonical map.

The non-standard case of non-birationality of the bicanonical map, *i.e.* the case where ϕ_2 is non-birational and the surface has no genus 2 fibration, has been studied by several authors.

Du Val, [16], classified the regular surfaces S of general type with $p_g \geq 3$, whose general canonical curve is smooth and hyperelliptic. Of course, for these surfaces, the bicanonical map is composed with an involution i such that S/i is rational. The families of surfaces exhibited by Du Val, presenting the non-standard case, are nowadays called the *Du Val examples*.

Other authors have later studied the non-standard case: the articles [8, 10, 12, 13, 25] and [3] treat the cases $\chi(\mathcal{O}_S) > 1$ or $q(S) \geq 2$ (*cf.* the expository paper [11] for more information on this problem).

Xiao Gang, [25], presented a list of possibilities for the non-standard case of non-birationality of the bicanonical morphism ϕ_2. For the case when ϕ_2 has degree 2 and the bicanonical image is a ruled surface, Theorem 2 of [25] extended Du Val's list to $p_g(S) \geq 1$ and added two extra families (this result is still valid assuming only that ϕ_2 is composed with an involution such that the quotient surface is a ruled surface). Recently G. Borrelli [3] excluded these two families, confirming that the only possibilities for this instance are the Du Val examples.

For irregular surfaces the following holds (see [25, Theorems 1, 3], [8, Theorem A], [12, Theorem 1.1], [13]):

Suppose that S is a smooth minimal irregular surface of general type having non-birational bicanonical map. If $p_g(S) \geq 2$ and S has no genus 2 fibration, then only the following (effective) possibilities occur:

- $p_g(S) = q(S) = 2,\ K_S^2 = 4;$
- $p_g(S) = q(S) = 3,\ K_S^2 = 6.$

In both cases ϕ_2 is composed with an involution i such that $\mathrm{Kod}(S/i) = 2$.

This paper completes this result classifying the minimal surfaces S with $p_g(S) = q(S) = 1$ such that ϕ_2 is composed with an involution i satisfying $\mathrm{Kod}(S/i) \geq 0$.

The main result is the following:

Theorem 1.1. *Let S be a smooth minimal irregular surface of general type with an involution i such that* $\mathrm{Kod}(S/i) \geq 0$ *and the bicanonical map ϕ_2 of S is composed with i. If $p_g(S) = q(S) = 1$, then only the following possibilities can occur*:

a) *S/i is regular, the Albanese fibration of S has genus* 2 *and*

(i) $\mathrm{Kod}(S/i) = 2$, $\chi(S/i) = 2$, $K_S^2 = 2$, $\deg(\phi_2) = 8$, *or*

(ii) $\mathrm{Kod}(S/i) = 1$, $\chi(S/i) = 2$, $2 \leq K_S^2 \leq 4$, $\deg(\phi_2) \geq 4$, *or*

(iii) *S/i is birational to a $K3$ surface,* $3 \leq K_S^2 \leq 6$, $\deg(\phi_2) = 4$;

b) *S has no genus 2 fibration and S/i is birational to a $K3$ surface.*

Moreover, there are examples for (i)*,* (ii) *with* $K_S^2 = 4$, (iii) *with* $K_S^2 = 3, 4$ *or 5 and for* b) *with* $K_S^2 = 6$ *and* ϕ_2 *of degree 2.*

Remark 1.2. Examples for (iii) were given by Catanese in [6]. The other examples will be presented in Section 5.

Note that surfaces of general type with $p_g = q = 1$ and $K^2 = 3$ or 8 were also studied by Polizzi in [19] and [20].

In the example in Section 5 for case b) of Theorem 1.1, S has $p_g = q = 1$ and $K^2 = 6$. This seems to be the first construction of a surface with these invariants. This example contradicts a result of Xiao Gang. More precisely, the list of possibilities in [25] rules out the case where S has no genus 2 fibration, $p_g(S) = q(S) = 1$ and S/i is birational to a $K3$ surface. In Lemma 7 of [25] it is written that R has only negligible singularities, but the possibility $\chi(K_{\widetilde{P}} + \widetilde{\delta}) < 0$ in formula (3) of page 727 was overlooked. In fact we will see that R ($\overline{B}$ in our notation) can have a non-negligible singularity.

An important technical tool that will be used several times is the *canonical resolution* of singularities of a surface. This is a resolution of singularities as described in [4].

The paper is organized as follows. Section 2 studies some general properties of a surface of general type S with an involution i. Section 3 states some properties of surfaces with $p_g = q = 1$. Section 4 contains the proof of Theorem 1.1. Crucial ingredients for this proof are the existence of the Albanese fibration of S and the formulas of Section 2. In Section 5 examples for Theorem 1.1 are obtained, via the construction of branch curves with appropriate singularities. The Computational Algebra System *Magma* is used to perform the necessary calculations (visit http://magma.maths.usyd.edu.au/magma for more information about Magma).

Notation and conventions. We work over the complex numbers; all varieties are assumed to be projective algebraic. We do not distinguish between line bundles and divisors on a smooth variety. Linear equivalence is denoted by $\equiv$. A *nodal curve* or (-2)-*curve* C on a surface is a curve isomorphic to $\mathbb{P}^1$ such that $C^2 = -2$. Given a surface X, $\mathrm{Kod}(X)$ means the *Kodaira dimension* of X. We say that a curve singularity is *negligible* if it is either a double point or a triple point which resolves to at most a double point after one blow-up. A (n, n) *point*, or *point of type* (n, n), is a point of multiplicity n with an infinitely near point also of multiplicity n. An *involution* of a surface S is an automorphism of S of order 2. We say that a map is composed with an involution i of S if it factors through the map $S \to S/i$. The rest of the notation is standard in Algebraic Geometry.

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2. Generalities on involutions

Let S be a smooth minimal surface of general type with an involution i. As S is minimal of general type, this involution is biregular. The fixed locus of i is the union of a smooth curve R'' (possibly empty) and of $t \geq 0$ isolated points $P_1, \ldots, P_t$. Let S/i be the quotient of S by i and $p : S \to S/i$ be the projection onto the quotient. The surface S/i has nodes at the points $Q_i := p(P_i)$, $i = 1, \ldots, t$, and is smooth elsewhere. If $R'' \neq \emptyset$, the image via p of R'' is a smooth curve B'' not containing the singular points Q_i, $i = 1, \ldots, t$. Let now $h : V \to S$ be the blow-up of S at $P_1, \ldots, P_t$ and set $R' = h^*R''$. The involution i induces a biregular involution $\widetilde{i}$ on V whose fixed locus is $R := R' + \sum_1^t h^{-1}(P_i)$. The quotient $W := V/\widetilde{i}$ is smooth and one has a commutative diagram:

$$\begin{array}{ccc} V & \xrightarrow{h} & S \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle p} \\ W & \xrightarrow{g} & S/i \end{array}$$

where $\pi : V \to W$ is the projection onto the quotient and $g : W \to S/i$ is the minimal desingularization map. Notice that

$$A_i := g^{-1}(Q_i), \quad i = 1, \ldots, t,$$

are (-2)-curves and $\pi^*(A_i) = 2 \cdot h^{-1}(P_i)$. Set $B' := g^*(B'')$. Because π is a double cover with branch locus $B' + \sum_1^t A_i$, there exists a line bundle L on W such that

$$2L \equiv B := B' + \sum_1^t A_i.$$

It is well known that (*cf.* [4, Chapter V, Section 22]):

$$p_g(S) = p_g(V) = p_g(W) + h^0(W, \mathcal{O}_W(K_W + L)),$$

$$q(S) = q(V) = q(W) + h^1(W, \mathcal{O}_W(K_W + L))$$

and

$$\begin{aligned} K_S^2 - t = K_V^2 &= 2(K_W + L)^2, \\ \chi(\mathcal{O}_S) = \chi(\mathcal{O}_V) &= 2\chi(\mathcal{O}_W) + \tfrac{1}{2} L(K_W + L). \end{aligned} \tag{2.1}$$

