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Stochastic Poisson-Sigma model

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Abstract. We produce a stochastic regularization of the Poisson-Sigma model of Cattaneo-Felder, which is an analogue regularization of Klauder's stochastic regularization of the Hamiltonian path integral [23] in field theory. We perform also semi-classical limits.

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1. Introduction

Let us consider a manifold M endowed with a Poisson structure, a bilinear map $\{.,.\}$ from the space of smooth functions on the manifold into the space of smooth functions on the manifold, anticommutative and satisfying the Jacobi relation. Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer [6, 7] have introduced the so-called program of deformation quantization. These authors get the following formal series:

$$f * g = \sum i^n h^n P_n(f, g). \tag{1.1}$$

The P_n's are differential operators. This series is diverging. The program of deformation quantization was carried out by Kontsevich [24]. We refer to the survey of Dito-Sternheimer about this topic [17].

Cattaneo-Felder [15] have established the link between Kontsevich formula and quantum field theory. Let us suppose that the manifold is $\mathbb{R}^d$. They consider the so-called Poisson-Sigma model. Let us recall how it is constructed: we consider the disk D, 3 points $\infty, 1, 2$ on the boundary of the disk. They consider the space of forms η on D and the space of maps X from D into $\mathbb{R}^d$. Let $\alpha_{i,j}$ be the components of the Poisson structure on $\mathbb{R}^d$. Let $(t, s) = S$ be the polar coordinates on D: $t \in [0, 1]$, $s \in S^1$. Cattaneo-Felder consider the action:

$$\sum_i \int_D \eta_i(S) \wedge dX_i + \sum_{i,j} \int_D \alpha_{i,j}(X)\eta_i \wedge \eta_j = S(X, \eta) \tag{1.2}$$

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where $X = (X_1, \dots, X_d)$ and where η_j are 1-forms on D. [15] consider the following formula for the non-perturbative $*$-product:

$$f *_h g(x) = \int_{\eta, X, X(\infty)=x} f(X(1))g(X(2)) \exp[iS/h] dDXdD\eta \qquad (1.3)$$

where the field X and the form η are chosen at random according to the formal Lebesgue measure on the configuration space. [15] perform the semi-classical analysis when $h \to 0$ and get the asymptotic expansion:

$$f *_h g(x) = \sum (ih)^n P_n(f, g). \qquad (1.4)$$

The objects of Cattaneo-Felder are formal (see [20, 21]) and use the heavy apparatus of quantum field theory. Our purpose is to add a stochastic regulator in (1.3) in order to define the functional integral rigorously. We get a stochastic product $f *_{st,h} g$.

Let us recall that in (1.3), we have to choose two kinds of objects at random: the field $X : D \to \mathbb{R}^d$ and the forms η over D. So we have to introduce stochastic regulators to define a random field X and to define random forms η.

In order to define the random field $X : D \to \mathbb{R}^d$, we will follow the procedure of Airault-Malliavin [2]. Airault-Malliavin [2] have defined the Brownian motion over a loop group. Let us recall that infinite dimensional processes over infinite dimensional manifolds have a long history: see works of Kuo [26], Belopolskaya-Daletskii [8] and Daletskii [16]. Albeverio-Léandre-Röckner [4] have defined the Ornstein-Uhlenbeck process over the free loop space, by using the theory of Dirichlet forms. Brzezniak-Elworthy [12] have given an abstract generalization of the works of Airault-Malliavin.

In this paper, we are concerned with a $(1 + 1)$-dimensional theory: this means we consider a diffusion process on the loop space. Various works in this direction were done by Brzezniak-Léandre [13, 14], Léandre [34, 35, 36]. Let us remark that in (1.3), there is the condition $X(\infty) = x$. [14, 34, 36] have introduced a convenient Brownian bridge in order to do the conditional expectation by $X(\infty) = x$. But there is another procedure to condition functionals: it is the Airault-Malliavin-Sugita procedure [1, 44]. In this work, we will follow this procedure.

In order to define random forms, we will employ the techniques of [37]. This means we will not choose our random forms on D according to the formal Lebesgue measure on the space of forms, but we will introduce a stochastic Gaussian regulator in order to define the probability measure on the space of forms.

