

Local vs. global hyperconvexity, tautness or k -completeness for unbounded open sets in \mathbb{C}^n

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Abstract. Some known localization results for hyperconvexity, tautness or k -completeness of bounded domains in \mathbb{C}^n are extended to unbounded open sets in \mathbb{C}^n .

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1. Introduction

For bounded domains in \mathbb{C}^n the following localization results are well known:

A bounded domain $D \in \mathbb{C}^n$ is taut, hyperconvex or Kobayashi complete if and only if D is locally taut, locally hyperconvex or locally Kobayashi complete, respectively.

The aim of this note is to extend these results to unbounded domains imposing the correct conditions at infinity. So we get:

1. *An unbounded open set $D \subset \mathbb{C}^n$, locally hyperconvex at every finite boundary point, for which ∞ is a plurisubharmonic (psh) peak point, is hyperconvex (cf. Proposition 2.2).*
2. *Let $D \subset \mathbb{C}^n$ be an unbounded open set for which ∞ is a barrier point. Then D is locally taut if and only if D is taut (cf. Corollary 3.5).*
3. *An unbounded open set $D \subset \mathbb{C}^n$, for which ∞ is a k' -point, is Kobayashi complete if and only if it is locally Kobayashi complete (cf. Proposition 3.6).*

Moreover, a characterization for unbounded Kobayashi hyperbolic domains is given, namely:

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4. An unbounded open set $D \subset \mathbb{C}^n$ is Kobayashi hyperbolic if and only if for all $b \in D$ one has $\liminf_{z \rightarrow b, w \rightarrow \infty} l_D(z, w) > 0$, where l_D denotes the Lempert function of D (cf. Proposition 3.1).

We apply the above results to characterize Hartogs domains.

The above localization results are presented for any open, not necessarily connected, subset of \mathbb{C}^n . To this end we will introduce several pointwise conditions like that of t -point, k' -point and k -point.

2. Barrier points and hyperconvex open sets

A boundary point a of an open set D in \mathbb{C}^n is said to be a (global) barrier point for D if there is a negative psh function φ on D with $\lim_{z \rightarrow a} \varphi(z) = 0$ (here and below $\infty \in \partial D$ if D is unbounded); φ is called a (global) barrier function for D at a .

We call a a local barrier point for D if there is an open neighborhood U of a (neighborhood in this paper always means an open neighborhood) such that a is a barrier point for $D \cap U$.

Proposition 2.1. *If ∞ is a barrier point of an (unbounded) open set D in \mathbb{C}^n , then D admits a bounded strictly psh barrier function at ∞ . In particular, any connected component of D is hyperbolic (i.e. the Kobayashi pseudodistance is a distance) [15] and possesses a Bergman metric [5].¹*

Proof. Let ψ be a barrier function for D at ∞ . We may assume that $-1 < \psi < 0$. Let $\{D_j\}$ be a sequence of open sets such that $D_j \subset\subset D_{j+1}$ and $\cup_{j=1}^\infty D_j = D$. Then there are balls $\mathbb{B}(0, r_j)$,² $j \in \mathbb{N}$, with

$$\alpha_j := \inf_{D \setminus \mathbb{B}(0, r_j)} \psi > \beta_j := \sup_{D_j} \psi.$$

Setting

$$\varphi_j(z) := \begin{cases} \psi(z), & z \in D \setminus \mathbb{B}(0, r_j) \\ \max\{\psi(z), r_j^{-2}(\alpha_j - \beta_j)\|z\|^2 + \beta_j\}, & z \in D \cap \mathbb{B}(0, r_j) \end{cases},$$

it is easy to check that the function $\varphi := \sum_{j=1}^\infty \frac{\varphi_j}{2^j}$ has the required properties. \square

Remark. We do not know whether the hyperbolicity of D already follows by the weaker assumption that ∞ is a local barrier point.

On the other hand, we shall see later (Remark (ii) after Example 4.3) that there are unbounded open sets for which ∞ is a local barrier point but not a global one.

¹ The result in [5] requires the bounded strictly psh function to be continuous. However, the proof there works without this assumption.

² $\mathbb{B}(a, r) := \mathbb{B}_n(a, r) := \{z \in \mathbb{C}^n : \|z - a\| < r\}$.

We say that an open subset $D \subset \mathbb{C}^n$ is *hyperconvex* if there is a negative psh exhaustion function φ of D , i.e. $\lim_{z \rightarrow \partial D} \varphi(z) = 0$.