Furthermore, from the papers [12] and [9], if S is a smooth minimal surface of general type with an involution i, then

$$\chi(\mathcal{O}_W(2K_W + L)) = h^0(W, \mathcal{O}_W(2K_W + L)), \tag{2.2}$$

$$\chi(\mathcal{O}_W) - \chi(\mathcal{O}_S) = K_W(K_W + L) - h^0(W, \mathcal{O}_W(2K_W + L)) \tag{2.3}$$

and the bicanonical map

$$\phi_2 \text{ is composed with } i \text{ if and only if } h^0(W, \mathcal{O}_W(2K_W + L)) = 0. \tag{2.4}$$

From formulas (2.1) and (2.3) one obtains the number t of nodes of S/i :

$$t = K_S^2 + 6\chi(\mathcal{O}_W) - 2\chi(\mathcal{O}_S) - 2h^0(W, \mathcal{O}_W(2K_W + L)). \tag{2.5}$$

Let P be a minimal model of the resolution W of S/i and $\rho : W \to P$ be the natural projection. Denote by $\overline{B}$ the projection $\rho(B)$ and by δ the "projection" of L.

Remark 2.1. Resolving the singularities of $\overline{B}$ we obtain exceptional divisors E_i and numbers $r_i \in 2\mathbb{N}^+$ such that $E_i^2 = -1$, $K_W = \rho^*(K_P) + \sum E_i$ and $B = \rho^*(\overline{B}) - \sum r_i E_i$.

Proposition 2.2. *With the previous notations, the bicanonical map ϕ_2 is composed with i if and only if*

$$\chi(\mathcal{O}_P) - \chi(\mathcal{O}_S) = K_P(K_P + \delta) + \frac{1}{2}\sum(r_i - 2).$$

Proof. From formulas (2.3), (2.4) and Remark 2.1 we get

$$\begin{aligned}\chi(\mathcal{O}_P) - \chi(\mathcal{O}_S) &= \frac{1}{2}K_W(2K_W + 2L)\\ &= \frac{1}{2}\left(\rho^*(K_P) + \sum E_i\right)\left(2\rho^*(K_P + \delta) + \sum(2 - r_i)E_i\right)\\ &= K_P(K_P + \delta) + \frac{1}{2}\sum(r_i - 2).\end{aligned}$$ □

3. Surfaces with $p_g = q = 1$ and an involution

Let S be a minimal smooth projective surface of general type satisfying $p_g(S) = q(S) = 1$.

Note that then $2 \le K_S^2 \le 9$: we have $K_S^2 \le 9\chi(\mathcal{O}_S)$ by the Myiaoka-Yau inequality (see [4, Chapter VII, Theorem (4.1)]) and $K_S^2 \ge 2p_g$ for an irregular surface (see [14]).

Furthermore, if the bicanonical map of S is not birational, then $K_S^2 \neq 9$. In fact, by [12], if $K_S^2 = 9$ and ϕ_2 is not birational, then S has a genus 2 fibration, while Théorème 2.2 of [24] implies that if S has a genus 2 fibration and $p_g(S) = q(S) = 1$, then $K_S^2 \leq 6$.

Since $q(S) = 1$ the Albanese variety of S is an elliptic curve E and the Albanese map is a connected fibration (see e. g. [1] or [4]).

Suppose that S has an involution i. Then i preserves the Albanese fibration (because $q(S) = 1$) and so we have a commutative diagram

$$\begin{array}{ccccc} V & \xrightarrow{h} & S & \longrightarrow & E \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle p} & & \downarrow \\ W & \longrightarrow & S/i & \longrightarrow & \Delta \end{array} \tag{3.1}$$

where Δ is a curve of genus ≤ 1. Denote by

$$f_A : W \to \Delta$$

the fibration induced by the Albanese fibration of S.

Recall that

$$\rho : W \to P$$

is the projection of W onto its minimal model P and

$$\overline{B} := \rho(B),$$

where $B := B' + \sum_1^t A_i \subset W$ is the branch locus of π.

Let

$$\overline{B'} := \rho(B'), \quad \overline{A_i} = \rho(A_i).$$

When $\overline{B}$ has only negligible singularities, the map ρ contracts only exceptional curves contained in fibres of f_A. In fact, there exists otherwise a (-1)-curve $J \subset W$ such that $JB = 2$ and so $\pi^*(J)$ is a rational curve transverse to the fibres of the (genus 1 base) Albanese fibration of S, which is impossible. Moreover, ρ contracts no curve meeting $\sum A_i$, because $h : V \to S$ is the contraction of isolated (-1)-curves. Therefore the singularities of $\overline{B}$ are exactly the singularities of $\overline{B'}$, *i.e.* $\overline{B'} \bigcap \sum \overline{A_i} = \emptyset$. In this case the image of f_A on P will be denoted by $\overline{f_A}$.

If $\Delta \cong \mathbb{P}^1$, then the double cover $E \to \Delta$ is ramified over 4 points p_j of Δ, thus the branch locus $B' + \sum_1^t A_i$ is contained in 4 fibres

$$F_A^j := f_A^*(p_j), \ j = 1, ..., 4,$$

of the fibration f_A. Hence, by Zariski's Lemma (see e. g. [4]), the irreducible components B_i' of B' satisfy $B_i'^2 \leq 0$. If $\overline{B}$ has only negligible singularities, then also $\overline{B'}^2 \leq 0$. As $\pi^*(F_A^j)$ is of multiplicity 2, each component of F_A^j which is not a component of the branch locus $B' + \sum_1^t A_i$ must be of even multiplicity.

4. The classification theorem

In this section we will prove Theorem 1.1. We will freely use the notation and results of Sections 2 and 3.

Proof of Theorem 1.1. Since $p_g(P) \leq p_g(S) = 1$, then $\chi(\mathcal{O}_P) \leq 2 - q(P) \leq 2$. Proposition 2.2 gives $\chi(\mathcal{O}_P) \geq 1$, because K_P is nef (*i.e.* $K_P C \geq 0$ for every curve C). So from Proposition 2.2 and the classification of surfaces (see e. g. [1] or [4]) only the following cases can occur:

1. P is of general type;
2. P is a surface with Kodaira dimension 1;
3. P is an Enriques surface, $\overline{B}$ has only negligible singularities;
4. P is a K3 surface, $\overline{B}$ has a 4-uple or (3, 3) point, and possibly negligible singularities.

We will show that: case 3 does not occur, in cases 1 and 2 the Albanese fibration has genus 2 and only in case 4 the Albanese fibration can have genus $\neq 2$.