If we do not look at the conditional expectation by $X(\infty) = x$, the action S becomes a stochastic integral, which belongs to all of the Sobolev spaces of the Malliavin Calculus [41]. We consider the measure

$$h \to E\left[f(X(1))g(X(2))h(X(\infty)) \exp[iS]\right].$$

By Malliavin Calculus, it has a smooth density. Moreover, the magic properties of the Airault-Malliavin equation tell us that the density of the law of $X(\infty)$ is strictly positive.

We have:

Theorem A.

$$f *_{st} g = E\big[f(X(1))g(X(2))\exp[iS]|X(\infty) = x\big] \tag{1.5}$$

defines a continuous bilinear map from $C_b^\infty(\mathbb{R}^d)$ *into* $C_b^\infty(\mathbb{R}^d)$.

We use the Malliavin Calculus to prove Theorem A.

We perform a semi-classical analysis when $h \to 0$: for that task, we choose a small leading Brownian motion as well as a small stochastic regularization of η. Such considerations were done in [32]. But S is only a stochastic integral: so, by improving a bit the techniques of [32], we have:

Theorem B.

$$f *_{st,h} g = E_h\big[f(X(1))g(X(2))\exp[iS/h]|X(\infty) = x\big] \tag{1.6}$$

has, when $h \to 0$ *an asymptotic expansion:*

$$f *_{st,h} g = \sum h^n Q_n(f, g) \tag{1.7}$$

where the Q_n*'s are differential operators acting on* f *and* g.

For that, we use the techniques of asymptotics of Wiener functionals by using the Malliavin Calculus: we refer to the surveys by Léandre [28], Kusuoka [27] and Watanabe [45] for this topic.

The reader interested in the relation existing between analysis over loop space and mathematical physics can consult the survey by Albeverio [3] and the two surveys by Léandre [29, 30].

2. The model without conditioning

Let $\Pi(x)$ be a linear map from $\mathbb{R}^n$ into $\mathbb{R}^d$, which depends smoothly from $x \in \mathbb{R}^d$: we suppose that the derivatives of all orders of Π are bounded and that $(\Pi(x)e_i)$ $i = 1, \dots, n$ spans uniformly $\mathbb{R}^d$ for the canonical basis $e_1, \dots, e_n$ of $\mathbb{R}^n$.

Let $H = H^{1,2}(S^1; \mathbb{R}^n)$ be the Hilbert space of maps from the circle S^1 into $\mathbb{R}^n$ such as:

$$\int_0^1 |h(s)|^2 ds + \int_0^1 |d/ds h(s)|^2 ds = \|h\|^2 < \infty\,. \tag{2.1}$$

We write $h = (h^1, \dots, h^n)$. Moreover,

$$h^j(0) = \langle h, e^j \rangle \tag{2.2}$$

where

$$e^j(s) = \big(0, \dots, 0, \lambda \exp[-s] + \mu \exp[s], 0, \dots, 0\big) \tag{2.3}$$

for some λ and some μ for $0 \le s \le 1$. Moreover $e^j(s)$ is smooth on $]0, 1[$ with half derivatives at all orders at 0 and 1: $e^j(0) = e^j(1)$ but $d/dse^j(0) \neq d/dse^j(1)$.

We have:

$$h^j(s) = \langle h, e^j(. - s)\rangle . \tag{2.4}$$

We consider the Brownian motion with values in H:

$$B_t(s) = (B_t^1(s), \dots, B_t^n(s)) . \tag{2.5}$$

The processes $B_.^j(.)$ are independent and $t \to B_t^j(s)$ is a Brownian motion with values in $\mathbb{R}$ submitted to the relation:

$$d\langle B_t^j(s), B_t^j(s')\rangle = e(s - s')dt \tag{2.6}$$

where $e^j(s)$ is the j^{th} coordinate of $e(s)$.

