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The equation $-\Delta u - \lambda \frac{u}{|x|^2} = |\nabla u|^p + cf(x)$: The optimal power

BOUMEDIENE ABDELLAOUI AND IRENEO PERAL

Abstract. We will consider the following problem

$$-\Delta u - \lambda \frac{u}{|x|^2} = |\nabla u|^p + c\,f, \quad u > 0 \text{ in } \Omega,$$

where $\Omega \subset \mathbb{R}^N$ is a domain such that $0 \in \Omega$, $N \geq 3$, $c > 0$ and $\lambda > 0$. The main objective of this note is to study the precise threshold $p_+ = p_+(\lambda)$ for which there is no *very weak supersolution* if $p \geq p_+(\lambda)$. The optimality of $p_+(\lambda)$ is also proved by showing the solvability of the Dirichlet problem when $1 \leq p < p_+(\lambda)$, for $c > 0$ small enough and $f \geq 0$ under some hypotheses that we will prescribe.

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1. Introduction

We consider the linear operator

$$\mathcal{L}_\lambda(\,\cdot\,) \equiv -\Delta(\,\cdot\,) - \lambda \frac{(\,\cdot\,)}{|x|^2} : W^{1,2}(\mathbb{R}^N) \to W^{-1,2}(\mathbb{R}^N), \quad N \geq 3 \text{ and } \lambda > 0.$$

By Hardy inequality $\mathcal{L}_\lambda$ is continuous and, moreover, is positive if $\lambda < \Lambda_N = (\frac{N-2}{2})^2$. We will restrict ourselves to the interval $0 < \lambda \leq \Lambda_N$ where the behavior of $\mathcal{L}_\lambda$ is quite peculiar. To have an idea of such a behavior we refer to the papers [11] and [13]. In [11] is proved, among others, the following result.

Let Ω be a bounded domain in $\mathbb{R}^N$ with $0 \in \Omega$. Consider the problem

$$\mathcal{L}_\lambda(u) = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{P}$$

with $f \in L^m(\Omega)$, $1 < m < \frac{2N}{N+2}$, and $\lambda < \lambda_{m,N} \equiv \frac{N(m-1)(N-2m)}{m^2}$. Then the weak solution u belongs to $W_0^{1,m^*}(\Omega)$, $m^* = \frac{mN}{N-m}$. Moreover the result is optimal.

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In particular, for general $f \in L^1(\Omega)$, problem (P) has no solution (see also [2]). Also in [11] is proved that, even if $f \in L^m(\Omega)$ with $m > \frac{N}{2}$ the solutions are unbounded. In this sense we see that the behavior of the solution is like the classical Laplacian case, only if $\lambda < \lambda_{m,N}$, that is, the summability of the solution depends explicitly on λ.

In [13] is studied the semilinear problem

$$\mathcal{L}_\lambda(u) = u^p \tag{SP}$$

and a new *critical exponent* is obtained. Precisely we can reformulate one of the main results in [13] as follows:

Let $0 < \lambda \leq \Lambda_N$. *There exists* $q^+(\lambda)$ *such that equation* (SP) *has a nontrivial solution in* $\mathcal{D}'(B_r(0))$ *with* $u^p, \dfrac{u}{|x|^2} \in L^1(B_r(0))$ *if and only if* $p \in (1, q^+(\lambda))$.

An explicit expression for $q^+(\lambda)$ is given.

In the two previous results the critical parameters are deeply related to the values

$$\alpha_{(\pm)} = \frac{N-2}{2} \pm \sqrt{\left(\frac{N-2}{2}\right)^2 - \lambda}, \tag{1.1}$$

which are the roots of the algebraic equation $\alpha^2 - (N-2)\alpha + \lambda = 0$. Such roots give the radial solutions, $u(r) = c_1|x|^{-\alpha_{(+)}} + c_2|x|^{-\alpha_{(-)}}$, $c_1, c_2 \in \mathbb{R}$, to the equation

$$-\Delta u - \lambda \frac{u}{|x|^2} = 0.$$

In this paper we will consider the quasilinear problem

$$-\Delta u = |\nabla u|^p + \lambda \frac{u}{|x|^2} + c\, f,\ x \in \Omega \subset \mathbb{R}^N,\ N \geq 3, \tag{1.2}$$

where Ω is a domain such that $0 \in \Omega$. We assume that $\lambda,\ c$ are positive real numbers and f is a nonnegative function under some extra hypotheses that we will precise later. According with the results in [11], the existence of a solution to the equation (1.2) it is not clear. Therefore, the main problem under consideration in this work is to get, for $\lambda > 0$ fixed, the optimal exponent $p_+(\lambda)$ in order to find a solution for (1.2). Notice that the variational technics are not useful in the quasilinear setting, then the difficulties are considerably bigger than in the semilinear case.

It is worthy to point out that this type of quasilinear problems appear in several contexts. For instance the case $p = 2$, and in the simplest case $\lambda = 0$, problem (1.2) is the stationary counterpart of the Kardar-Parisi-Zhang model (see [18]) and of some flame propagation models (see [7]). Moreover, equation (1.2) can be read as the Hamilton-Jacobi equation

$$|\nabla u|^p + \lambda \frac{u}{|x|^2} + c\, f = 0$$

with the viscosity term given by the Laplacian.[1] See for instance [21] for details and applications of this topic.

The paper is organized as follows. In Section 2 we identify the critical exponent $p_+(\lambda)$ and prove the nonexistence result for $p \geq p_+(\lambda)$. This nonexistence result is the strongest possible: we prove nonexistence for *very weak solutions* in the sense of the Definition 2.1, *i.e.* just the class to give sense to distributional solutions. As we will see, for all $\lambda > 0$, $p_+(\lambda) < 2$, then the classical case $p = 2$ falls in the nonexistence interval. We would like to point out that in [2] has been studied the natural quadratic term related to $\mathcal{L}_\lambda$ in order to have existence.

In Section 3 we analyze the nonexistence result by proving a blow-up result of the solutions of approximate problems. Sections 4 and 5 are devoted to the existence results that, in particular, show the optimality of $p^+(\lambda)$. In Section 4 we study the existence in the case $0 < \lambda < \Lambda_N$ while in Section 5 we study the existence in the critical case $\lambda = \Lambda_N$. Finally, in Subsection 5.1 we present some open questions.

2. Nonexistence results: exponent $p_+(\lambda)$

The main result in this section is to find a necessary and sufficient condition on p in a such way that problem (1.2) has not positive supersolution in a very weak sense. In the whole section, we use the concept of *very weak* (*sub, super*) *solution* which, roughly speaking, is the more general setting for which the equation has a meaning in distributional sense.

Definition 2.1. We say that $u \in L^1_{\text{loc}}(\Omega)$ is a very weak super-solution (sub-solution) to equation (1.2) if $\frac{u}{|x|^2} \in L^1_{\text{loc}}(\Omega)$, $|\nabla u|^p \in L^1_{\text{loc}}(\Omega)$ and $\forall \phi \in C_0^\infty(\Omega)$ such that $\phi \geq 0$, we have

$$\int_\Omega (-\Delta\phi) u\, dx \geq (\leq) \int_\Omega \left(|\nabla u|^p + \lambda \frac{u}{|x|^2} + f \right) \phi\, dx.$$

If u is a very weak super and subsolution, then we say that u is a very weak solution.

If $\lambda > \Lambda_N \equiv (\frac{N-2}{2})^2$, the non-existence result of positive very weak solution to problem (1.2) is a consequence of the optimality of Λ_N as constant in the Hardy inequality. See for instance [1]. Then, hereafter we will assume $0 < \lambda \leq \Lambda_N$.
We begin by the following elementary result which gives a lower estimate of u near the origin.

Lemma 2.2. *Assume that $u \geq 0$ in Ω, $u \not\equiv 0$, $u \in L^1_{\text{loc}}(\Omega)$ and $\frac{u}{|x|^2} \in L^1_{\text{loc}}(\Omega)$. If u satisfies $-\Delta u - \lambda \frac{u}{|x|^2} \geq 0$ in the sense of distributions, then there exists a positive constant C and a small ball $B_R(0) \subset \Omega$ such that $u(x) \geq C|x|^{-\alpha_{(-)}}$ in $B_R(0)$, where $\alpha_{(-)}$ is defined in* (1.1).

[1] As a consequence of the nonexistence results in this paper, the reader could check without difficulty that the *vanishing viscosity method* by P. Lax does not produce a solution for the first order equation.

Proof. By using the strong maximum principle and comparison result it is not difficult to obtain that $u \geq \eta$ in a small ball $B_r(\Omega)$. Fixed $R > 0$, let $w \in W^{1,2}(B_R(0))$ be the unique positive solution to problem

$$\begin{cases} -\Delta w - \lambda \dfrac{w}{|x|^2} = 0 & \text{in } B_R(0), \\ w = \eta & \text{on } \partial B_R(0). \end{cases} \tag{2.1}$$

By a direct computation we obtain that $w(r) = Cr^{-\alpha_{(-)}}$ with $\alpha_{(-)} = \frac{N-2}{2} - \sqrt{(\frac{N-2}{2})^2 - \lambda}$ and $C = \frac{\eta}{R^{-\alpha_{(-)}}}$. Since u is a super-solution to problem (2.1), then using the weak comparison principle we conclude that $u \geq w$ in $B_R(0)$, thus $u \geq C|x|^{-\alpha_{(-)}}$ in $B_R(0)$ and the result follows. □

We will use the following necessary condition for existence.