Each of cases 1,..., 4 will be studied separately. We start by considering:

Case 1. As P is of general type, $K_P^2 \geq 1$ and K_P is nef, Proposition 2.2 gives $\chi(\mathcal{O}_P) = 2$, $K_P^2 = 1$, $K_P\delta = 0$ and $\overline{B}$ has only negligible singularities. The equality $K_P\overline{B'} = K_P 2\delta = 0$ implies $\overline{B'}^2 < 0$ when $B' \neq 0$. In the notation of Remark 2.1 one has $K_W \equiv \rho^*(K_P) + \sum E_i$ and $B' = \rho^*(\overline{B'}) - 2\sum E_i$. So

$$K_S^2 = K_V^2 + t = \frac{1}{4}(2K_V)^2 + t = \frac{1}{4}\pi^*(2K_W + B)^2 + t$$
$$= \frac{1}{2}(2K_W + B)^2 + t = \frac{1}{2}(2K_W + B')^2 = \frac{1}{2}(2K_P + \overline{B'})^2 = \frac{1}{2}(4 + \overline{B'}^2).$$

Since $K_S^2 \geq 2p_g(S)$ for an irregular surface (see [14]), $\overline{B'}^2 < 0$ is impossible, hence $B' = 0$ and $K_S^2 = 2$. By [5] minimal surfaces of general type with $p_g = q = 1$ and $K^2 = 2$ have Albanese fibration of genus 2. This is case (i) of Theorem 1.1. We will see in Section 5 an example for this case.

Finally the fact that $\deg(\phi_2) = 8$ follows immediately because ϕ_2 is a morphism onto $\mathbb{P}^2$ and $(2K_S)^2 = 8$.

Next we exclude:

Case 3. Using the notation of Remark 2.1 of Section 2, we can write $K_W \equiv \rho^*(K_P) + \sum E_i$ and $2L \equiv \rho^*(2\delta) - 2\sum E_i$, for some exceptional divisors E_i. Hence

$$L(K_W + L) = \frac{1}{2}L(2K_W + 2L)$$
$$= \frac{1}{2}(\rho^*(\delta) - \sum E_i)(2\rho^*(K_P) + \rho^*(2\delta)) = \frac{1}{2}\delta(2K_P + 2\delta) = \delta^2$$

and then, from (2.1), $\delta^2 = -2$. Now (2.4) and (2.5) imply $t = K_S^2 + 4$, thus

$$\overline{B'}^2 = \overline{B}^2 + 2t = (2\delta)^2 + 2t = -8 + 2t = 2K_S^2 > 0.$$

This is a contradiction because we have seen that $\overline{B'}^2 \leq 0$ when $\overline{B'}$ has only negligible singularities. Thus case 3) does not occur.

Now we focus on:

Case 2. Since we are assuming that $\mathrm{Kod}(P) = 1$, P has an elliptic fibration (*i.e.* a morphism $f_e : P \to C$ where C is a curve and the general fibre of f_e is a smooth connected elliptic curve). Then K_P is numerically equivalent to a rational multiple of a fibre of f_e (see e. g. [1] or [4]). As $K_P\delta \geq 0$, Proposition 2.2, together with $\chi(\mathcal{O}_P) \leq 2$, yield $K_P\delta = 0$ or 1.

Denote by F_e (respectively F_A) a general fibre of f_e (respectively f_A) and let $\overline{F_A} := \rho(F_A)$. If $K_P\delta = 0$, then $F_e\overline{B} = 0$, which implies that the fibration f_e lifts to an elliptic fibration on S. This is impossible because S is a surface of general type. So $K_P\delta = 1$ and, since $p_g(P) \leq p_g(S) = 1$, the only possibility allowed by Proposition 2.2 is

$p_g(P) = 1$, $q(P) = 0$ and $\overline{B}$ has only negligible singularities.

Now $q(P) = 0$ implies that the elliptic fibration f_e has a rational base, thus the canonical bundle formula (see e. g. [4, Chapter V, Section 12]) gives $K_P \equiv \sum(m_i - 1)F_i$, where m_iF_i are the multiple fibres of f_e. From

$$2 = 2\delta K_P = \overline{B'}K_P = \overline{B'}\sum(m_i - 1)F_i, \qquad \overline{B'}F_i \equiv 0 \pmod 2$$

we get

$$K_P \equiv \frac{1}{2}F_e.$$

Since $\overline{B}$ has only negligible singularities, $\overline{B'}^2 \leq 0$ and then

$$2K_S^2 = (2K_W + B')^2 = \rho^*\left(2K_P + \overline{B'}\right)^2 = 8 + \overline{B'}^2 \leq 8. \tag{4.1}$$

Therefore $2 \leq K_S^2 \leq 4$. If $K_S^2 = 2$, then the Albanese fibration of S is of genus 2, by [5]. So, to prove statement a), (ii) of Theorem 1.1, we must show that for $K_S^2 = 3$ or 4 the Albanese fibration of S has genus 2. We will study each of these cases separately.

First we consider

• $\mathbf{K_S^2 = 4}$

Let $\overline{F_A^i} := \rho(F_A^i)$, $i = 1, \ldots, 4$.

Claim 4.1. If f_A is not a genus 2 fibration then

$$\overline{F_A^j} = 2\overline{B'},$$

for some $j \in \{1, \ldots, 4\}$.

Proof. By formula (4.1) $\overline{B'}^2 = 0$, and so $\overline{B'}$ contains the support of $x \geq 1$ of the $\overline{F_A^i}$'s. The facts $K_P\overline{F_A} > 0$ (because $g(\overline{F_A}) \geq 2$) and $K_P\overline{B'} = 2$ imply $x = 1$, *i.e.* $\overline{F_A^j} = k\overline{B'}$, for some $j \in \{1, \ldots, 4\}$ and $k \in \mathbb{N}^+$. If $k = 1$ then $\overline{F_A}K_P = 2$, thus $\overline{F_A}$ is of genus 2 and S is as in case (ii) of Theorem 1.1.

Suppose now $k \geq 2$. Then each irreducible component of the divisor

$$D := \overline{F_A^1} + \ldots + \overline{F_A^4}$$

whose support is not in $\sum_1^{14}\overline{A_i}$ is of multiplicity greater than 1. The fibration $\overline{f_A}$ gives a cover $F_e \to \mathbb{P}^1$ of degree $\overline{F_A}F_e$, for a general fibre F_e of the elliptic fibration f_e. The Hurwitz formula (see e. g. [17]) says that the ramification degree r of this cover is $2\overline{F_A}F_e$. Let $p_1, \ldots, p_n$ be the points in $F_e \cap D$ and α_i be the intersection number of F_e and D at p_i. Of course $F_eD = 4\overline{F_A}F_e = \sum_1^n \alpha_i$ and then $F_e \bigcap \sum \overline{A_i} = \emptyset$ implies $\alpha_i \geq 2$, $i = 1, \ldots, n$. We have

$$2\overline{F_A}F_e = r \geq \sum_1^n(\alpha_i - 1) = \sum_1^n \alpha_i - n = 4\overline{F_A}F_e - n,$$

i.e. $n \geq 2\overline{F_A}F_e$. The only possibility is $n = 2\overline{F_A}F_e$ and $\alpha_i = 2\ \forall i$, which means that every component Γ of D such that $\Gamma F_e \neq 0$ is exactly of multiplicity 2. In particular an irreducible component of $\overline{B'}$ is of multiplicity 2, thus $k = 2$, *i.e.* $\overline{F_A^j} = 2\overline{B'}$. □

Claim 4.2. There is a smooth rational curve C contained in a fibre F_C of the elliptic fibration f_e, and not contained in fibres of $\overline{f_A}$, such that

$$m := \widehat{C}\sum_1^t A_i \leq 3, \tag{4.2}$$

where $\widehat{C}$ is the strict transform of C in W.