We consider the Airault-Malliavin equation [2, 12]:

$$dx_t(s)(x) = \Pi(x_t(s)(x))d_t B_t(s)\,;\; x_0(s)(x) = x . \tag{2.7}$$

It is a family of Stratonovitch equations. We have shown that $s \to x_t(s)(x)$ is $1/2 - \epsilon$ Hölder by Gronwall lemma and Kolmogorov lemma [39]: we have an improvement of this result. Namely:

Proposition 2.1. *$x \to (s \to x_1(s)(x))$ is almost-surely smooth for the Hölder topology.*

Proof. This comes from the fact that $s \to \frac{D^{(r)}}{Dx^{(r)}} x_1(s)(x)$ is almost surely Hölder $1/2 - \epsilon$ in s (see [32] for an analogous statement). Namely, the stochastic differential equation of $\frac{D}{Dx} x_t(s)(x)$ is

$$d\frac{D}{D(x)} x_t(s)(x) = D\Pi(x_t(s)(x)) \frac{D}{D(x)} x_t(s)(x) d_t B_t(s) \tag{2.8}$$

and we get by induction the differential equation of $\frac{D^{(r)}}{Dx^{(r)}} x_t(s)(x)$. □

Let us write for Δs small:

$$B_t(s + \Delta s) = B_t(s) + \Delta_s B_t(s). \tag{2.9}$$

We have (See [35] Part III):

Property H(1). If s' does not belong to $]s, s + \Delta s[$:

$$d\langle \Delta_s B_.(s), B_.(s')\rangle = dtC(s, s')\Delta s + O(\Delta s^2). \tag{2.10}$$

Property H(2). if $]s, s + \Delta s[\cap]s', s' + \Delta s'[= \emptyset$, we have:

$$d\langle \Delta_s B_.(s), \Delta_{s'} B_.(s')\rangle = dtC(s, s')\Delta s \Delta s' + O(\Delta s + \Delta s')^3 . \tag{2.11}$$

Let us consider a sequence of intervals $]s_i, s_i + \Delta s_i[$, two intervals being either disjoint or equal. We denote by $|I|$ the number of intervals and by $\|I\|$ the number of distinct intervals. Let us consider some points r_j of the circle which do not belong to the union of the previous open intervals. Let $\alpha_t(i)$ be some processes, which are $B_.(r_j)$ measurable, previsible, and which are semi-martingales. We suppose that the local characteristic [22] of each $\alpha_t(i)$ have bounded Sobolev norms in the sense of the Malliavin Calculus [41] for the Gaussian space spanned by the $B_.(r_j)$. We put iteratively:

$$I^{i+1}(t) = \int_0^t I^i(u)\alpha_u(i) d_u \Delta_{s_i} B_u(s_i) \tag{2.12}$$

and we get an iterated Stratonovitch integral $I^I(t)$. Let F be a measurable functional for the Gaussian space spanned by the $B_.(r_j)$: we suppose that F has bounded Sobolev norms in the sense of Malliavin Calculus for the space spanned by the $B_.(r_j)$. We denote by I' the set of indices obtained by selecting from I an interval only one time. The cardinal of I' is therefore $\|I\|$.We have the main lemma:

Lemma 2.2.

$$E[FI^I(t)] \le C \prod_{i \in I'} \Delta s_i \tag{2.13}$$

where C can be estimated in terms of the Sobolev norms of F and of the $\alpha_.(i)$.

Proof. We apply the Clark-Ocone formula to F [41]. We select the Itô term in $I^I(t)$ and the finite energy term in $I^I(t)$. We conclude by applying Itô formula and Properties H(1) and H(2) and property H(3):

Property H(3).

$$d_t\langle \Delta_s B_.(s), \Delta_s B_.(s)\rangle = C(s)\Delta s dt + O(\Delta s^2)dt . \tag{2.14}$$

The statement follows by induction on $|I|$. □

Remark 2.3. We remark that we have analogue estimates if we consider a product $\prod_{i \in I} I^i(t)$ of single integrals or if we consider double iterated integrals in the product. Namely, we can come back to the situation of Lemma 2.2 by using the Itô-Stratonovitch formula. We put:

$$I_t^1(s,\Delta s)(x)=\frac{D}{D(x)}x_t(s)(x)\int_0^t D\Pi(x_u(s)(x))\frac{D^{-1}}{D(x)}x_u(s)(x)d_u\Delta_s B_u(s) \quad (2.15)$$

and

$$I_t^2(s,\Delta s)(x)$$
$$=\frac{D}{D(x)}x_t(s)(x)\int_0^t\frac{D^{-1}}{D(x)}x_u(s)(x)\big\langle D\Pi(x_u(s)(x)), I_u^1(s,\Delta s)(x), d_u\Delta_s B_u(s)\big\rangle. \quad (2.16)$$

By using the rule of differentiation of stochastic differential equations along a parameter [10, 25], we have that:

$$x_t(s+\Delta s)(x)=x_t(s)(x)+I_t^1(s,\Delta s)(x)+I_t^2(s,\Delta s)(x)+O(\Delta s^{3/2})\,. \quad (2.17)$$

The error term is uniform in x over each compact set of $\mathbb{R}^d$.