Lemma 2.3. *Consider the equation*

$$-\Delta w - \lambda \frac{w}{|x|^2} = g \text{ in } \Omega, \tag{2.2}$$

with $g \in L^1_{\text{loc}}(\Omega)$, $g(x) \geq 0$ and $\lambda \leq \Lambda_N$. If (2.2) *has a very weak supersolution then $|x|^{-\alpha_{(-)}} g \in L^1_{\text{loc}}(\Omega)$ where $\alpha_{(-)}$ is defined by* (1.1).

Proof. Assume that w is a very weak supersolution to (2.2) then it is sufficient to check the conclusion in balls containing the origin, $B_R(0)$. For $g_n \equiv T_n(g)$ we solve the problem

$$\begin{cases} -\Delta w_n - \lambda \dfrac{w_n}{|x|^2} = g_n & \text{in } B_R(0), \\ w_n = 0 & \text{on } \partial B_R(0). \end{cases} \tag{2.3}$$

Using comparison argument as in [13] we obtain: i) $\{w_n\}_{n\in\mathbb{N}}$ in nondecreasing and ii) $w_n \leq w$. Consider ϕ the solution to problem

$$\begin{cases} -\Delta \phi - \lambda \dfrac{\phi}{|x|^2} = 1 & \text{in } B_R(0), \\ \phi = 0 & \text{on } \partial B_R(0). \end{cases}$$

One can check that $\phi(x) \backsimeq c|x|^{-\alpha_{(-)}}$ in a neighborhood of $x = 0$. Then by taking ϕ as a test function in problem (2.3) we conclude that

$$\int_{B_R(0)} w_n dx = \int_{B_R(0)} g_n \phi dx \geq C_2 \int_{B_R(0)} g_n |x|^{-\alpha_-} dx,$$

then the result follows by the monotone convergence theorem . □

Remark 2.4. It is easy to check that if in problem (1.2) we replace $|x|^{-2}$ by a weight $g \in L^m(\Omega)$ with $m > \frac{N}{2}$, then there exists $0 < \lambda_0$ such that for $0 < \lambda < \lambda_0$ problem (1.2) has a weak solution for suitable f. The behavior of the problem with the Hardy singular potential is quite different.

To find the optimal exponent we search a solution in the form $u(x) = A|x|^{-\beta}$ of the equation. Hence by a direct computation we obtain that $\beta = \frac{2-p}{p-1}$ and

$$\beta^p A^{p-1} = \beta(N - \beta - 2) - \lambda.$$

Since the left hand side is positive, then the right hand side must be positive, but the second member is positive if and only if

$$\alpha_{(-)} < \beta < \alpha_{(+)} \text{ where } \alpha_{(\pm)} \text{ are defined by (1.1).}$$

Since $\alpha_{(-)} < \beta < \alpha_{(+)}$ is equivalent to

$$p_-(\lambda) \equiv \frac{2 + \alpha_{(+)}}{\alpha_{(+)} + 1} < p < \frac{2 + \alpha_{(-)}}{\alpha_{(-)} + 1} \equiv p_+(\lambda),$$

hence the heuristic guess as optimal exponent seems to be $p_+(\lambda)$.
The main result on nonexistence in this direction is the following.

Theorem 2.5. *Assume that $f \geq 0$. Let $p_+(\lambda) = \frac{2+\alpha_{(-)}}{1+\alpha_{(-)}}$, where $\alpha_{(-)}$ is defined in* (1.1). *If $p \geq p_+(\lambda)$, then equation* (1.2) *has no positive very weak super-solution. In the case where $f \equiv 0$, the unique non negative very weak super-solution is $u \equiv 0$.*

Proof. We divide the proof into tree steps.

First step: $p > p_+(\lambda)$

Assume by contradiction that equation (1.2) has a very weak super-solution u, then $-\Delta u - \lambda \frac{u}{|x|^2} \gneq 0$. Then there exists a positive constant C and a small ball $B_r(0) \subset \mathbb{R}^N$ such that $u(x) \geq C|x|^{-\alpha_{(-)}}$ in $B_r(0)$. Let $\phi \in \mathcal{C}_0^\infty(B_r(0))$, therefore, using $|\phi|^{p'}$ as a test function in (1.2) and by Hölder, Young inequalities we obtain that

$$c_1\lambda \int_{B_r(0)} \frac{u|\phi|^{p'}}{|x|^2} dx \leq \int_{B_r(0)} |\nabla\phi|^{p'} dx \tag{2.4}$$

where c_1 is a positive constant that is independent of u and ϕ. Using the lower estimate for u in $B_r(0)$ that provides Lemma 2.2, we obtain that

$$c_2\lambda \int_{B_r(0)} \frac{|\phi|^{p'}}{|x|^{2+\alpha_{(-)}}} dx \leq \int_{B_r(0)} |\nabla\phi|^{p'} dx.$$

We recall that $p > p_+(\lambda)$, hence we obtain that $2 + \alpha_{(-)} > p'$ and then we reach a contradiction with the classical Hardy inequality for $W_0^{1,p'}(B_r(0))$. Then the result follows.

Second step: $p = p_+(\lambda)$ **and** $\lambda < \Lambda_N$

Again we argue by contradiction. Assume that equation (1.2) has a very weak super-solution u. As above, by Lemma 2.2 there is a positive constant c_0 such that

$$u(x) \geq \frac{c_0}{|x|^{\alpha_{(-)}}} \text{ in some ball } B_\eta(0) \subset\subset \Omega, \tag{2.5}$$

without loss of generality we assume that $\eta = e^{-1}$. Using Lemma 2.3 we obtain that

$$\int_{B_\eta(0)} |\nabla u|^{p_+(\lambda)}|x|^{-\alpha_{(-)}}dx < \infty \text{ and } \int_{B_\eta(0)} \frac{u}{|x|^{2+\alpha_{(-)}}}dx < \infty. \tag{2.6}$$

Let $w(x) = |x|^{-\alpha_{(-)}}(\log(\frac{1}{|x|}))^\beta$ where β is a positive small constant that we will choose bellow. Since $\lambda < \Lambda_N$, $w \in W^{1,2}(B_\eta(0))$ and then, in particular, $w \in W^{1,p_+(\lambda)}(B_\eta(0))$. By a direct computation we obtain that

$$\begin{aligned} &-\Delta w - \lambda\frac{w}{|x|^2} \\ &\quad = \frac{\beta}{|x|^{2+\alpha_{(-)}}}\left(\log\left(\frac{1}{|x|}\right)\right)^{\beta-1}\left[(N-2-2\alpha_{(-)}) + (1-\beta)\left(\log\left(\frac{1}{|x|}\right)\right)^{-1}\right] \end{aligned}$$

Notice that $|\nabla w| = |x|^{-\alpha_{(-)}-1}(\alpha_{(-)}\log(\frac{1}{|x|}) + \beta(\log(\frac{1}{|x|}))^{\beta-1}))$, thus

$$\begin{aligned} |\nabla w|^{p_+(\lambda)}&\left(\alpha_{(-)}\log\left(\frac{1}{|x|}\right) + \beta\left(\log\left(\frac{1}{|x|}\right)\right)^{-1}\right)^{1-p_+(\lambda)} \\ &= |x|^{-\alpha_{(-)}-2}\left(\alpha\log\left(\frac{\eta}{|x|}\right) + \beta\left(\log\left(\frac{1}{|x|}\right)\right)^{-1}\right). \end{aligned}$$

Since $|x| \leq e^{-1}$, by choosing β small enough, we conclude that

$$-\Delta w - \lambda\frac{w}{|x|^2} \leq \beta^{\frac{1}{2}}|\nabla w|^{p_+(\lambda)}h(x)$$

where $h(x) = \left(\alpha_{(-)}\log(\frac{1}{|x|}) + \beta(\log(\frac{1}{|x|}))^{-1}\right)^{1-p_+(\lambda)}$, which is bounded in the ball $B_\eta(0)$. Consider $u_1 \equiv c_1 u$, then

$$-\Delta u_1 - \lambda\frac{u_1}{|x|^2} \geq c_1^{1-p}|\nabla u_1|^{p_+(\lambda)}.$$

Let c_0 be a fixed constant satisfying (2.5) when $\eta = e^{-1}$ and take $c_1 > 0$ such that $c_1 c_0 \geq 1$. Then for β suitable small we have

$$c_1^{1-p_+(\lambda)} \geq ||h||_\infty \beta^{\frac{1}{2}}.$$

Since $c_1 c_0 \geq 1$ we obtain that $u_1(x) \geq w(x)$ for $|x| = e^{-1}$ and moreover

$$-\Delta u_1 - \lambda\frac{u_1}{|x|^2} \geq \beta^{\frac{1}{2}} h(x)|\nabla u_1|^{p_+(\lambda)}.$$

Claim. $u_1 \geq w$

We call $v = w - u_1$. By using the regularity of w and by (2.6) we obtain that $v \in W^{1,p_+(\lambda)}(B_\eta(0))$, $v \leq 0$ on $\partial B_\eta(0)$ and

$$\int_{B_\eta(0)} \frac{|v|}{|x|^{2+\alpha_{(-)}}}dx < \infty, \quad \int_{B_\eta(0)} |\nabla v|^{p_+(\lambda)}|x|^{-\alpha_{(-)}}dx < \infty. \tag{2.7}$$