Proof. Since $\overline{A_i}F_e = \overline{A_i}2K_P = 0$, then each $\overline{A_i}$ is contained in a fibre of f_e, and in particular the elliptic fibration f_e has reducible fibres. Denote by C an irreducible component of a reducible fibre F_C of f_e, by ξ the multiplicity of C in F_C and by $\widehat{C}$ the strict transform of C in W. If the intersection number of C and the support of $F_C - \xi C$ is greater than 3 then, from the configurations of singular fibres of an elliptic fibration (see e. g. [4, Chapter V, Section 7]), F_C must be of type I_0^*, *i.e.* it has the following configuration: it is the union of four disjoint (-2)-curves $\theta_1, \dots, \theta_4$ with a (-2)-curve θ, with multiplicity 2, such that $\theta\theta_i = 1$, $i = 1, \dots, 4$.

So if $\widehat{C}\sum_1^t A_i > 3$, the fibre F_C containing C is of type I_0^* with $\widehat{C}\sum_1^t A_i = 4$. Since the number of nodes of S/i is $t = K_S^2 + 10 = 14 \not\equiv 0 \pmod 4$, there must be a reducible fibre such that for every component $C \not\subset \sum_1^t \overline{A_i}$, $\widehat{C}\sum_1^t A_i \leq 3$. As $f_e \neq \overline{f_A}$ and the $\overline{A_i}$'s are contained in fibres of f_e and in fibres of $\overline{f_A}$, we can choose C not contained in fibres of $\overline{f_A}$. □

Let C be as in Claim 4.2 and consider the resolution $\widetilde{V} \to V$ of the singularities of $\pi^*(\widehat{C})$. Let $G \subset \widetilde{V}$ be the strict transform of $\pi^*(\widehat{C})$. Notice that G has multiplicity 1, because C transverse to the fibres of $\overline{f_A}$ implies $C \not\subset \overline{B}$. Recall that E denotes the basis of the Albanese fibration of S.

Claim 4.3. The Albanese fibration of $\widetilde{V}$ induces a cover $G \to E$ with ramification degree

$$r := K_{\widetilde{V}}G + G^2.$$

Proof. Let $G_1, \dots, G_h$ be the connected (hence smooth) components of G. The curve C is not contained in fibres of $\overline{f_A}$, thus G is not contained in fibres of the Albanese fibration of $\widetilde{V}$. This fibration induces a cover $G_i \to E$ with ramification degree, from the Hurwitz formula,

$$r_i = 2g(G_i) - 2 = K_{\widetilde{V}}G_i + G_i^2.$$

This way we have a cover $G \to E$ with ramification degree

$$r = \sum r_i = K_{\widetilde{V}}(G_1 + \cdots + G_h) + \left(G_1^2 + \cdots + G_h^2\right) = K_{\widetilde{V}}G + G^2. \qquad \square$$

We are finally in position to show that $g(F_A) = 2$.

Let $n := \widehat{C}B'$. We have

$$\begin{aligned} 2K_V\pi^*(\widehat{C}) &= \pi^*(2K_W + B' + \sum A_i)\pi^*(\widehat{C}) \\ &= 2(2K_W + B' + \sum A_i)\widehat{C} = 4K_W\widehat{C} + 2(B' + \sum A_i)\widehat{C} \\ &= 4(-2 - \widehat{C}^2) + 2(n+m) = -8 - 2\pi^*(\widehat{C})^2 + 2(n+m), \end{aligned}$$

i.e.

$$K_V\pi^*(\widehat{C}) + \pi^*(\widehat{C})^2 = n + m - 4.$$

Suppose that $g(F_A) \neq 2$. Let $\Lambda \subset V$ be the double Albanese fibre induced by $\overline{F_A^j} = 2\overline{B'}$ (as in Claim 4.1) and $\widetilde{\Lambda} \subset \widetilde{V}$ be the total transform of Λ. From

$$G\widetilde{\Lambda} = \pi^*(\widehat{C})\Lambda \geq \pi^*(\widehat{C})\pi^*(B') = 2n$$

one has $r \geq n$. Then

$$n + m - 4 = K_V \pi^*(\widehat{C}) + \pi^*(\widehat{C})^2 \geq K_{\widetilde{V}} G + G^2 = r \geq n$$

and so $m \geq 4$, which contradicts Claim 4.2.

So if $K_S^2 = 4$, then the Albanese fibration of S is of genus 2.

We will now consider the possibility

• $\mathbf{K_S^2 = 3}$

In this case a general Albanese fibre Λ has genus 2 or 3 (see [7]). Suppose then $g(\Lambda) = 3$. Surfaces S with $K_S^2 = g(\Lambda) = 3$ are studied in detail in [7]. There (see also [18]) it is shown that the relative canonical map γ, given by $|K_S + n\Lambda|$ for some n, is a morphism.

We know that $K_P \overline{B'} = 2$ and $\overline{B'}^2 = -2$, by (4.1). We have already seen that $\overline{B}$ has only negligible singularities (which means $r_i = 2\ \forall i$, in the notation of Remark 2.1) and then ρ contracts no curve meeting $\sum A_i$. Let R' be the support of $\pi^*(B')$.

Claim 4.4. We have

$$K_V R' = 1.$$

Proof.

$$\begin{aligned} 2K_V \cdot 2R' &= \pi^*(2K_W + B)\pi^*(B') = 2(2K_W + B)B' \\ &= 2(2K_W + B')B' = 2\left(2\rho^*(K_P) + \rho^*(\overline{B'})\right)\left(\rho^*(\overline{B'}) - \sum 2E_i\right) \\ &= 2(2K_P + \overline{B'})\overline{B'} = 2(4-2) = 4, \end{aligned}$$

thus $K_V R' = 1$. □

As the map

$$\gamma \circ h : V \longrightarrow \gamma(S)$$

is a birational morphism, $\gamma \circ h(R')$ is a line (plus possibly some isolated points). This way there exists a smooth rational curve $\beta \subset B'$ such that

$$K_V \widetilde{\beta} = 1,$$

where $\widetilde{\beta} \subset R'$ is the support of $\pi^*(\beta)$. The adjunction formula gives $\widetilde{\beta}^2 = -3$, thus $\beta^2 = -6$. Notice that $\widetilde{\beta}$ is the only component of R' which is not contracted by the map $\gamma \circ h$.

Let

$$\alpha := B' - \beta \subset W,$$

$$\overline{\beta} := \rho(\beta),\ \overline{\alpha} := \rho(\alpha) \subset P.$$

When α is non-empty, the support of $\pi^*(\alpha)$ is an union of (-2)-curves, since it is contracted by $\gamma \circ h$. Equivalently α is a disjoint union of (-4)-curves.