Let us consider a 1-form on $[0,1]\times S^1$, $\eta=\eta_1 ds+\eta_2 dt$. We put a Gaussian measure on the set of η: η_1 and η_2 are independent. On the space of η we consider a Gaussian measure whose reproducing Hilbert space is defined as follows: we consider the space of function taking values in $\mathbb{R}^{2d}$ endowed with the Sobolev norm $\int_{S^1}\langle(-\frac{d^2}{ds^2}+1)\eta(s),\eta(s)\rangle ds=\|\eta\|^2_{H^d}$ and the space of forms endowed with the Hilbert norm $\int_0^1\|\frac{\partial}{\partial t}\eta_t(.)\|^2_{H^d}dt$. The random forms which are obtained in that way are almost surely Hölder. Let us consider $N=2^{N_0}$. We consider the polygonal approximation $s\to x_t^N(s)(x)$ of $s\to x_t(s)(x)$. We consider a coordinate $x_.^{N,j}(s)(x)$ of it. We put:

$$A_t^{N,j}(x)=\int_{S^1}\eta_2^j(s,t)d_s x_t^{N,j}(s)(x)\,. \quad (2.18)$$

We have:

Proposition 2.4. *When $N\to\infty$, $A_t^{N,j}(x)$ tends in all of the L^p to a real random variable*

$$\int_{S^1}\eta_2^j(s,t)d_s x_t^j(s)(x)\,. \quad (2.19)$$

Moreover, the stochastic integral defined in (2.19) *depends almost surely smoothly on x and in all of the L^p.*

Proof. We omit to write the index j, doing as if the diffusion $x_t(s)(x)$ was one dimensional. We write:

$$A_t^N(x)=\sum A_i^N=\sum\int_{s_i}^{s_{i+1}}\eta_2(s,t)d_s x_t^N(s)(x)\,. \quad (2.20)$$

Let us decompose A_i^N as a sum:

$$A_i^N = B_i^N + C_i^N \tag{2.21}$$

where

$$B_i^N = \eta_2(s_i, t)\Delta_{s_i} x_t(s_i)(x) \tag{2.22}$$

and where:

$$C_i^N = \int_{s_i}^{s_{i+1}} (\eta_2(s,t) - \eta_2(s_i,t)) \frac{ds}{s_{i+1} - s_i} \Delta_{s_i} x_t(s_i)(x)\,. \tag{2.23}$$

First step. Convergence of $\sum B_i^N$ in all the L^p.

We write $\Delta s_i = 1/N$. Moreover,

$$\begin{aligned} B_i^N &= B_{i,1}^N + B_{i,2}^N + \text{error} \\ &= \eta_2(s_i,t) I_t^1(s_i, \Delta s_i)(x) + \eta_2(s_i,t) I_t^2(s_i, \Delta s_i)(x) + \text{error}\,. \end{aligned} \tag{2.24}$$

Let us study first the convergence of $\sum B_{i,1}^N$ in all the L^p. Let $N' = 2^{N_0'}$ be an integer larger than N. We write:

$$D_i^N = B_{i,1}^N - \sum_{[s_{i'}, s_{i'+1}] \subseteq [s_i, s_{i+1}]} B_{i',1}^{N'}\,. \tag{2.25}$$

In $B_{i,1}^N$ and in $I_t^1(s_i, \Delta s_i)$, we get:

$$d_t \Delta_{s_i} B_t(s_i) = \sum_{[s_{i'}, s_{i'+1}] \subseteq [s_i, s_{i+1}]} d_t \Delta_{s_{i'}} B_t(s_{i'}) \tag{2.26}$$

and we apply Lemma 2.2 in order to get the estimate:

$$E\Big[\prod_{i_j \in I} D_{i_j}^N\Big] = o(1) \prod_{i_j \in I'} \Delta s_{i_j} = o(1) C(I)\,. \tag{2.27}$$

But there are at most CN^r set of multi-indices I such that $|I| = p$ and $\|I\| = r$. Therefore the result.

Let us study the behaviour of $\sum B_{i,2}^N$ in (2.24).