By a direct computation it follows that

$$-\Delta v - \lambda\frac{v}{|x|^2} \leq p_+(\lambda)h(x)\beta^{\frac{1}{2}}|\nabla w|^{p_+(\lambda)-2}\nabla w \nabla v \equiv a(x)\nabla v$$

where the vector field $a(x) = -\beta^{\frac{1}{2}} p_+(\lambda)\frac{x}{|x|^2} \in L^q(B_\eta(0))$ for all $q < N$. Notice that with the regularity of the vector field, a, we can not apply the comparison argument used in [6]. To overcame this lack of regularity we proceed as follows. Using Kato type inequality (see [19] and the extension in [14]) we get,

$$-\Delta v_+ - \lambda\frac{v_+}{|x|^2} + p_+(\lambda)\beta^{\frac{1}{2}}\left\langle\frac{x}{|x|^2}, \nabla v_+\right\rangle \leq 0 \text{ and } \int_{B_\eta(0)} |\nabla v_+|^{p_+}|x|^{-\alpha_{(-)}}dx < \infty,$$

and since $\frac{\alpha_{(-)}}{p_+(\lambda)} < \frac{N-2}{2}$, then by Hardy-Sobolev inequality applied to v_+ we obtain

$$\int_{B_\eta(0)} \frac{v_+^{p_+(\lambda)}}{|x|^{p_+(\lambda)+\alpha_{(-)}}}dx < \infty. \tag{2.8}$$

Define $\gamma = \frac{\beta^{\frac{1}{2}} p_+(\lambda)}{2}$ and consider the weight $|x|^{-2\gamma}$. Then for suitable β, $2\gamma < N - 2$ and hence $|x|^{-2\gamma}$ is an admissible weight in order to have Caffarelli-Kohn-Nirenberg inequalities (see [15]). Thus there results that[2]

$$\begin{aligned} &- \operatorname{div}(|x|^{-2\gamma}\nabla v_+) - \lambda\frac{v_+}{|x|^{2(\gamma+1)}} \\ &\quad = |x|^{-2\gamma}\left(-\Delta v_+ + p_+(\lambda)\left\langle\frac{x}{|x|^2}, \nabla v_+\right\rangle - \lambda\frac{v_+}{|x|^2}\right) \leq 0. \end{aligned} \tag{2.9}$$

[2] A detailed study of these equations related to the Caffarelli-Kohn-Nirenberg inequalities can be seen in [3] and the references therein.

Moreover, there exits $\sigma_1 > 2+\alpha_{(-)}$, depending only on N and λ such that

$$\int_{B_\eta(0)} \frac{v_+}{|x|^{\sigma_1}} dx < \infty. \tag{2.10}$$

Indeed,

$$\begin{aligned}\int_{B_\eta(0)} \frac{v_+}{|x|^{\sigma_1}} dx &= \int_{B_\eta(0)} \frac{v_+}{|x|^{\frac{p_+(\lambda)+\alpha_{(-)}}{p_+(\lambda)}}} \frac{1}{|x|^{\sigma_1 - \frac{p_+(\lambda)+\alpha_{(-)}}{p_+(\lambda)}}} dx \\ &\le \left(\int_{B_\eta(0)} \frac{v_+^{p_+(\lambda)}}{|x|^{p_+(\lambda)+\alpha_{(-)}}} dx\right)^{\frac{1}{p_+(\lambda)}} \left(\int_{B_\eta(0)} \frac{1}{|x|^{p'_+(\sigma_1 - \frac{p_+(\lambda)+\alpha_{(-)}}{p_+(\lambda)})}} dx\right)^{\frac{1}{p'_+}}.\end{aligned}$$

Denote $\theta(\sigma_1) = p'_+(\sigma_1 - \frac{p_+(\lambda)+\alpha_{(-)}}{p_+(\lambda)})$. Since $p_+(\lambda) = \frac{2+\alpha_{(-)}}{1+\alpha_{(-)}}$, then $p'_+ = 2+\alpha_{(-)}$, the conjugate of $p_+(\lambda)$, hence there result that $\theta(\sigma_1) = (2+\alpha_{(-)})(\sigma_1 - 1) - \alpha_{(-)}(1+\alpha_{(-)})$. By a direct computation we get $\theta(2+\alpha_{(-)}) = 2(1+\alpha_{(-)}) = N - 2\sqrt{\Lambda_N - \lambda} < N$, then there exists $\sigma_1 > 2+\alpha_{(-)}$ such that $\theta(\sigma_1) < N$ and then $\int_{B_\eta(0)} |x|^{-(p'_+(\sigma_1 - \frac{p_+(\lambda)+\alpha_{(-)}}{p_+(\lambda)}))} dx < \infty$. Thus (2.10) holds.
The idea should be to use φ, the solution to problem

$$\begin{cases} -\text{div}(|x|^{-2\gamma}\nabla\varphi) - \lambda \dfrac{\varphi}{|x|^{2(\gamma+1)}} = \dfrac{1}{|x|^{2(\gamma+1)}} & \text{in } B_\eta(0), \\ \varphi = 0 & \text{on } \partial B_\eta(0), \end{cases}$$

as a test function in (2.9). A direct calculation shows that

$$\varphi(x) = \frac{1}{|x|^a} - \frac{1}{\eta^a} \text{ where } a = \frac{N-2(\gamma+1)}{2} - \sqrt{\left(\frac{N-2(\gamma+1)}{2}\right)^2 - \lambda},$$

that has not the required regularity to be used directly as a test function in (2.9). Therefore, we consider the approximating sequence,

$$\varphi_n(x) = \frac{1}{(|x|+\frac{1}{n})^a} - \frac{1}{(\eta+\frac{1}{n})^a},$$

then $\varphi_n \in \mathcal{C}^1(B_\eta(0))$, $\varphi_n = 0$ on $\partial B_\eta(0)$,

$$\begin{aligned}\nabla\varphi_n(x) = -\frac{a}{(|x|+\frac{1}{n})^{a+1}} \frac{x}{|x|} \text{ and } &-\text{div}(|x|^{-2\gamma}\nabla\varphi_n)) \\ &= |x|^{-2\gamma}\left(\frac{a(N-1-2\gamma)}{|x|(|x|+\frac{1}{n})^{a+1}} - \frac{a(a+1)}{(|x|+\frac{1}{n})^{a+2}}\right).\end{aligned}$$

Notice that

$$\int_{B_\eta(0))} |x|^{-2\gamma} |\nabla v_+||\nabla \varphi_n| dx < \infty \text{ and } \int_{B_\eta(0))} \frac{v_+\,\varphi_n}{|x|^{2(\gamma+1)}} dx < \infty,$$

then choosing φ_n as a test function in (2.9) we obtain that

$$\int_{B_\eta(0))} v(-\text{div}(|x|^{-2\gamma} \nabla \varphi_n)) dx - \lambda \int_{B_\eta(0))} \frac{v_+ \varphi_n}{|x|^{2(\gamma+1)}} dx \leq 0. \tag{2.11}$$

By the definition of w_n we have,

$$\frac{v_+ \varphi_n}{|x|^{2(\gamma+1)}} \leq \frac{v_+}{|x|^{a+2(\gamma+1)}} + \frac{2}{\eta^a} \frac{v_+}{|x|^{2(\gamma+1)}}.$$

Since $a + 2(\gamma + 1) \to 2 + \alpha_{(-)}$ as $\gamma \to 0$, then by choosing β small we find γ small such that $a + 2(\gamma + 1) < \sigma_1$. Hence $\frac{v_+\varphi_n}{|x|^{2(\gamma+1)}} \leq \frac{C\,v_+}{|x|^{\sigma_1}}$. Then by definition of σ_1 and using the dominated convergence theorem, we easily prove that

$$\int_{B_\eta(0))} \frac{v_+ \varphi_n}{|x|^{2(\gamma+1)}} dx \to \int_{B_\eta(0))} \frac{v_+}{|x|^{a+2(\gamma+1)}} dx - \frac{1}{\eta^a} \int_{B_\eta(0))} \frac{v_+}{|x|^{2(\gamma+1)}} dx \text{ as } n \to \infty.$$

We deal now with the first term in (2.11),

$$\begin{aligned} \left| v_+ \text{div}(|x|^{-2\gamma} \nabla \varphi_n) \right| &= \left| \left(\frac{a(N-1-2\gamma)v_+}{|x|^{1+2\gamma}(|x| + \frac{1}{n})^{a+1}} - \frac{a(a+1)v_+}{|x|^{2\gamma}(|x| + \frac{1}{n})^{a+2}} \right) \right| \\ &\leq \frac{a(N-1-2\gamma)v_+}{|x|^{1+2\gamma}(|x| + \frac{1}{n})^{a+1}} + \frac{a(a+1)v_+}{|x|^{2\gamma}(|x| + \frac{1}{n})^{a+2}}. \end{aligned}$$

As above it is not difficult to see that

$$\frac{a(N-1-2\gamma)v_+}{|x|^{1+2\gamma}(|x| + \frac{1}{n})^{a+1}} + \frac{a(a+1)v_+}{|x|^{2\gamma}(|x| + \frac{1}{n})^{a+2}} \leq \frac{a(N+a-2\gamma)v_+}{|x|^{\sigma_1}}$$

and then by the dominated convergence theorem we obtain

$$\int_{B_\eta(0))} v_+ \text{div}(|x|^{-2\gamma} \nabla \varphi_n) dx \to \int_{B_\eta(0))} \frac{a(N-a-2(\gamma+1))v_+}{|x|^{2(1+\gamma)+a}} dx \text{ as } n \to \infty.$$

Hence passing to the limit in (2.11) and taking into account that $a(N - a - 2(\gamma + 1)) - \lambda = 0$, there result that

$$\begin{aligned} &\int_{B_\eta(0))} v(-\text{div}(|x|^{-2\gamma} \nabla \varphi_n)) dx - \lambda \int_{B_\eta(0))} \frac{v_+ \varphi_n}{|x|^{2(\gamma+1)}} dx \to \\ &\to \frac{1}{\eta^a} \int_{B_\eta(0))} \frac{v_+}{|x|^{2(1+\gamma)}} dx, \quad \text{as } n \to \infty, \end{aligned}$$

thus, according with (2.11), $\int_{B_\eta(0))} \frac{v_+}{|x|^{2(1+\gamma)}} dx \leq 0$, hence $v_+ \equiv 0$ and then $u_1 \geq w$.