Claim 4.5. We have
$$K_W^2 \geq -2.$$

Proof. Consider the Chern number c_2 and the second Betti number b_2. It is well known that, for a surface X,
$$c_2(X) = 12\chi(\mathcal{O}_X) - K_X^2, \quad b_2(X) = c_2(X) - 2 + 4q(X).$$

Therefore
$$b_2(W) = 22 - K_W^2, \quad b_2(V) = b_2(S) + t = 11 + 13 = 24.$$

The inequality $K_W^2 \geq -2$ follows from the fact $b_2(V) \geq b_2(W)$. □

From Claim 4.5, we conclude that the resolution of $\overline{B'}$ blows-up at most two double points, thus
$$B'^2 \geq -2 + 2(-4) = -10 = \beta^2 + (-4).$$

This implies that α is a smooth (-4)-curve when $\alpha \neq 0$.

Claim 4.6. Only the following possibilities can occur:

- $\overline{\beta}$ has one double point and no other singularity, or
- $\overline{\alpha}$, $\overline{\beta}$ are smooth, $\overline{\alpha}\overline{\beta} = 2$.

Proof. Recall that $B' = \alpha + \beta$ is contained in fibres of f_A and, since $\overline{B'}$ has only negligible singularities, then also $\overline{B'} = \overline{\alpha} + \overline{\beta}$ is contained in fibres of $\overline{f_A}$. In particular $\overline{\alpha}^2, \overline{\beta}^2 \leq 0$.

If $\overline{\alpha}$ is singular, then it has arithmetic genus $p_a(\overline{\alpha}) = 1$ and $\overline{\alpha}^2 = 0$. But then $\overline{\alpha}$ has the same support of a fibre of $\overline{f_A}$, which is a contradiction because $\overline{f_A}$ is not elliptic. Therefore $\overline{\alpha}$ is smooth.

Since $K_P\overline{\alpha} \geq 0$, $K_P\overline{B'} = 2$ implies $K_P\overline{\beta} \leq 2$. We know that β is a smooth rational curve and $\beta^2 = -6$, thus $K_W\beta = 4$. If $\overline{\beta}$ is smooth, then one must have $\overline{\alpha}\overline{\beta} > 1$. From Claim 4.5 the only possibility in this case is $\overline{\alpha}\overline{\beta} = 2$. If $\overline{\beta}$ is singular, then $\overline{\beta}^2 \leq 0$ implies that $\overline{\beta}$ has one ordinary double point and no other singularity. □

Let $D := \overline{\beta}$ if $\overline{\beta}$ is singular. Otherwise let $D := \overline{\alpha} + \overline{\beta}$.

The 2-connected divisor $\widetilde{D} := \frac{1}{2}(\rho \circ \pi)^*(D)$ has arithmetic genus $p_a(\widetilde{D}) = 1$. We know that $(K_V + n\Lambda)\widetilde{D} = 1$ (because $K_V R' = 1$) and that $\widetilde{D}$ contains a component A such that $(K_V + n\Lambda)A = 0$ (because D has at least one negligible singularity). These two facts imply, from [10, Proposition A.5, (ii)], that the relative canonical map γ has a base point in $\widetilde{D}$. As mentioned above, γ is a morphism, which is a contradiction.

Finally the assertion about $\deg(\phi_2)$ in Case 2.: we have proved that S has a genus 2 fibration, so it has an hyperelliptic involution j. The bicanonical map ϕ_2 factors through both i and j, thus $\deg(\phi_2) \geq 4$.

This finishes the proof of case a), (ii) of Theorem 1.1.

We end the proof of Theorem 1.1 with Case a), (iii): A surface of general type with a genus 2 fibration and $p_g = q = 1$ satisfies $K^2 \leq 6$ (see [24]). Denote by j the map such that $\phi_2 = j \circ i$. The quotient S/i is a $K3$ surface thus, from [15], $\deg(j) \leq 2$. Analogously to Case 2, deg $(\phi_2) \geq 4$, thus $\deg(j) = 2$, $\deg(\phi_2) = 4$ and then $K_S^2 \neq 2$ (see Case 1).

It follows from [24, page 66] that, if the genus 2 fibration of S has a rational basis, then $K_S^2 = 3$. It is shown in [19] that, in these conditions, $\deg(\phi_2) = 2$. We then conclude that the genus 2 fibration of S is the Albanese fibration.

Examples for case a), (iii) with $K_S^2 = 3, 4$ or 5 were given by Catanese in [6]. The existence of the other cases is proved in the next section. □

5. Examples

In this section we will construct smooth minimal surfaces of general type S with $p_g(S) = q(S) = 1$ having an involution i such that the bicanonical map ϕ_2 of S is composed with i and:

1) $K_S^2 = 6$, $g = 3$, $\deg(\phi_2) = 2$, S/i is birational to a $K3$ surface;

2) $K_S^2 = 4$, $g = 2$, $\mathrm{Kod}(S/i) = 1$;

3) $K_S^2 = 2$, $g = 2$, $\mathrm{Kod}(S/i) = 2$,

where g denotes the genus of the Albanese fibration of S.

Example 5.1. In [23] Todorov gives the following construction of a surface of general type S with $p_g(S) = 1$, $q(S) = 0$ and $K_S^2 = 8$. Consider a Kummer surface Q in $\mathbb{P}^3$, *i.e.* a quartic having as singularities only 16 nodes (ordinary double points). Let $G \subset Q$ be the intersection of Q with a general quadric, $\widetilde{Q}$ be the minimal resolution of Q and $\widetilde{G} \subset \widetilde{Q}$ be the pullback of G. The surface S is the minimal model of the double cover $\pi : V \to \widetilde{Q}$ ramified over $\widetilde{G} + \sum_1^{16} A_i$, where $A_i \subset \widetilde{Q}$, $i = 1, \ldots, 16$, are the (-2)-curves which contract to the nodes of Q.

It follows from the double cover formulas (*cf.* [4, Chapter V, Section 22]) that the imposition of a quadruple point to the branch locus decreases K^2 by 2 and the Euler characteristic χ by 1.

We will see that we can impose a quadruple point to the branch locus of the Todorov construction, thus obtaining S with $K_S^2 = 6$. In this case I claim that $p_g(S) = q(S) = 1$. In fact, let W be the surface $\widetilde{Q}$ blown-up at the quadruple point, E be the corresponding (-1)-curve, B be the branch locus and L be the line bundle such that $2L \equiv B$. From formula (2.3) in Section 2, one has

$h^0(W, \mathcal{O}_W(2E + L)) = 0$ (thus the bicanonical map of V factors through π), hence also $h^0(W, \mathcal{O}_W(E + L)) = 0$ and then

$$p_g(S) = p_g(W) + h^0(W, \mathcal{O}_W(E + L)) = 1.$$

We will see that $\deg(\phi_2) = 2$, hence $\phi_2(S)$ is a $K3$ surface and so S has no genus 2 fibration.

First we need to obtain an equation of a Kummer surface. The Computational Algebra System *Magma* has a direct way to do this, but I prefer to do it using a beautiful construction that I learned from Miles Reid.