In $I_t^2(s_i, \Delta s_i)$, we write:

$$d_u \Delta_{s_i} B_u(s_i) d_v \Delta_{s_i} B_v(s_i) = \sum_{[s_j, s_{j+1}], [s_{j'}, s_{j'+1}] \subseteq [s_i, s_{i+1}]} d_u \Delta_{s_j} B_u(s_j) d_v \Delta_{s_{j'}} B_v(s_{j'})\,. \tag{2.28}$$

We select in the decomposition (2.27) the sum where we have $s_j = s_{j'}$, and we get a decomposition of $B^N_{i,2}$ into $D^N_{i,1} + D^N_{i,2}$ where in $D^N_{i,1} = \sum D^N_{i,j,1}$ we consider only the diagonal terms.

We write:

$$\sum \eta_2(s_i,t)D^N_{i,1} - \sum \eta_2(s_j,t)B^{N'}_{j,2} = \sum_{[s_j,s_{j+1}]\subseteq[s_i,s_{i+1}]} (\eta_2(s_i,t)-\eta_2(s_j,t))D^N_{i,j,1} + \sum_{[s_j,s_{j+1}]\subseteq[s_i,s_{i+1}]} \eta_2(s_j,t)(D^N_{i,j,1} - B^{N'}_{j,2})\,. \tag{2.29}$$

The first term tends trivially to 0 in all of the L^p. By applying the remark following Lemma 2.2, the second term tends to zero in all of the L^p when $N' \to \infty$.

Let us consider $\sum \eta_2(s_i,t)D^N_{i,2}$. Let us show that it tends in all the L^p to zero. Let $I = \{i_1, \ldots, i_{|I|}\}$ with $\|I\|$ given. According to Lemma 2.2. we have:

$$E\Big[\prod_{i_j\in I} D^N_{i,2}\Big] = O(N^{-\|I\|})\,. \tag{2.30}$$

Hence, we can write $D^N_{i,2} = \sum_{j\neq j'} D^N_{j,j',2}$. If we distribute in $(D^N_{i,2})^r$, there are at most $C(N'/N)^k$ products $\prod D^N_{j_l,j_{l'},2}$ where the cardinal described by j_l, $j_{l'}$ is k. But k is at least equal to 2. Therefore (2.30). We conclude as in (2.27).

Second step. Convergence of $\sum C^N_i$ in all of the L^p.
We write

$$\sum C^N_i = \sum \alpha^N(s_i)I^1_t(s_i,\Delta s_i)(x) + \text{error} \tag{2.31}$$

where $\alpha^N(s_i)$ is independent of the system of $I^1_t(s_i,\Delta s_i)$ and tends to 0 in all of the L^p. Therefore the sum tends to 0 by the previous considerations in all the L^p.

In order to show that the stochastic integral defined by (2.19) depends almost surely and in all of the L^p from x, we can apply the previous considerations to $\frac{D^r}{Dx^r}A^{N,j}_t(x)$ and show that it converges in all the L^p to $\frac{D^r}{Dx^r}\int_{S^1}\eta^j_2(s,t)d_sx^j_t(s)(x)$. The Sobolev imbedding theorem allows to conclude. □

Let us introduce the stochastic Poisson-Sigma action defined as follows:

$$S(x_.(.)(x),\eta) = \sum_j \int_{[0,1]\times S^1} \eta^j \wedge dx^j_.(.)(x) + \sum_{i,j}\int_{[0,1]\times S^1} \alpha_{i,j}(x_.(.)(x))\eta^i\wedge\eta^j\,. \tag{2.32}$$

Proposition 2.4 allows us to deduce the following theorem:

Theorem 2.5. *The random variable $S(x_.(.)(x), \eta)$ is real, and is almost surely differentiable in x. For all r, all $p \geq 1$:*

$$\sup_x E\left[\left|\frac{D^r}{Dx^r} S(\eta_., x_.(.)(x))\right|^p\right] < \infty . \tag{2.33}$$

This allows us to state the following theorem: let $C_b^\infty(\mathbb{R}^d)$ be the Fréchet space of smooth functions f on $\mathbb{R}^d$ with bounded derivatives at each order endowed with the set of semi-norms:

$$\|f\|_{r,\infty} = \sup_x \left|\frac{D^r}{Dx^r} f(x)\right| . \tag{2.34}$$

Theorem 2.6. *The map which sends (f, g) to:*

$$E[f(x_1(1)(x))g(x_1(2)(x)) \exp[iS(x_.(.)(x)), \eta)]] \tag{2.35}$$

is a continuous bilinear application from $C_b^\infty(\mathbb{R}^d)$ into $C_b^\infty(\mathbb{R}^d)$. 1 and 2 denote in (2.35) two different points of S^1.