To finish the proof in this case we use the same argument as in the first step. More precisely for all $\phi \in \mathcal{C}_0^\infty(B_r(0))$, $0 < r << \eta$ we have

$$c_1 \int_{B_r(0)} \frac{u_1|\phi|^{p'_+}}{|x|^2} dx \leq \int_{B_r(0)} |\nabla\phi|^{p'_+} dx \tag{2.12}$$

where $c_1 > 0$ is independent of ϕ. Using the result of the claim and by the fact that $p'_+ = \alpha_{(-)} + 2$ we obtain that,

$$c_2 \int_{B_r(0)} \frac{|\phi|^{p'_+}}{|x|^{p'_+}} \left(\log\left(\frac{1}{|x|}\right)\right)^\beta dx \leq \int_{B_r(0)} |\nabla\phi|^{p'_+} dx$$

a contradiction with Hardy inequality in $W_0^{1,p'_+}(B_r(0))$. Hence the result follows.

Third step: $p = p_+(\lambda)$ **and** $\lambda = \Lambda_N$

Assume by contradiction that problem (1.2) has a positive very weak super-solution u. In this case $\alpha_{(-)} = \frac{N-2}{2}$ and $p_+(\lambda) = \frac{N+2}{N}$, hence by Lemma 2.2 we obtain that $u(x) \geq c|x|^{-\alpha_{(-)}}$ and by Lemma 2.3

$$\int_{B_\eta(0)} |\nabla u|^{p_+(\lambda)} |x|^{-\alpha_{(-)}} dx < \infty.$$

We consider $\phi \in \mathcal{C}_0^\infty(B_\eta(0))$ such that $\phi \geq 0$ and $\phi = 1$ in $B_{\eta_1}(0)$, then by the regularity of u we obtain $\int_{B_\eta(0)} |\nabla(\phi u)|^{p_+(\lambda)}|x|^{-\alpha_{(-)}}dx$. Since $\frac{\alpha_{(-)}}{p_+(\lambda)} = \frac{N(N-2)}{2(N+2)} < N$, we can apply Caffarelli-Kohn-Nirenberg inequalities to obtain that

$$C_1 \int_{B_\eta(0)} (\phi u)^{p_+(\lambda)} |x|^{-\alpha_{(-)}} dx \leq \int_{B_\eta(0)} |\nabla(\phi u)|^{p_+(\lambda)} |x|^{-\alpha_{(-)}} dx < \infty.$$

$$\int_{B_{\eta_1}(0)} u^{p_+(\lambda)} |x|^{-\alpha_{(-)}} dx < \infty \text{ for some } \eta_1 < \eta$$

Therefore we conclude that $u \in \mathcal{D}^{1,p^+}_{\alpha_{(-)}}(B_{\eta_1}(0))$, which is defined as the completion of $\mathcal{C}^\infty(\overline{B_\eta(0)})$ with respect to the norm

$$||\phi||^{p_+(\lambda)}_{\mathcal{D}^{1,p^+}_{\alpha_{(-)}}} = \int_{B_{\eta_1}(0)} |\phi|^{p_+(\lambda)} |x|^{-\alpha_{(-)}} dx + \int_{B_{\eta_1}(0)} |\nabla\phi|^{p_+(\lambda)} |x|^{-\alpha_{(-)}} dx.$$

It is not difficult to see that for all $\phi \in \mathcal{D}^{1,p^+}_{\alpha_{(-)}}(B_\eta(0))$ we have

$$\begin{aligned} C_2 \int_{B_{\eta_1}(0)} & \frac{|\phi|^{p_+(\lambda)}}{|x|^{\alpha_{(-)}+p_+(\lambda)}} dx \\ & \leq \int_{B_{\eta_1}(0)} |\phi|^{p_+(\lambda)} |x|^{-\alpha_{(-)}} dx + \int_{B_{\eta_1}(0)} |\nabla\phi|^{p_+(\lambda)} |x|^{-\alpha_{(-)}} dx \end{aligned}$$

where $C_2 > 0$ is independent of ϕ, in particular,

$$\int_{B_{\eta_1}(0)} \frac{u^{p_+(\lambda)}}{|x|^{\alpha_{(-)}+p_+(\lambda)}} dx < \infty. \tag{2.13}$$

Using the fact that $u(x) \geq c|x|^{-\alpha_{(-)}}$ and since $\alpha_{(-)} + p_+(\lambda) + \alpha_{(-)} p_+(\lambda) = N$, we reach a contradiction with (2.13). Hence the nonexistence result follows. □

Remark 2.6.

1. Notice that $p_+(\lambda) < 2$, for all $\lambda \in (0, \Lambda_N]$, hence for $p = 2$ we easily obtain the nonexistence result by the first step in Theorem 2.5. Moreover, $p_+(\lambda) \to \frac{N+2}{N}$ if $\lambda \to \Lambda_N$ and $p_+(\lambda) \to 2$ if $\lambda \to 0$. As a consequence, we find a discontinuity with the known results for $\lambda = 0$. See, for instance, [17].
2. If $1 < p \leq \frac{N}{N-1}$, then problem (1.2) has non very weak positive solution in $\mathbb{R}^N$. This follows using the results in [6] and [17]. For the reader convenient we include a proof.
   We argue by contradiction. Assume that (1.2) has a positive solution u with $1 < p \leq \frac{N}{N-1}$. It is not difficult to see, using the strong maximum principle, that for any compact set $K \subset \Omega$ there exists a positive constant $c(K)$ such that $u \geq c(K)$. Let $\phi \in \mathcal{C}_0^\infty(\Omega)$, then using $|\phi|^{p'}$ as a test function in (1.2) we obtain that

   $$p' \int_{\mathbb{R}^N} |\nabla u||\nabla \phi||\phi|^{p'-1} dx \geq \int_{\mathbb{R}^N} |\nabla u|^p |\phi|^{p'} dx + \lambda \int_{\mathbb{R}^N} \frac{u}{|x|^2} |\phi|^{p'} dx.$$

   Using Young inequalities we conclude that

   $$\int_{\mathbb{R}^N} |\nabla \phi|^{p'} dx \geq c_1 \lambda \int_{\mathbb{R}^N} \frac{u}{|x|^2} |\phi|^{p'} dx.$$

   Since $p' > N$, then $Cap_{1,p'}(K) = 0$ for any compact set of $\mathbb{R}^N$. Thus, there exists a sequence $\{\phi_n\} \subset \mathcal{C}_0^\infty(\mathbb{R}^N)$ such that $\phi \geq \chi_K$ and $||\nabla \phi_n||_{L^{p'}(\mathbb{R}^N)} \to 0$ as $n \to \infty$. Hence by substituting in the last inequality we reach a contradiction.
3. Despite the previous remark, in bounded domains there are no restriction on p from below. This follows by the fact that the relative q-capacity, $q > N$, of a ball with respect to a concentric bigger ball is not zero. See [23], page 106.

3. Blow up result

As a consequence of the non existence result, we obtain the next blow-up behavior for approximated problems.

Theorem 3.1. *Assume that $p \geq p_+(\lambda)$. If $u_n \in W_0^{1,p}(\Omega)$ is a solution to problem*

$$\begin{cases} -\Delta u_n = |\nabla u_n|^p + \lambda a_n(x) u_n + \alpha f & \text{in } \Omega, \\ u_n > 0 & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

with $f \geq 0$, $f \neq 0$ and $a_n(x) = \frac{1}{|x|^2 + \frac{1}{n}}$, then $u_n(x_0) \to \infty$, $\forall x_0 \in \Omega$.

To prove Theorem 3.1, we need the following lemma that extends Lemma 5.2 in [6].

Lemma 3.2. *Assume that $\{u_n\}$ is a sequence of positive functions such that $\{u_n\}$ is uniformly bounded in $W^{1,p}_{\text{loc}}(\Omega)$ for some $1 < p \leq 2$ with $u_n \rightharpoonup u$ weakly in $W^{1,p}_{\text{loc}}(\Omega)$ and that $u_n \leq u$ for all $n \in \mathbb{N}$. Assume that $-\Delta u_n \geq 0$ in $\mathcal{D}'(\Omega)$ and if $p < 2$, that the sequence $\{T_k(u_n)\}$ is uniformly bounded in $W^{1,2}_{\text{loc}}(\Omega)$ for k fixed. Then $\nabla T_k(u_n) \to \nabla T_k(u)$ strongly in $(L^2_{\text{loc}}(\Omega))^N$.*

Proof. Notice that the hypothesis on the boundedness of $\{T_k(u_n)\}$ in $W^{1,2}_{\text{loc}}(\Omega)$ is needed just when $p < 2$. Therefore by hypothesis we conclude that

$$||\nabla T_k(u)||_{L^2(K)} \leq ||\nabla T_k(u_n)||_{L^2(K)} \text{ for all bounded regular domain } K \subset\subset \Omega.$$

Let $\phi \in \mathcal{C}_0^\infty(\Omega)$ be a positive function, then since $u_n \leq u$ we get

$$\int_\Omega -\Delta u_n (T_k(u_n)\phi) dx \leq \int_\Omega -\Delta u_n (T_k(u)\phi) dx.$$

Notice that

$$\int_\Omega -\Delta u_n (T_k(u_n)\phi) dx = \int_\Omega \phi |\nabla T_k(u_n)|^2 dx + \int_\Omega T_k(u_n) \nabla\phi \nabla u_n dx. \tag{3.2}$$