We want a quartic surface $Q \in \mathbb{P}^3$ whose singularities are exactly 16 nodes. Projecting from one of the nodes to $\mathbb{P}^2$, one realizes the "Kummer" surface as a double cover

$$\psi : X \longrightarrow \mathbb{P}^2$$

with branch locus the union of 6 lines L_i (see [17, page 774]), each one tangent to a conic C (the image of the projection point) at a point p_i. The surface X contains 15 nodes (from the intersection of the lines) and two (-2)-curves (the pullback $\psi^*(C)$) disjoint from these nodes. To obtain a Kummer surface we have just to contract one of these curves.

Denote also by L_i the defining polynomial of each line L_i. An equation for X is $z^2 = L_1 \cdots L_6$ in the weighted projective space $\mathbb{P}(3, 1, 1, 1)$, with coordinates (z, x_1, x_2, x_3). We will see that this equation can be written in the form $AB+DE = 0$, where the system $A = B = D = E = 0$ has only the trivial solution and B, E are the defining polynomials of one of the (-2)-curves in $\psi^*(C)$. Now consider the surface X' given by $Bs = D$, $Es = -A$ in the space $\mathbb{P}(3, 1, 1, 1, 1)$ with coordinates (z, s, x_1, x_2, x_3). There is a morphism $X \to X'$ which restricts to an isomorphism

$$X - \{B = E = 0\} \longrightarrow X' - \{[0 : 1 : 0 : 0 : 0]\}$$

and which contracts the curve $\{B = E = 0\}$ to the point $[0 : 1 : 0 : 0 : 0]$. This is an example of *unprojection* (see [21]).

The variable z appears isolated in the equations of X', therefore eliminating z we obtain the equation of the Kummer Q in $\mathbb{P}^3$ with variables (s, x_1, x_2, x_3). All this calculations will be done using Magma.

In what follows a line preceded by > is an input line, something preceded by // is a comment. A \ at the end of a line means continuation in the next line. The other lines are output ones.

```
> K<e>:=CyclotomicField(6);//e denotes the 6th root of unity.
> //We choose a conic C with equation x1x3-x2^2=0 and fix the
> //p_i's: (1:1:1), (e^2:e:1), (e^4:e^2:1), (e^6:e^3:1),
> //(e^8:e^4:1), (e^10:e^5:1).
> R<z,s,x1,x2,x3>:=PolynomialRing(K,[3,1,1,1,1]);
```

```
> g:=&*[e^(2*i)*x1-2*e^i*x2+x3:i in [0..5]];
> //g is the product of the defining polynomials
> //of the tangent lines L_i to C at p_i.
> X:=z^2-g;
> X eq (z+x1^3-x3^3)*(z-x1^3+x3^3)+4*(x1*x3-4*x2^2)^2*\
> (-x1*x3+x2^2);//The decomposition AB+DE.
true
> i:=Ideal([s*(z-x1^3+x3^3)-4*(x1*x3-4*x2^2)^2,\
> s*(x1*x3-x2^2)-(z+x1^3-x3^3)]);
> j:=EliminationIdeal(i,1);
> j;
Ideal of Graded Polynomial ring of rank 5 over K
Lexicographical Order Variables: z, s, x1, x2, x3
Variable weights: 3 1 1 1 1 Basis:
[-1/2*s^2*x1*x3+1/2*s^2*x2^2+s*x1^3-s*x3^3+2*x1^2*x3^2-
16*x1*x2^2*x3+32*x2^4]
> 2*Basis(j)[1];
-s^2*x1*x3+s^2*x2^2+2*s*x1^3-2*s*x3^3+4*x1^2*x3^2-
32*x1*x2^2*x3+64*x2^4
> //This is the equation of the Kummer Q.
```

We want to find a quadric H such that $H \bigcap Q$ is a reduced curve $\overline{B'}$ having an ordinary quadruple point pt as only singularity. Since the computer is not fast enough while working with more than 5 or 6 variables, we first need to think what the most probable case is.

Like we have seen in Section 3, the branch locus $B' + \sum_1^{16} A_i$ is contained in 4 fibres $F_A^1, \ldots, F_A^4$ of a fibration f_A of W, where W is the resolution of Q blown-up at pt and the A_i's are the (-2)-curves which contract to the nodes of Q.

Of course we have a quadric intersecting Q at a curve with a quadruple point pt : the tangent space T to Q at pt counted twice. But this one is double, so we need to find an irreducible one (and these two induce f_A), the curve $\overline{B'}$. These curves $2T$ and $\overline{B'}$ are good candidates for $\overline{F_A^1}$ and $\overline{F_A^2}$ (in the notation of Sections 3 and 4). If this configuration exists, then the 16 nodes must be contained in the other two fibres, $\overline{F_A^3}$ and $\overline{F_A^4}$. These fibres are divisible by 2, because $\overline{F_A^1} = 2T$, and are double outside the nodes. Since in a $K3$ surface only 0, 8 or 16 nodes can have sum divisible by 2, it is reasonable to try the following configuration: each of $\overline{F_A^3}$ and $\overline{F_A^4}$ contain 8 nodes with sum divisible by 2 and is double outside the nodes.

It is well known (see e. g. [17]) that the Kummer surface Q has 16 double hyperplane sections T_i such that each one contains 6 nodes of Q and that any two of them intersect in 2 nodes. The sum of the 8 nodes contained in

$$N := (T_1 \cup T_2) \backslash (T1 \cap T_2)$$

is divisible by 2. Magma will give 3 generators h_1, h_2, h_3 for the linear system of quadrics through these nodes.

```
> K<e>:=CyclotomicField(6);
> P3<s,x1,x2,x3>:=ProjectiveSpace(K,3);
> F:=-s^2*x1*x3+s^2*x2^2+2*s*x1^3-2*s*x3^3+4*x1^2*x3^2-\
> 32*x1*x2^2*x3+64*x2^4;
> Q:=Scheme(P3,F);/*The Kummer*/; SQ:=SingularSubscheme(Q);
> T1:=Scheme(P3,x1-2*x2+x3); T2:=Scheme(P3,s);
> N:=Difference((T1 join T2) meet SQ, T1 meet T2);
> s:=SetToSequence(RationalPoints(N));
> //s is the sequence of the 8 nodes.
> L:=LinearSystem(P3,2);
>//This will give the h_i's:
> LinearSystem(L,[P3!s[i] : i in [1..8]]);
Linear system on Projective Space of dimension 3
  Variables: s, x1, x2, x3 with 3 sections:

s*x1-2*x1^2-4*x1*x2-2*x1*x3+8*x2^2+4*x2*x3+2*x3^2
s*x2-2*x1^2-4*x1*x2+4*x2*x3+2*x3^2
s*x3-2*x1^2-4*x1*x2+2*x1*x3-8*x2^2+4*x2*x3+2*x3^2
```