3. A stochastic star product

Let us recall that, if the Malliavin Calculus has a lot of precursors (see the work of Hida, Elworthy, Fomin, Albeverio . . .), the main novelty of the Malliavin Calculus was to complete the differential operations known at that time on the Wiener space in all of the L^p. This allowed Malliavin to recover Hörmander's theorem by probabilistic methods [38]. The first ones who have applied the Malliavin Calculus to other Gaussian spaces than the traditional Wiener space are Nualart and Sanz [42] in order to study the Brownian sheet. Here, we apply the Malliavin Calculus in our situation.

We consider the space $H(B)$ of maps from [0, 1] into H, $h_t(.)(B)$, such that

$$\int_0^1 \left\|\frac{\partial}{\partial t} h_t(.)(B)\right\|_B^2 dt < \infty \tag{3.1}$$

and the space $H(\eta)$ of maps from [0, 1] into H^d, $h_t(.)(\eta)$, such that

$$\int_0^1 \left\|\frac{\partial}{\partial t} h_t(.)(\eta)\right\|_{H^d}^2 dt < \infty . \tag{3.2}$$

$H(B)$ is the Hilbert reproducing space of the Gaussian field $B_.(.)$ and $H(\eta)$ is the Hilbert reproducing space of the Gaussian field η.

If F is a functional which is $B_.(.)$ and η measurable, we take its derivative in the direction of $H(B)$ and $H(\eta)$. $\nabla^r F$ is therefore a random element of $(H(B) \oplus H(\eta))^{\otimes r}$. We consider its L^p norm and we get:

$$\|F\|_{r,p} = E[\|\nabla^r F\|^p]^{1/p} \tag{3.3}$$

which is the collection of Sobolev norms in the sense of the Malliavin Calculus [41]. F is said to be smooth in the Malliavin sense if $\|F\|_{r,p} < \infty$ for all r and p.

Lemma 3.1. *$\frac{D^r}{Dx^r} x_t(s)(x)$ and $\frac{D^{-1}}{Dx} x_t(s)(x)$ are smooth in the sense of Malliavin. Moreover their Sobolev norms are bounded in $s, t \in [0, 1]$ and x,and the kernel of their derivatives are $B_.(s)$- measurable.*

Proof. This result is classical [41] if we consider these functionals as $B_.(s)$-measurable. But

$$d/dt h_t(s)(B) = d/dt\langle h_t(.)(B), e(. - s)\rangle\ . \tag{3.4}$$

Therefore the result. □

Proposition 3.2. *$\frac{D^r}{Dx^r} A_t^{N,j}(x)$ tends to $\frac{D^r}{Dx^r} \int_{S^1} \eta_2^j(s, t)\, d_s\, x_t^j(s)(x)$ in all the Sobolev spaces and the Sobolev norms of this last stochastic integral are bounded in $x \in R^d$.*

Proof. If we do not take the derivatives of $d_u \Delta_s B_u(s)$ and $d_v \Delta_s B_v(s) d_u \Delta_s B_u(s)$ in (2.15) and in (2.16), the result goes by the same methods as the proof of Proposition 2.4, by applying Lemma 3.1. Let us take the derivatives of $d_u \Delta_s B_u(s)$in (2.15) and (2.16). They are given by $\frac{\partial}{\partial u}\Delta_s h_u(s)(B) = \frac{\partial}{\partial u}\langle h_u(.)(B), e(.-s-\Delta s) - e(.-s)\rangle_H$ and therefore the treatment leads to simpler considerations than in the statement of Proposition 2.4. □

We deduce from Proposition 3.2 that $\frac{D^r}{Dx^r} S(x_.(.)(x), \eta)$ is bounded in x in all the Sobolev spaces. We get, since the stochastic Poisson-Sigma action $S(x_.(.)(x), \eta)$ is real, that:

Proposition 3.3. *Let $\mu(x)$ be the measure on R^d which sends $h \in C_b(\mathbb{R}^d)$ to:*

$$E[f(x_1(1)(x))g(x_1(2)(x))h(x_1(\infty)(x))\exp[iS(x_.(.)(x)), \eta)]] \tag{3.5}$$

where f and g belong to $C_b^\infty(\mathbb{R}^d)$. $\mu(x)$ has a density $q(x, y)$ with respect to the Lebesgue measure and the uniform norm of $\frac{D^r}{Dx^r}\frac{D^{r'}}{Dy^{r'}} q(x, y)$ can be estimated in terms of the uniform norms of the derivatives of f and g.