On the other hand, as $u_n \leq u$,

$$\begin{aligned} \int_\Omega -\Delta u_n (T_k(u)\phi) dx &= \int_\Omega \phi \nabla u_n \nabla T_k(u) dx + \int_\Omega T_k(u) \nabla\phi \nabla u_n dx \\ &= \int_\Omega \phi \nabla T_k(u_n) \nabla T_k(u) dx + \int_\Omega T_k(u) \nabla\phi \nabla u_n dx \\ &\leq \frac{1}{2}\int_\Omega \phi |\nabla T_k(u_n)|^2 dx + \frac{1}{2}\int_\Omega \phi |\nabla T_k(u)|^2 dx \\ &\quad + \int_\Omega T_k(u) \nabla\phi \nabla u_n dx \end{aligned}$$

Thus by the above computation and (3.2) there result

$$\frac{1}{2}\int_\Omega \phi|\nabla T_k(u_n)|^2 dx \le \frac{1}{2}\int_\Omega \phi|\nabla T_k(u)|^2 dx + \int_\Omega (T_k(u) - T_k(u_n))\nabla\phi\nabla u_n dx.$$

Hence we conclude that

$$\limsup_{n\to\infty} \int_\Omega \phi\Big(|\nabla T_k(u_n)|^2 - |\nabla T_k(u)|^2\Big)dx \le 0 \text{ for all positive test function } \phi.$$

We set $w_n = \phi T_k(u_n)$ and $w = \phi T_k(u)$ where ϕ is a positive test function, then $w_n \rightharpoonup w$ weakly in $W^{1,2}_{\text{loc}}(\Omega)$. Notice that $w_n \to w$ strongly in $L^2_{\text{loc}}(\Omega)$. Therefore using the above computation we get easily that

$$0 \le \limsup_{n\to\infty}(||\nabla w_n||_{L^2_{\text{loc}}(\Omega)} - ||\nabla w||_{L^2_{\text{loc}}(\Omega)}) \le 0.$$

Thus using the definition of the weak limit and using the strong convergence of w_n to w in $L^2_{\text{loc}}(\Omega)$ we get the desired result. □

Remark 3.3. From an anonymous referee we learnt that the previous lemma is related to a result by F. Murat in [22]. We thank for the information, add the reference and, for the convenience of the reader, we maintain the proof.

Lemma 3.4. *Let g be a positive function such that $g \in L^\rho(\Omega)$ with $\rho > \frac{N}{2}$ and $s > 0$. Assume that w_1, w_2 are positive functions such that $w_1, w_2 \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ verifying*

$$\begin{cases} -\Delta w_1 \le \dfrac{|\nabla w_1|^p}{1+s|\nabla w_1|^p} + g & \text{in } \Omega, \\ w_1 = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

and

$$\begin{cases} -\Delta w_2 \ge \dfrac{|\nabla w_2|^p}{1+s|\nabla w_2|^p} + g & \text{in } \Omega, \\ w_2 = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.4}$$

then $w_2 \ge w_1$ in Ω.

Proof. Consider $w = w_1 - w_2$, then $w \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$. We will prove that $w^+ = 0$. By (3.3) and (3.4) it follows that

$$-\Delta w \le \frac{|\nabla w_1|^p}{1+s|\nabla w_1|^p} - \frac{|\nabla w_2|^p}{1+s|\nabla w_2|^p}.$$

Now for $x, y \in \mathbb{R}^N$ we define the function ρ by setting

$$\rho(t) = T(t|x| + (1-t)|y|) \text{ where } T(t) = \frac{|t|^p}{1+s|t|^p}.$$

For $x = \nabla w_1$ and $y = \nabla w_2$ we have

$$\frac{|\nabla w_1|^p}{1+s|\nabla w_1|^p} - \frac{|\nabla w_2|^p}{1+s|\nabla w_2|^p} = \rho(1) - \rho(0) = \rho'(\theta)\,.$$

Since

$$|\rho(1)-\rho(0)| \le \Big||\nabla w_1| - |\nabla w_2|\Big||T'(\theta)| \le |\nabla w_1 - \nabla w_2||T'(\theta)|$$

and $|T'(t)| = p\frac{|t|^{p-1}}{(1+s|t|^p)^2} \le C$, we conclude that

$$\left|\frac{|\nabla w_1|^p}{1+s|\nabla w_1|^p} - \frac{|\nabla w_2|^p}{1+s|\nabla w_2|^p}\right| \le C|\nabla w|.$$

Hence it follows that

$$-\Delta w \le C|\nabla w|, \qquad w \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega).$$

Using Kato inequality we get

$$-\Delta w_+ \le C|\nabla w_+|, \qquad 0 \le w_+ \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega).$$

Therefore, using the maximum principle in Lemma 4.6 of [6] it follows that $w_+ \equiv 0$ and then we obtain the result. □

Now we are able to prove the blow-up result.

Proof of Theorem 3.1. Without loss of generality, we can assume that $f \in L^\infty(\Omega)$ and that λ is small enough. Assume the existence of $x_0 \in \Omega$ such that $u_n(x_0) \le C$ for all n. Using the extended maximum principle obtained in [12], there exists a structural positive constant C' (independent of u_n), such that

$$C \ge u_n(x_0) \ge C'(\Omega)\delta(x_0)\int_\Omega (\lambda a_n(x)u_n + |\nabla u_n|^p + f)\delta(x)\,dx, \tag{3.5}$$

where $\delta(x) = \text{dist}(x, \partial\Omega)$. Let $\phi \in \mathcal{C}_0^\infty(\Omega)$ be a positive function, by using $T_k(u_n)\phi$ as a test function in (3.1), we can prove that $T_k(u_n)$ is uniformly bounded in $W^{1,2}_{\text{loc}}(\Omega)$.

For n fixed, we consider $v_j \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ the minimal positive solution to problem,

$$\begin{cases} -\Delta v_j = \lambda a_n(x) v_j + \dfrac{|\nabla v_j|^p}{1+\frac{1}{j}|\nabla v_j|^p} + \alpha f & \text{in } \Omega, \\ v_j = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.6}$$

Using an iteration argument as in [6], it follows that $v_j \le v_{j+1}$ and $v_j \le u_n$ for every n. Define

$$w_n = \lim_{j\to\infty} v_j \le u_n.$$

Claim. The following statements hold:

a) $\{T_k(w_n)\}$ is bounded in $W^{1,2}_{\rm loc}(\Omega)$.
b) $w_n \in W^{1,p}_{\rm loc}(\Omega)$.
c) w_n is a supersolution to problem (3.1).
d) $w_n \le w_{n+1}$.

Assume the claim holds. As above there exists a positive structural constant C', such that

$$C \ge w_n(x_0) \ge C'(\Omega)\delta(x_0) \int_\Omega (\lambda a_n(x) w_n + |\nabla w_n|^p + f)\delta(x)\, dx.$$

Since $\{w_n\}$ is a monotone sequence we conclude that

$$a_n(x) w_n \nearrow \frac{w}{|x|^2} \text{ in } L^1_{\rm loc}(\Omega) \text{ and } \int_\Omega |\nabla w_n|^p \delta(x) \le C'.$$

Thus $\{w_n\}$ is bounded in $W^{1,p}_{\rm loc}(\Omega)$, hence using (1) in the claim and Lemma 3.2, we conclude that

$$T_k(w_n) \to T_k(w) \text{ strongly in } W^{1,2}_{\rm loc}(\Omega).$$

Since w_n is a supersolution to (3.1), then by letting $n \to \infty$ we obtain that w satisfies to

$$-\Delta w \ge |\nabla w|^p + \lambda \frac{w}{|x|^2} + c\, f$$

a contradiction with Theorem 2.5.

Proof of the claim. a) and d) follows directly from equation (3.6) by application of the corresponding inequality of type (3.5), and the fact that a_n is a nondecreasing sequence. To prove b) we consider separately two cases: i) $p \le 2$ and ii) $p > 2$. For $p \le 2$ we have that

$$T_k(v_j) \to T_k(w_n) \text{ as } j \to \infty, \text{ strongly in } W^{1,2}_{\rm loc}(\Omega). \tag{3.7}$$

To obtain the convergence, we use a nonlinear test function as in [8]. (See too [10] and [9]). Consider $\phi(s) = s\, e^{\frac{1}{4}s^2}$, in such a way that $\phi'(s) - |\phi(s)| \ge \frac{1}{2}$. For $\psi \in C_0^\infty(\Omega)$, $\psi \ge 0$, take $\phi(T_k(v_j) - T_k(w_n))\psi(x)$ as test function in equation (3.6). We obtain from the left hand side,

$$\begin{aligned}&\int_\Omega \nabla v_j \phi'(T_k(v_j) - T_k(w_n))\nabla(T_k(v_j) - T_k(w_n))\psi\, dx\\ &\quad = \int_\Omega |\nabla(T_k(v_j) - T_k(w_n))|^2 \phi'(T_k(v_j) - T_k(w_n))\psi dx + o(1).\end{aligned}$$

We set $H(\nabla v_j) = \frac{|\nabla v_j|^p}{1+\frac{1}{j}|\nabla v_j|^p}$. Then the right hand side could be estimated by,

$$\int_\Omega H(\nabla v_j)\phi(T_k(v_j) - T_k(w_n))\psi\, dx$$
$$\leq \delta \int_\Omega |\nabla T_k(v_j) - \nabla T_k(w_n)|^2\, |\phi(T_k(v_j) - T_k(w_n))|\psi\, dx + o(1)$$

where $\delta \leq 1$. Since

$$\int\limits_\Omega \Big(\lambda a_n(x)v_j + \alpha f\Big)\phi(T_k(v_j) - T_k(w_n))\psi(x)dx \to 0 \text{ as } m \to \infty,$$

we conclude the required convergence and in particular the almost everywhere convergence up to a subsequence.