Now we want to find a quadric H in the form $h_1 + bh_2 + ch_3$, for some b, c (or, less probably, in the form $bh_2 + ch_3$) such that the projection of $H \bigcap Q$ to $\mathbb{P}^2$ (by elimination) is a curve with a quadruple point. To find a quadruple point we just have to impose the annulation of the derivatives up to order 3 and ask Magma to do the rest.

```
> R<s,b,c,x1,x2,x3>:=PolynomialRing(Rationals(),6);
> F:=-s^2*x1*x3+s^2*x2^2+2*s*x1^3-2*s*x3^3+4*x1^2*x3^2-\
> 32*x1*x2^2*x3+64*x2^4;
> h1:=s*x1-2*x1^2-4*x1*x2-2*x1*x3+8*x2^2+4*x2*x3+2*x3^2;
> h2:=s*x2-2*x1^2-4*x1*x2+4*x2*x3+2*x3^2;
> h3:=s*x3-2*x1^2-4*x1*x2+2*x1*x3-8*x2^2+4*x2*x3+2*x3^2;
> H:=h1+b*h2+c*h3;
> I:=ideal<R|[F,H]>;
> I1:=EliminationIdeal(I,1);
> q0:=Evaluate(Basis(I1)[1],x3,1);//We work in the affine plane.
> R4<B,C,X1,X2>:=PolynomialRing(Rationals(),4);
> h:=hom<R->R4|[0,B,C,X1,X2,0]>;
> q:=h(q0);q1:=Derivative(q,X1);q2:=Derivative(q,X2);
> q3:=Derivative(q1,X1);q4:=Derivative(q1,X2);q5:=Derivative\
> (q2,X2);q6:=Derivative(q3,X1);q7:=Derivative(q3,X2);
> q8:=Derivative(q4,X2);q9:=Derivative(q5,X2);
> A4:=AffineSpace(R4);
> S:=Scheme(A4,[q,q1,q2,q3,q4,q5,q6,q7,q8,q9]);
> Dimension(S);
0
> PointsOverSplittingField(S);
```

This last command gives the points of S, as well as the necessary equations to define the field extensions where they belong. There are various solutions. One of them gives the desired quadruple point. The confirmation is as follows:

```
> R<x>:=PolynomialRing(Rationals());
> K<r13>:=ext<Rationals()|x^4 + x^3 + 1/4*x^2 + 3/32>;
> P3<s,x1,x2,x3>:=ProjectiveSpace(K,3);
> F:=-s^2*x1*x3+s^2*x2^2+2*s*x1^3-2*s*x3^3+4*x1^2*x3^2-\
> 32*x1*x2^2*x3+64*x2^4;
> b:=64/55*r13^3-272/55*r13^2-96/55*r13-46/55;
> c:=-2176/605*r13^3+448/605*r13^2+624/605*r13-361/605;
> H:=(s*x1-2*x1^2-4*x1*x2-2*x1*x3+8*x2^2+4*x2*x3+2*x3^2)+\
> b*(s*x2-2*x1^2-4*x1*x2+4*x2*x3+2*x3^2)+\
> c*(s*x3-2*x1^2-4*x1*x2+2*x1*x3-8*x2^2+4*x2*x3+2*x3^2);
> Q:=Scheme(P3,F);
> C:=Scheme(Q,H);
> IsReduced(C);
false
> RC:=ReducedSubscheme(C);
> #SingularPoints(RC);//# means ‘‘number of’’.
1
> HasSingularPointsOverExtension(RC);
false
> pt:=Representative(SingularPoints(RC));
> pt in SingularSubscheme(Q);//pt is not a node of Q.
false
> T:=DefiningPolynomial(TangentSpace(Q,pt));
> T2:=Scheme(Q,T^2);
> #RationalPoints(T2 meet C);
1
> pt in RationalPoints(T2 meet C);
true
> HasPointsOverExtension(T2 meet C);
false
```

This way $T2$ and C generate a pencil with a quadruple base point and the curve $\overline{B'}$ is a general element of this pencil.

Finally, it remains to be shown that the degree of the bicanonical map ϕ_2 is 2. As $(2K_S)^2 = 24$, it suffices to show that $\phi_2(S)$ is of degree 12. Since, in the notation of diagram (3.1), $h^*|2K_S| = \pi^*|2K_W + B'|$ then $\phi_2(S)$ is the image of W via the map $\tau : W \to \phi_2(S)$ given by $|2K_W + B'|$. The projection of this linear system on Q is the linear system of the quadrics whose intersection with Q has a double point at pt. In order to easily write this linear system, we will translate the point pt to the origin (in affine coordinates).

```
> QA:=AffinePatch(Scheme(P3,F),4);
> p:=Representative(RationalPoints(AffinePatch(Cluster(pt),4)));
> A3<x,y,z>:=Ambient(QA);
> psi:=map<A3->A3|[x-p[1],y-p[2],z-p[3]]>;Q0:=psi(QA);
> FA:=DefiningPolynomial(Q0);
> j:=[Evaluate(Derivative(FA,A3.i),Origin(A3)):i in [1,2,3]];
> J:=LinearSystem(A3,[j[1]*x+j[2]*y+j[3]*z,x^2,x*y,x*z,y^2,y*z,\
```

```
> z^2]);
> P6:=ProjectiveSpace(K,6);
> tau:=map<A3->P6|Sections(J)>;
> Degree(tau(Q0));
12
```

Example 5.2. Here we will construct a surface of general type S, with $p_g = q = 1$ and $K^2 = 4$, as the minimal model of a double cover of a surface W such that $\mathrm{Kod}(W) = p_g(W) = 1$ and $q(W) = 0$.

Step 1. Construction of W.

Consider five distinct lines $L_1, \ldots, L_5 \subset \mathbb{P}^2$ meeting in one point p_0. Let $p_1 \in L_4$, $p_2, p_3 \in L_5$ be points distinct from p_0. Choose three distinct non-degenerate conics, C_1, C_2, C_3, tangent to L_4 at p_1 and passing through p_2, p_3. Define

$$D := L_1 + \ldots + L_4 + C_1 + C_2.$$

Denote by $p_4, \ldots, p_{15}$ the 12 nodes of D contained in $L_1 + L_2 + L_3$. To resolve the $(3,3)$ point of D at p_1 we must do two blow-ups: one at p_1 and other at an infinitely near point p_1'. Let $\mu : X \to \mathbb{P}^2$ be the blow-up with centers $p_0, p_1, p_1', p_2, \ldots, p_{15}$ and $E_0, E_1, E_1', E_2, \ldots, E_{15}$ be the corresponding exceptional divisors (with self-intersection -1). Consider

$$D' := \mu^*(D) - 4E_0 - 2E_1 - 4E_1' - 2\sum_{2}^{15} E_i.$$

Let $\psi : \widetilde{X} \to X$ be the double cover of X with branch locus D'. The surface $\widetilde{X}$ is the canonical resolution of the double cover of $\mathbb{P}^2$ ramified over D. Let W be the minimal model of $\widetilde{X}$ and ν be the corresponding morphism.

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\ \nu\ } & W \\ {\scriptstyle\psi}\big\downarrow & & \\ X & \xrightarrow{\ \mu\ } & \mathbb{P}^2 . \end{array}$$

Notice that ν contracts two (-1)-curves contained in $(\mu \circ \psi)^*(L_4)$.