Proof. This comes from the fact that $\frac{D^r}{Dx^r} \exp[iS(x_.(.)(x), \eta)]$ and $\frac{D^r}{Dx^r} x_1(s)(x)$ have bounded Sobolev norms in the sense of the Malliavin Calculus in x and from the Malliavin Calculus [41]. □

Proof of Theorem A. $x_1(\infty)(x)$ is given by a diffusion on $\mathbb{R}^d$. Its law has a smooth density $p_1(x, y) > 0$ with bounded derivatives of all orders in x and y. By using the Airault-Malliavin-Sugita procedure [1, 44], we get :

$$\frac{\mu(x,x)}{p_1(x,x)} = E[f(x_1(1)(x))g(x_1(2)(x))\exp[iS(x_.(.)(x),\eta)]|x_1(\infty)(x)=x]\,. \quad (3.6)$$

Then the result follows, since $p_1(x, x) > c > 0$. □

4. Semi-classical analysis

Following [40] and [18], let us put $\epsilon = h^{1/2}$. We replace $B_.(.)$ by $\epsilon B_.(.)$ and η by $\epsilon\eta$. We get a random field $x_.(.)(\epsilon)(x)$.

By using the classical rules of differentiation of $x_t(s)(\epsilon)(x)$ along the parameter ϵ and x [10, 39, 25] and considerations analog to Lemma 3.1, we get:

Lemma 4.1. *$\frac{D^{r'}}{D\epsilon^{r'}}\frac{D^r}{Dx^r}x_t(s)(\epsilon)(x)$ and $\frac{D^{r'}}{D\epsilon^{r'}}\frac{D^{-1}}{Dx}x_t(s)(\epsilon)(x)$ are smooth in the sense of Malliavin for the total Gaussian space. Moreover, their Sobolev norms are bounded in $s, t \in [0, 1], \epsilon \in [0, 1]$ and x in $\mathbb{R}^d$ and the kernels of their derivatives are $B_.(s)$-measurable.*

We get by adding the new parameter ϵ:

Proposition 4.2. *$\frac{D^{r'}}{D\epsilon^{r'}}\frac{D^r}{Dx^r}A_t^{N,j}(\epsilon)(x)$ tends to $\frac{D^{r'}}{D\epsilon^{r'}}\frac{D^r}{Dx^r}\int_{S^1}\epsilon\eta_2^j(s,t)d_sx_t^j(s)(\epsilon)(x))$ in all of the Sobolev spaces of the Malliavin Calculus. The Sobolev norms in the sense of Malliavin Calculus of the last stochastic integral are bounded in $x \in \mathbb{R}^d$ and $\epsilon \in [0, 1]$. Moreover, they are 0 if $r' = 0$ or $r' = 1$.*

We get:

Proposition 4.3. *Let $\mu_\epsilon(x)$ be the measure on $\mathbb{R}^d$ which to $h \in C_b(\mathbb{R}^d)$ assigns:*

$$E\Big[f(x_1(1)(\epsilon)(x))g(x_1(2)(\epsilon)(x))\exp[i/\epsilon^2 S(x_.(.)(\epsilon)(x)),\epsilon\eta)]h(x_1(\infty)(\epsilon)(x))\Big] \quad (4.1)$$

where f and g belong to $C_b^\infty(\mathbb{R}^d)$. $\mu_\epsilon(x)$ has a density $q_\epsilon(x)$ ($\epsilon > 0$) and when $\epsilon \to 0$:

$$q_\epsilon(x,x) = \epsilon^{-d}\sum_{i=1}^n h^i\tilde{Q}_i(f,g)(x) + O(h^n) \quad (4.2)$$

where $\tilde{Q}_i$ are differential operators in f and g.