In the case $p \geq 2$ the result is directly obtained as follows,

$$\int_\Omega |\nabla v_j|^2 dx = \int_\Omega (-\Delta v_j)v_j dx \leq \int_\Omega (-\Delta v_j)u_n dx$$
$$\leq \left(\int_\Omega |\nabla v_j|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega |\nabla u_n|^2 dx\right)^{\frac{1}{2}},$$

then $v_j \rightharpoonup w_n$ weakly in $W_0^{1,2}(\Omega)$ as $j \to \infty$. By using the last inequality and the weak lower semi-continuity of the norm there result that

$$\int_\Omega |\nabla w_n|^2 dx \leq \liminf_{j\to\infty} \int_\Omega |\nabla v_j|^2 dx \leq \limsup_{j\to\infty} \int_\Omega |\nabla v_j|^2 dx.$$

Moreover, taking into account that $-\Delta v_j \geq 0$,

$$\int_\Omega |\nabla v_j|^2 dx = \int_\Omega (-\Delta v_j)v_j dx \leq \int_\Omega (-\Delta v_j)w_n dx$$
$$\leq \left(\int_\Omega |\nabla v_j|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega |\nabla w_n|^2 dx\right)^{\frac{1}{2}},$$

hence

$$\limsup_{j\to\infty} \int_\Omega |\nabla v_j|^2 dx \leq \int_\Omega |\nabla w_n|^2 dx.$$

Then we conclude the strong convergence in $W_0^{1,2}(\Omega)$. In particular we have the almost everywhere convergence of the gradients and therefore to conclude the proof of b) it is sufficient to observe that

$$\begin{aligned} C \geq u_n(x_0) &\geq C'(\Omega)\delta(x_0) \int\limits_\Omega \left(\lambda a_n(x)v_j + \frac{|\nabla v_j|^p}{1+\frac{1}{j}|\nabla v_j|^p} + f\right)\delta(x)\, dx \\ &\geq C'(\Omega)\delta(x_0) \int\limits_\Omega (\lambda a_n(x)w_n + |\nabla w_n|^p + f)\delta(x)\, dx, \end{aligned} \tag{3.8}$$

by Fatou's lemma. To prove c), we use a nonnegative test function in problem (3.6) and we pass to the limit by Fatou's lemma.

4. Existence result: $1 < p < p_+(\lambda)$ and $\lambda < \Lambda_N$

We consider $\alpha_{(+)}$ and $\alpha_{(-)}$ defined in (1.1). Joint to the critical exponent $p_+(\lambda) \equiv \frac{2+\alpha_{(-)}}{1+\alpha_{(-)}}$ we define

$$p_-(\lambda) \equiv \frac{2+\alpha_{(+)}}{1+\alpha_{(+)}}, \qquad \text{that verifies} \qquad p_-(\lambda) \le p_+(\lambda).$$

We have the following result.

Theorem 4.1. *Assume that $p_-(\lambda) < p < p_+(\lambda)$ where $p_-(\lambda)$, $p_+(\lambda)$ are given above. Then problem* (1.2) *with $f \equiv 0$ has a very weak solution $u > 0$ in $\mathbb{R}^N$.*

Proof. We search a solution in the form $u(x) = A|x|^{-\beta}$. Hence by a direct computation we obtain that $\beta = \frac{2-p}{p-1}$ and

$$\beta^p A^{p-1} = \beta(N - \beta - 2) - \lambda.$$

To have $A > 0$ we need $\beta \in (\alpha_{(-)}, \alpha_{(+)})$ which is equivalent to $p_-(\lambda) < p < p_+(\lambda)$. Notice that $u \in L^1_{\text{loc}}(\Omega)$, $\frac{u}{|x|^2} \in L^1_{\text{loc}}$ and since $\frac{N}{N-1} < p_-(\lambda) < p$, $|\nabla u|^p \in L^1_{\text{loc}}$. (Compare with Remark 2.6 2.). Hence the result follows. ☐

Remark 4.2. The solution w in Theorem 4.1 is in the space $W^{1,2}_{\text{loc}}(\mathbb{R}^N)$ if and only if $p > \frac{N+2}{N}$. Notice that for all $\lambda \in [0, \Lambda_N)$, $\frac{N+2}{N} \in (p_-(\lambda), p_+(\lambda))$ and if $\lambda = \Lambda_N$ then $\frac{N+2}{N} = p_-(\lambda) = p_+(\lambda)$.

We deal now with the existence of solutions to Dirichlet problem in bounded domain.

Theorem 4.3. *Assume that $1 < p < p_+(\lambda)$ where $p_+(\lambda) = \frac{2+\alpha_{(-)}}{1+\alpha_{(-)}}$. There exists c_0 such that if $c < c_0$ and $f(x) \le \frac{1}{|x|^2}$, then problem*

$$\begin{cases} -\Delta u = |\nabla u|^p + \lambda \dfrac{u}{|x|^2} + c f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

has a very weak positive solution u.

Proof. Assume that for $c > 0$ and $f(x) \leq \frac{1}{|x|^2}$, we are able to find a positive supersolution $\overline{w} \in W^{1,p}(\Omega)$ to problem (4.1) such that

$$\exists s > 0, \text{ for which } \frac{\overline{w}^{1+s}}{|x|^2}, \overline{w}^{\frac{(2-p)s+p}{2-p}} \in L^1(\Omega). \tag{4.2}$$

Consider $a_n(x) = \frac{1}{|x|^2+\frac{1}{n}} \uparrow |x|^{-2}$, $f_n = \min\{f, n\} \uparrow f$, then problem

$$\begin{cases} -\Delta u_n = \lambda a_n(x) u_n + \dfrac{|\nabla u_n|^p}{1+\frac{1}{n}|\nabla u_n|^p} + c f_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.3}$$

has a minimal positive solution $u_n \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$. By Lemma 3.4 and using the comparison principle in [6], we get that $u_n \leq u_{n+1}$ and $u_n \leq \overline{w}$ for every n. Hence $\overline{u} = \lim_{n\to\infty} u_n \leq w$. Define $\phi_n = (1+u_n)^s - 1$, where s is as in (4.2). Then using ϕ_n as a test function in (4.3), there result

$$\int_\Omega \frac{|\nabla u_n|^2}{(1+u_n)^{1-s}} dx \leq C_1, \quad \int_\Omega |\nabla u_n|^p (1+u_n)^s dx \leq C_2,$$

therefore, in particular

$$\frac{1}{k}\int_\Omega |\nabla T_k u_n|^2 \leq C_3, \quad \int_\Omega |\nabla u_n|^p \leq C_4.$$

Let us consider $\phi(s) = s\, \mathrm{e}^{\frac{1}{4}s^2}$ and consider $\phi(T_k u_n - T_k \overline{u})$ as a test function in (4.3) then by the convergence arguments used in [9], we obtain

$$\nabla T_k u_n \to \nabla T_k \overline{u} \text{ as } n \to \infty \text{ strongly in } W_0^{1,2}(\Omega).$$

In particular $\nabla u_n \to \nabla \overline{u}$ almost everywhere in Ω.

Let $G_k(t) = t - T_k(t)$, then using $\psi_n \equiv (1+G_k(u_n))^s - 1$, as test function in (4.3), there result that

$$\limsup_{k\to\infty} \int_{u_n \geq k} |\nabla u_n|^p dx \leq \limsup_{k\to\infty} \int_\Omega |\nabla G_k(u_n)|^p (1+G_k(u_n))^s dx = 0,$$

uniformly in n. Vitali's lemma allow us to conclude that

$$\nabla u_n \to \nabla \overline{u}, \quad n \to \infty, \text{ strongly in } L^p(\Omega).$$

Hence $\overline{u}$ is a very weak solution to problem (4.1).

It is worthy to point out that for the values of p for which a super-solution in $W_0^{1,2}(\Omega)$ exists (in particular if $1 < p \leq p_-(\lambda)$), the proof is easier and, moreover, the solution $\overline{u} \in W_0^{1,2}(\Omega)$. In this last case it suffices to take u_n as test function and to use Lemma 5.3 in [6].

To find the required super-solution we will consider two cases:

i) $p_- < p < p_+(\lambda)$
ii) $1 < p \le p_-$.

Case i): $p_- < p < p_+(\lambda)$

Consider u the radial solution obtained in Theorem 4.1, then $\frac{u^{1+s}}{|x|^2}$, $u^{\frac{(2-p)s+p}{2-p}} \in L^1(\Omega)$ for all $0 < s < \frac{p(N-1)-N}{2-p} < 1$. Define $v(x)$ to be the unique solution to problem

$$\begin{cases} -\Delta v = 0 & \text{in } \Omega, \\ v = u & \text{on } \partial\Omega, \end{cases}$$

Notice that $v \in \mathcal{C}^\infty(\Omega)$ and $0 < c_1 \le v \le c_2$ for some positive constant c_1 and c_2.