We have $K_X \equiv -\mu^*(3L) + E_1' + \sum_0^{15} E_i$, where L denotes a general line of $\mathbb{P}^2$. Hence, using the double cover formulas (*cf.* (2.1)),

$$K_{\widetilde{X}} \equiv \psi^*\left(K_X + \frac{1}{2}D'\right) \equiv \psi^*(\mu^*(L) - E_0 - E_1') \equiv \psi^*(\widehat{L_4} + (E_1 - E_1') + E_1'),$$

where $\widehat{L_4} \subset X$ is the strict transform of L_4. Since $\widehat{L_4}$ and $E_1 - E_1'$ are (-2)-curves contained in the branch locus D', then $\frac{1}{2}\psi^*(\widehat{L_4})$ and $\frac{1}{2}\psi^*(E_1 - E_1')$ are (-1)-curves in $\widetilde{X}$, thus

$$K_W \equiv \nu(\psi^*(E_1')).$$

The divisor $2\nu(\psi^*(E_1')) \equiv 2K_W$ is a (double) fibre of the elliptic fibration of W induced by the pencil of lines through p_0. So $p_g(W) = 1$ and W has Kodaira dimension 1.

From (2.1) one has

$$\chi(\mathcal{O}_W) = 2 + \frac{1}{8}D'(2K_X + D') = 2 + \frac{1}{8}(28 - 28) = 2.$$

Step 2. The branch locus in W.

Since the strict transforms $\widehat{L_1}, \ldots, \widehat{L_4} \subset X$ are in the branch locus D', then there are curves $l_1, \ldots, l_4 \subset \widetilde{X}$ such that

$$(\mu \circ \psi)^*(L_1 + \cdots + L_4) = 2l_1 + \cdots + 2l_4 + 4\psi^*(E_0) + \psi^*(E_1 - E_1') + 2\psi^*(E_1') + \sum_4^{15} A_i,$$

where each $A_i := \psi^*(E_i)$ is a (-2)-curve. But also $E_1 - E_1'$ is in the branch locus, thus $\psi^*(E_1 - E_1') \equiv 0 \pmod 2$ and then

$$\sum_4^{15} A_i \equiv 0 \pmod 2.$$

The strict transform $\widehat{L_5}$ is a (-2)-curve which do not intersect D' thus

$$\psi^*(\widehat{L_5}) = A_{16} + A_{17},$$

with A_{16}, A_{17} disjoint (-2)-curves.

Denote by $\widehat{C_3} \subset X$ the strict transform of the conic C_3. We have

$$(\mu \circ \psi)^*(C_3 + L_4 + L_5) = \psi^*(\widehat{C_3}) + 2l_4 + A_{16} + A_{17} + 2\psi^*(E_0 + \cdots + E_3) + 2\psi^*(E_1') \equiv 0 \pmod 2.$$

With this we conclude that

$$\psi^*(\widehat{C_3}) + \sum_4^{17} A_i \equiv 0 \pmod 2.$$

Notice that $F \cdot \nu\left(\psi^*(\widehat{C_3})\right) = 4$ for a fibre F of the elliptic fibration of W, thus $K_W \cdot \nu\left(\psi^*(\widehat{C_3})\right) = 2$.

Step 3. Construction of S.

Let $\pi : V \to W$ be the double cover with branch locus

$$B := \nu\left(\psi^*(\widehat{C_3}) + \sum_4^{17} A_i\right)$$

and S be the minimal model of V. From the double cover formulas (2.1) we obtain

$$2K_V^2 = (2K_W + B)^2 = 4K_W^2 + 4K_W B + B^2 = 4\cdot 0 + 4\cdot 2 + (-28) = -20$$

and, by contraction of the (-1)-curves $\frac{1}{2}\pi^*(\nu(A_i))$,

$$K_S^2 = K_V^2 + 14 = -10 + 14 = 4.$$

Let $L :\equiv \frac{1}{2}B$. Formulas (2.1) give

$$\chi(\mathcal{O}_S) = 2\chi(\mathcal{O}_W) + \frac{1}{2}L(K_W + L) = 4 - 3 = 1.$$

Using now formula (2.3) we obtain $h^0(W, \mathcal{O}_W(2K_W + L)) = 0$, which means that the bicanonical map of V factors through π.

Because K_W is effective then also $h^0(W, \mathcal{O}_W(K_W + L)) = 0$ and

$$p_g(S) = p_g(W) + h^0(W, \mathcal{O}_W(K_W + L)) = 1.$$

Hence $q(S) = 1$ and then, as we noticed in the beginning of Section 4, the curve $\nu\left(\psi^*(\widehat{C_3})\right)$ is contained in the fibration of W which induces the Albanese fibration of S. As $\nu\left(\psi^*(\widehat{C_3})\right)^2 = 0$, we conclude that the Albanese fibration of S is the one induced by the pencil $|\widehat{C_3}|$. It is of genus 2 because $\widehat{C_3}D' = \widehat{C_3}(\widehat{L_1}+\widehat{L_2}+\widehat{L_3}) = 6$.

Example 5.3. Now we will obtain a surface of general type S, with $p_g = q = 1$ and $K^2 = 2$, as the minimal model of a double cover of a surface of general type W such that $K_W^2 = p_g(W) = 1$ and $q(W) = 0$.

Step 1. Construction of W.

Let $p_0, \dots, p_3 \in \mathbb{P}^2$ be distinct points and L_i be the line through p_0 and p_i, $i = 1, 2, 3$. For each $j \in \{1, 2, 3\}$ let C_j be the conic through p_1, p_2, p_3 tangent to the L_i's except for L_j. Denote by D a general element of the linear system generated by $3C_1 + 2L_1$, $3C_2 + 2L_2$ and $3C_3 + 2L_3$. The singularities of D are a $(3, 3)$-point at p_i, tangent to L_i, $i = 1, 2, 3$, and a double point at p_0. Let L_4 be a line through p_0 transverse to D.

Denote by W' the canonical resolution of the double cover of $\mathbb{P}^2$ with branch locus

$$D + L_1 + \dots + L_4$$

and by W the minimal model of W'. The formulas of [4, Chapter V, Section 22] give $\chi(W) = 2$ and $K_W^2 = 1$ (notice that the map $W' \to W$ contracts three (-1)-curves contained in the pullback of $L_1 + L_2 + L_3$). Since $K^2 \geq 2p_g$ for an irregular surface ([14]), W is regular and then $p_g(W) = \chi(W) - 1 = 1$.

Step 2. The branch locus in W.

The pencil of lines through p_0 induces a (genus 2) fibration of W. Let F_i be the fibre induced by L_i, $i = 1, \ldots, 4$. The fibre F_4 is the union of six disjoint (-2)-curves (corresponding to the nodes of $D - p_0$) with a double component (the strict transform of L_4). Each F_i, $i = 1, 2, 3$, is the union of two (-2)-curves with a double component (*cf.* [24, Section 1]). Thus $F_1 + \cdots + F_4$ contain disjoint (-2)-curves $A_1, \ldots, A_{12}$ such that

$$\sum_1^{12} A_i \equiv 0 \pmod 2.$$

Step 3. Construction of S.

Let V be the double cover of W with branch locus $\sum_1^{12} A_i$ and S be the minimal model of V. From (2.1) we obtain $\chi(\mathcal{O}_S) = 1$ and $K_V^2 = -10$. The A_i's lift to (-1)-curves in V, thus $K_S^2 = -10 + 12 = 2$. We have $1 = p_g(W) \leq p_g(S)$, hence $q(S) \neq 0$ and then $2 = K_S^2 \geq 2p_g(S)$. So $p_g(S) = q(S) = 1$.

The genus 2 fibration of W induces the Albanese fibration of S.

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