Proof. $q_\epsilon(x, x) = \epsilon^{-d}\tilde{q}_\epsilon(x, 0)$ where $\tilde{q}_\epsilon(x, y)$ is the density of the measure ν_ϵ:

$$E\left[f(x_1(1)(\epsilon)(x))g(x_1(2)(\epsilon)(x))\exp[i/\epsilon^2 S((x_.(.)(\epsilon)(x),\epsilon\eta)]h\left(\frac{x_1(\infty)(\epsilon)(x)-x}{\epsilon}\right)\right]. \tag{4.3}$$

The result follows by standard arguments of Malliavin Calculus depending on a parameter [28, 27, 45] because, in all the Sobolev spaces of Malliavin Calculus, when $\epsilon \to 0$, $x_1(1)(\epsilon)(x) \to x$, $x_1(2)(\epsilon)(x) \to x$, $\frac{x_1(\infty)(\epsilon)(x)-x}{\epsilon}$ tends to the nondegenerate Gaussian variable $\Pi(x)B_1(\infty)$ and

$$\epsilon^{-2}S(x_.(.)(\epsilon)(x), \epsilon\eta) \to \sum_{i,j}\alpha_{i,j}(x)\int_D \eta_i \wedge \eta_j + \sum_i \int_D \eta_i \wedge dB_i\,. \tag{4.4}$$

Let us write the measure ν_ϵ

$$h \to E[G_\epsilon h(Z_\epsilon)] \tag{4.5}$$

G_ϵ depends smoothly on ϵ in all of the Sobolev spaces of the Malliavin Calculus as well as Z_ϵ. Moreover Z_ϵ satisfies uniformly in ϵ Malliavin's nondegeneracy condition: $\sup_\epsilon E[\langle \nabla Z_\epsilon, \nabla Z_\epsilon\rangle^{-p}] < \infty$ for all positive integers p.

We have:

$$\frac{d^r}{d\epsilon^r}\nu_\epsilon h = \sum_{|(r')|\le r} E\left[\tilde{G}_{(r')}(\epsilon)\frac{\partial^{(r')}}{\partial y^{(r')}}h(Z_\epsilon)\right]. \tag{4.6}$$

But by Malliavin's condition of nondegeneracy, we can remove the derivative of h in (4.6) and we get

$$\frac{d^r}{d\epsilon^r}\nu_\epsilon f = E[\overline{G}_r(\epsilon)h(Z_\epsilon)] \tag{4.7}$$

Therefore $\tilde{q}(x, 0)$ is smooth in ϵ.

At $\epsilon = 0$, the Malliavin matrix of the Gaussian Z_0 is deterministic and $\overline{G}_r(0)$ contains only expressions in the Gaussian terms which are of the same parity as r. If r is odd

$$E[\overline{G}_r(0)|\Pi(x)B_1(\infty) = 0] = 0 \tag{4.8}$$

because we consider centered Gaussian variables. $\tilde{q}_\epsilon(x, 0)$ has therefore an asymptotic expansion $\sum \epsilon^i \overline{Q}_i(f, g)(x)$ but only the even powers of ϵ remain in this asymptotic expansion: namely the odd exponent leads to the expectation of odd functionals of some centered Gaussian measures, which are 0. The introduction of derivatives of f and g is due to the asymptotic expansion of $f(x_1(1)(\epsilon)(x))$ and of $g(x_1(2)(\epsilon)(x))$ when $\epsilon \to 0$ because $x_1(1)(\epsilon)(x)$ and $x_1(2)(\epsilon)(x)$ go to x in all the Sobolev spaces of the Malliavin Calculus when $\epsilon \to 0$. □

On the other hand, $p_\epsilon(x, x)$ has an asymptotic expansion:

$$p_\epsilon(x, x) = \epsilon^{-d} \sum_{i=1}^{n} h^i c_i(x) + O(h^n) \tag{4.9}$$

where the coefficients belong to $C_b^\infty(\mathbb{R}^d)$ and $c_0(x) > c > 0$.

Proof of theorem B. We get

$$f *_{st,h} g(x) = \frac{\sum h^i \tilde{Q}_i(f, g)(x) + O(h^n)}{\sum_{i=0}^{n} h^i c_i(x) + O(h^n)}. \tag{4.10}$$

The result holds because $c_0(x) > c > 0$. □

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