We set $\overline{w} = t(u - v)$, $t > 0$, it is clear that $\overline{w} \in W_0^{1,p}(\Omega)$, $w \ge 0$ in Ω and

$$\begin{aligned} -\Delta\overline{w} - \lambda\frac{\overline{w}}{|x|^2} &= t(-\Delta u - \lambda\frac{u}{|x|^2}) + t\lambda\frac{v}{|x|^2} = t|\nabla u|^p + t\lambda\frac{v}{|x|^2} \\ &\ge t\left(\frac{1}{(1+\varepsilon)^{p-1}}|\nabla\overline{w}|^p t^{-p} - \left(\frac{1+\frac{1}{\varepsilon}}{1+\varepsilon}\right)^{p-1}|\nabla v|^p\right) + t\lambda\frac{v}{|x|^2} \end{aligned}$$

where in the last estimate we have used the following elemental inequality,

$$|a+b|^p \le (1+\varepsilon)^{p-1}|a|^p + \left(1+\frac{1}{\varepsilon}\right)^{p-1}|b|^p.$$

Taking $t = \frac{1}{1+\varepsilon}$, we conclude that

$$-\Delta\overline{w} - \lambda\frac{\overline{w}}{|x|^2} \ge |\nabla\overline{w}|^p + \frac{\lambda}{1+\varepsilon}\frac{v}{|x|^2} - \frac{1}{\varepsilon^{p-1}(1+\varepsilon)}|\nabla v|^p.$$

Hence choosing ε large enough there exists a positive constant c_0 such that

$$\frac{\lambda}{1+\varepsilon}\frac{v}{|x|^2} - \frac{1}{\varepsilon^{p-1}(1+\varepsilon)}|\nabla v|^p \ge \frac{c_0}{|x|^2}.$$

Since $|x|^2 f(x) < 1$, therefore $\overline{w} \in W_0^{1,p}(\Omega)$ is a super-solution to problem (4.1) if $c < c_0$. Hence we conclude.

Case ii): $1 < p \le p_-$

We start by getting a super-solution in a ball, *i.e.*, $\Omega = B_R(0)$. Without loss of generality we will assume $R = 1$.

Since $p \le p_-$, there exists $\beta \in (\alpha_{(-)}, \alpha_{(+)})$, close to $\alpha_{(-)}$, such that $p(\beta + 1) < \beta + 2$.

Define $\overline{w}(x) \equiv A(|x|^{-\beta} - 1)$. Then $\overline{w} \in W_0^{1,p}(B_R(0))$ and

$$-\Delta \overline{w} - \lambda \frac{\overline{w}}{|x|^2} = A(\beta(N-\beta-2) - \lambda)|x|^{-\beta-2} + \frac{A}{|x|^2}.$$

Since $\beta \in (\alpha_{(-)}, \alpha_{(+)})$, then $\beta(N-\beta-2) - \lambda > 0$. Hence choosing $A^{p-1} = \frac{\beta(N-\beta-2)-\lambda)}{\beta^p}$ we obtain that

$$-\Delta \overline{w} - \lambda \frac{\overline{w}}{|x|^2} \ge |\nabla \overline{w}|^p + \frac{A}{|x|^2}.$$

Namely if $c_0 = A$, $\overline{w}$ is a super-solution to (4.1) in $B_1(0)$ for all $c < c_0$.

In the case of a general domain Ω that contains the origin we consider a ball $B_R(0)$ such that $\Omega \subset B_{\frac{R}{2}}(0)$. We have the corresponding super-solution in $B_R(0)$ found above, for which we perform the same arguments as in the first case. $\square$

5. The case where $\lambda \equiv \Lambda_N$ and $p < \frac{N+2}{N}$

Assume that $\lambda = \Lambda_N$ and $p < p_+ \equiv \frac{N+2}{N}$ and consider the function

$$w(x) = \left|\frac{x}{R}\right|^{-\frac{N-2}{2}} \left(\log\left(\frac{R}{|x|}\right)\right)^{1/2} - A,$$

where $A = (\frac{r}{R})^{-\frac{N-2}{2}}(\log(\frac{R}{|r|}))^{1/2}$, then $w(x) = 0$ if $|x| = r$. It is not difficult to see that $w \in W_0^{1,q}(B_r(0))$ for all $q < 2$ and

$$-\Delta w - \Lambda_N \frac{w}{|x|^2} = \frac{1}{4}\frac{w}{|x|^2}\left(\log\left(\frac{R}{|x|}\right)\right)^{-2} + \frac{A\Lambda_N}{|x|^2}$$

Since $|\nabla w(x)| = R^{\frac{N-2}{2}}(\log(\frac{R}{|x|}))^{\frac{1}{2}}|x|^{-\frac{N}{2}}(\frac{N-2}{2} + \frac{1}{2}(\log(\frac{R}{|x|}))^{-1})$ and $p < \frac{N+2}{N}$, then for a suitable positive constant c,

$$|\nabla w|^p \le c\frac{w}{|x|^2}\left(\log\left(\frac{R}{|x|}\right)\right)^{-2}.$$

Hence, up to a positive constant c_1, $c_1 w$ is a super-solution to problem

$$\begin{cases} -\Delta w = |\nabla w|^p + \Lambda_N \dfrac{w}{|x|^2} + c_0 f & \text{in } B_1(r), \\ w = 0 & \text{on } \partial B_r(0). \end{cases} \tag{5.1}$$

where $|x|^2 f$ is bounded and c_0 is small.

To prove the existence of a solution, we consider the approximated problems

$$\begin{cases} -\Delta u_n = \Lambda_N a_n(x) u_n + \dfrac{|\nabla v_n|^p}{1+\frac{1}{n}|\nabla v_n|^p} + cf_n & \text{in } B_r(0), \\ v_k = 0 & \text{on } \partial B_r(0), \end{cases} \tag{5.2}$$

where $a_n(x) = \min\{n, \frac{1}{|x|^2}\}$ and $f_n(x) = T_n(f(x))$. It is easy to check that $\Lambda_N < \lambda_1(a_n)$, the principal eigenvalue of the Laplacian with weight a_n. Then by similar arguments to the used above we prove that there exists a minimal solution u_n of (5.2). Since, in particular, $w \in W_0^{1,1}(B_r(0))$, by Theorem 4.3 in [6] we conclude that $\{u_n\}$ is increasing in n and that $u_n \le w$ in $B_r(0)$. Hence $u_n \uparrow u$ pointwise and $u \le w$. It is easy to see that $u \in L^q(B_r(0))$ for all $q < 2^*$.

Consider $H(B_r(0))$, the completion of $\mathcal{C}_0^\infty(B_r(0))$ with respect to the norm

$$||\phi||^2_{H(B_r(0))} = \int_{B_r(0)} |\nabla \phi|^2 dx - \Lambda_N \int_{B_r(0)} \frac{\phi^2}{|x|^2} dx.$$

It is well known that $H(B_r(0))$ is a Hilbert space and $W_0^{1,2}(B_r(0)) \subset H(B_r(0)) \subset W_0^{1,q}(B_r(0))$ for all $q < 2$.

We could check that $w \notin H(B_r(0)$, however $\{u_n\}$ is bounded in $H(B_r(0))$. Indeed, take u_n as a test function in (5.2), then

$$\begin{aligned} ||u_n||^2_{H(B_r(0))} &= \int_{B_r(0)} |\nabla u_n|^2 dx - \Lambda_N \int_{B_r(0)} \frac{u_n^2}{|x|^2} dx \\ &\le \int_{B_r(0)} |\nabla u_n|^p u_n dx + c_0 \int_{B_r(0)} \frac{u_n}{|x|^2} dx \\ &\le \int_{B_r(0)} |\nabla u_n|^p w dx + c_0 \int_{B_r(0)} \frac{w}{|x|^2} dx. \end{aligned}$$

Using Hölder, Young and the improved Hardy-Sobolev inequalities (see [5] and [24]) we obtain that

$$\begin{aligned} \int_{B_r(0)} |\nabla u_n|^p w dx &= \int_{B_r(0)} |\nabla u_n|^p \left(\log\left(\frac{R}{|x|}\right)\right)^{-p} \left(\log\left(\frac{R}{|x|}\right)\right)^{p} w dx \\ &\le \delta \int_{B_r(0)} |\nabla u_n|^2 \left(\log\left(\frac{R}{|x|}\right)\right)^{-2} dx \\ &\quad + C(\delta) \int_{B_r(0)} w^{\frac{2}{2-p}} \left(\log\left(\frac{R}{|x|}\right)\right)^{\frac{2p}{2-p}} dx \\ &\le \delta ||u_n||^2_{H(B_r(0))} + C(\delta) \int_{B_r(0)} w^{\frac{2}{2-p}} \left(\log\left(\frac{R}{|x|}\right)\right)^{\frac{2p}{2-p}} dx. \end{aligned}$$

Since $p < \frac{N+2}{N}$, then $\int_{B_r(0)} w^{\frac{2}{2-p}} (\log(\frac{R}{|x|}))^{\frac{2p}{2-p}} dx < \infty$. Hence choosing δ small we conclude that $||u_n||^2_{H(B_r(0))} \leq C$ and then $u_n \rightharpoonup u$ weakly in $H(B_r(0))$ thus $||u||^2_{H(B_r(0))} \leq ||u_n||^2_{H(B_r(0))}$.

We will prove that $u_n \to u$ strongly in $H(B_r(0))$. Hence we have just to prove that

$$\lim_{n\to\infty} ||u_n||^2_{H(B_r(0))} = ||u||^2_{H(B_r(0))}.$$

Consider the linear form $F_n : H(B_r(0) \to \mathbb{R}$, $F_n \equiv -\Delta u_n - \Lambda_N a_n(x) u_n$. By the regularity of u_n we find that $F_n \in H^*(B_r(0))$, the dual space of $H((B_r(0)))$. Since $u_n \leq u$ and by the fact that $-\Delta u_n - \Lambda_N a_n(x) u_n \geq 0$, we get

$$\begin{aligned} ||u_n||^2_{H(B_r(0))} &\leq \int_{H(B_r(0))} (-\Delta u_n - \Lambda_N a_n(x) u_n)\, u_n dx = \langle F_n, u_n \rangle \\ &\leq \int_{H(B_r(0))} (-\Delta u_n - \Lambda_N a_n(x) u_n)\, u dx = \langle F_n, u \rangle. \end{aligned}$$

If $\{F_n\}$ is uniformly bounded in $H^*(B_r(0))$ we are done because then $F_n \rightharpoonup F$ in the weak-star topology of $H^*(B_r(0))$ and in particular if $\phi \in \mathcal{C}_0^\infty(B_r(0))$, we obtain

$$\langle F_n, \phi \rangle \to \int_{B_r(0)} \left(\nabla u \nabla \phi - \Lambda_N \frac{u\,\phi}{|x|^2} \right) dx,$$

then $F = -\Delta u - \Lambda_N \dfrac{u}{|x|^2} \in H^*(B_r(0)$. Thus, by density,

$$\langle F_n, u \rangle \to \langle F, u \rangle = ||u||^2_{H(B_r(0)}$$

and as a byproduct the strong convergence and that u is a solution to problem (5.1) follows easily.

Hence to finish we have just to prove that $\{F_n\}$ is uniformly bounded in $H^*(B_r(0))$.

Consider $\phi \in \mathcal{C}_0^\infty(B_r(0))$, then

$$\begin{aligned} |\langle F_n, \phi \rangle| &= \left| \int_{B_r(0)} \phi(-\Delta u_n - \Lambda_N a_n(x) u_n) dx \right| \\ &\leq \int_{B_r(0)} |\nabla u_n|^p |\phi| dx + c_0 \int_{B_r(0)} |f||\phi| dx. \end{aligned}$$

Using the hypothesis on f we obtain that

$$\int_{B_r(0)} |f||\phi| dx \leq C(f) ||\phi||_{H(B_r(0))}.$$

For the other term, using the fact that $\{u_n\}$ is bounded in $H(B_r(0))$, there results that

$$\int_{B_r(0)} |\nabla u_n|^p |\phi| dx = \int_{B_r(0)} |\nabla u_n|^p \left(\log\left(\frac{R}{|x|}\right)\right)^{-p} \left(\log\left(\frac{R}{|x|}\right)\right)^{p} |\phi| dx$$

$$\leq \left(\int_{B_r(0)} |\nabla u_n|^2 \left(\log\left(\frac{R}{|x|}\right)\right)^{-2} dx\right)^{\frac{p}{2}} \left(\int_{B_r(0)} |\phi|^{\frac{2}{2-p}} \left(\log\left(\frac{R}{|x|}\right)\right)^{\frac{2p}{2-p}} dx\right)^{\frac{2-p}{2}}$$

$$\leq C \left(\int_{B_r(0)} |\phi|^{\frac{2}{2-p}} \left(\log\left(\frac{R}{|x|}\right)\right)^{\frac{2p}{2-p}} dx\right)^{\frac{2-p}{2}}.$$

Since $p < \frac{N+2}{2}$, then $\frac{2}{2-p} < 2^*$. Therefore using the properties $H(B_r(0))$, there exists a constant $C_1 > 0$ such that

$$\int_{B_r(0)} |\phi|^{\frac{2}{2-p}} \left(\log\left(\frac{R}{|x|}\right)\right)^{\frac{2p}{2-p}} dx \leq C_1 ||\phi||_{H(B_r(0))}.$$

As a conclusion we obtain that

$$|\langle F_n, \phi\rangle| \leq C ||\phi||_{H(B_r(0))}.$$

Remark 5.1.

1. As above we can consider the case of a general domain Ω that contains the origin and proving that $t(w - v)$ is a supersolution where w is defined above, v is a harmonic function such that $v = w$ on the boundary of Ω and $t > 0$. Then the existence result follows using the same computation as in the proof of Theorem 4.3. Hence we have that problem

$$\begin{cases} -\Delta u = \Lambda_N \frac{u}{|x|^2} + |\nabla u|^p + cf & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

   has a positive solution $u \in H(\Omega)$ if $|x|^2 f$ is bounded and c is small.
2. A proper definition of the *gradient* associated to the operator $-\Delta - \frac{\lambda}{|x|^2} I$ provides existence of solution, indeed in [2] is studied the example,

$$-\Delta u - \Lambda_N \frac{u}{|x|^2} = \left|\nabla u + \left(\frac{N-2}{2}\right) \frac{u}{|x|^2} x\right|^2 |x|^{\frac{N-2}{2}} + \lambda f(x)$$

   in Ω, $u = 0$ on $\partial\Omega$, $\Lambda_N = (\frac{N-2}{2})^2$ and f under some hypotheses of summability.

5.1. Some open problems

The following questions seem to be open problems with some interest.

1. Fixed $1 < p < p_+(\lambda)$ to obtain the optimal class of functions according their summability, in order to have existence of a *very weak solution* of Dirichlet problem with data in a such class.
2. Assume that for λ fixed and for f in a determined class we are able to find a very weak solution, u. What is the regularity of u in terms of the regularity of f?
3. Results on uniqueness or nonuniqueness. We recall that for $\lambda = 0$ there are some results on multiplicity of unbounded solutions, for instance in [16] (for a ball) and in [4], where all the solutions are characterized in any bounded domain.

References

[1] B. ABDELLAOUI and I. PERAL, *Some results for semilinear elliptic equations with critical potential*, Proc. Roy. Soc. Edinburgh **132A** (2002), 1–24.

[2] B. ABDELLAOUI and I. PERAL, *A note on a critical problem with natural growth in the gradient*, J. Eur. Math. Soc. Sect. A **8** (2006), 157–170.

[3] B. ABDELAHOUI and I. PERAL, *Nonexistence results for quasilinear elliptic equations related to Caffarelli-Kohn-Nirenberg inequalities*, Commun. Pure Appl. Anal. **2** (2003), 539–566.

[4] B. ABDELLAOUI, A. DALL'AGLIO and I. PERAL, *Some remarks on elliptic problems with critical growth on the gradient*, J. Differential Equations **222** (2006), 21–62.

[5] B. ABDELLAOUI, E. COLORADO and I. PERAL, *Some improved Caffarelli-Kohn-Nirenberg inequalities*, Cal. Var. Partial Differential Equations **23** (2005), 327–345.

[6] N. E. ALAA and M. PIERRE, *Weak solutions of some quasilinear elliptic equations with data measures*, SIAM J. Math. Anal. **24** (1993), 23-35.

[7] H. BERESTYCKI, S. KAMIN and G. SIVASHINSKY, *Metastability in a flame front evolution equation*, Interfaces Free Bound. **3** (2001) 361–392.

[8] L. BOCCARDO, T. GALLOUET and F. MURAT, "A Unified Presentation of Two Existence Results for Problems with Natural Growth", Research Notes in Mathematics, Vol. 296, 1993, 127–137, Longman.

[9] L. BOCCARDO, T. GALLOUËT and L. ORSINA, *Existence and nonexistence of solutions for some nonlinear elliptic equations*, J. Anal. Math. **73** (1997), 203–223.

[10] L. BOCCARDO, F. MURAT and J.-P. PUEL, *Existence des solutions non bornées pour certains équations quasi-linéaires*, Port. Math., **41** (1982), 507–534.

[11] L. BOCCARDO, L. ORSINA and I. PERAL, *A remark on existence and optimal summability of solutions of elliptic problems involving Hardy potential*, Discrete Cont. Dyn. Syst. **16** (2006), 513–523.

[12] H. BREZIS and X. CABRÉ, *Some simple nonlinear PDE's without solution,* Boll. Unione. Mat. Ital. Sez. B **8** (1998), 223–262.

[13] H. BREZIS, L. DUPAIGNE and A. TESEI, *On a semilinear equation with inverse-square potential*, Selecta Math. **11** (2005), 1–7.

[14] H. BREZIS and A. PONCE, *Kato's inequality when Δu is a measure*, C.R. Math. Acad. Sci. Paris **338** (2004), 599–604.

[15] L. CAFFARELLI, R. KOHN and L. NIRENBERG, *First order interpolation inequalities with weights*, Compositio Math. **53** (1984), 259–275.

[16] V. FERONE and F. MURAT, *Nonlinear problems having natural growth in the gradient: an existence result when the source terms are small*, Nonlinear Anal. **42** (2000), 1309–1326.

[17] K. HANSSON, V. G. MAZ'YA and I. E. VERBITSKY, *Criteria of solvability for multidimensional Riccati equations*, Ark. Mat. **37** (1999), 87–120.

[18] M. KARDAR, G. PARISI and Y. C. ZHANG, *Dynamic scaling of growing interfaces*, Phys. Rev. Lett. **56** (1986), 889–892.

[19] T. KATO, *Schrödinger operators with singular potentials*, Israel J. Math. **13** (1972), 135–148.

[20] J. L. KAZDAN and R. J. KRAMER, *Invariant criteria for existence of solutions to second-order quasilinear elliptic equations*, Comm. Pure Appl. Math. **31** (1978), 619–645.

[21] P. L. LIONS, "Generalized Solutions of Hamilton-Jacobi Equations", Pitman Res. Notes Math., Vol. 62, 1982.

[22] F. MURAT, *L'injection du cone positif de H^{-1} dans $W^{-1,q}$ est compacte pour tout* $q < 2$, J. Math. Pures Appl. **60** (1981) 309–322.

[23] V. G. MAZ'JA "Sobolev Spaces", Springer Verlag, Berlin, 1985.

[24] Z.Q. WANG and M. WILLEM, *Caffarelli-Kohn-Nirenberg inequalities with remainder terms*, J. Funct. Anal. **203** (2003), 550–568.

Département Mathématiques
Faculté des Sciences
Université Abou-Bekr Belkaid
Imana B.P 119 Tlemcen, Algeria
boumediene.abdellaoui@uam.es

Departamento de Matemáticas
U. Autónoma de Madrid
28049 Madrid, Spain
ireneo.peral@uam.es