In the literature sometimes the definition of hyperconvexity requires the exhaustion function φ to be continuous. But it is well known (see [19]) that both definitions are equivalent (see also Remark (ii) after the proof of Lemma 2.3).

Note that D is hyperconvex if and only if any of its connected components is hyperconvex. Indeed, it suffices to consider the case when D has infinitely many connected components D_1, D_2, \dots . Assume that every D_j is hyperconvex with respect to the negative psh exhaustion function φ_j . Then replace $\varphi_j, j \in \mathbb{N}$, by $\tilde{\varphi}_j := \max\{\varphi_j, -j^{-1}\}$. Hence, D is hyperconvex with respect to the psh function $\tilde{\varphi}$ on D , where $\tilde{\varphi}|_{D_j} := \tilde{\varphi}_j$.

Denote by $g_D(a, \cdot)$ the *pluricomplex Green function* of D with pole at $a \in D$, i.e.

$$g_D(a, \cdot) := \sup\{u(\cdot)\},$$

where the supremum is taken over all psh $u : D \mapsto [-\infty, 0]$ such that $u(\cdot) - \log \|\cdot - a\|$ is bounded from above near a .

It is clear that if a and z belong to the same connected component \tilde{D} of D , then $g_D(a, z) = g_{\tilde{D}}(a, z) < 0$. Otherwise, $g_D(a, z) = 0$. Recall that $g_D(a, \cdot)$ is a psh function.

Moreover, if D is a bounded domain in \mathbb{C}^n and $a \in D$, then it is well-known (cf. [11]) that

$$D \text{ is hyperconvex} \quad \text{if and only if} \quad \lim_{z \rightarrow b} g_D(a, z) = 0, \quad b \in \partial D.$$

We mention that from the proof of Proposition 2.2 below it will follow that the same result remains true for an unbounded open set.

Recall that a boundary point a of an open set D in \mathbb{C}^n is said to be a *psh peak point* for D if there exists a psh function φ on D with $\lim_{z \rightarrow a} \varphi(z) = 0$ and $\limsup_{z \rightarrow b} \varphi(z) < 0$ for any $b \in \partial D \setminus \{a\}$; φ is called a *psh peak function* for D at a . Observe the local character of that notion. Moreover, any psh peak point is a barrier point.

In [9] it is shown that any bounded locally hyperconvex domain is hyperconvex. Now we extend this result to the case of unbounded open sets.

Proposition 2.2. *Let D be an unbounded open set in \mathbb{C}^n which is locally hyperconvex at any finite boundary point (i.e. for any finite $a \in \partial D$ there is a neighborhood U of a such that any connected component of $D \cap U$ is hyperconvex). If ∞ is a psh peak point, then D is hyperconvex.*

To prove this proposition, we shall use the following

Lemma 2.3. *Let $D_2 \subset D_1 \subset D$ be open sets in \mathbb{C}^n with $D_1 \neq D$ and $\partial D_2 \cap D \subset D_1$. Let ψ be a negative psh function on D such that*

$$\alpha := \inf_{D \cap \partial D_1} \psi > \beta := \sup_{D \cap \partial D_2} \psi.$$

For $a \in D_2$, set $d(a) := \inf_{D \cap \partial D_2} g_{D_1}(a, \cdot)$. Then

$$g_D(a, z) \geq g_{D_1}(a, z) + \frac{\alpha}{\beta - \alpha} d(a) \text{ if } z \in D_1,$$

and

$$g_D(a, z) \geq \frac{\psi(z)}{\beta - \alpha} d(a) \text{ if } z \in D \setminus D_1.$$

Proof. We may assume that $d(a) > -\infty$. Then the function $u(a, z) := \frac{\psi(z) - \alpha}{\beta - \alpha} d(a)$, $z \in D$, satisfies

$$u(a, z) \leq g_{D_1}(a, z), \quad z \in D \cap \partial D_2,$$

and

$$u(a, z) \geq 0 \geq \limsup_{D_1 \ni \zeta \rightarrow z} g_{D_1}(a, \zeta), \quad z \in D \cap \partial D_1.$$

Therefore, it follows that

$$v(a, z) := \begin{cases} g_{D_1}(a, z), & z \in D_2 \\ \max\{g_{D_1}(a, z), u(a, z)\}, & z \in D_1 \setminus D_2 \\ u(a, z), & z \in D \setminus D_1 \end{cases}$$

is a psh function in the second variable with a logarithmic pole at a . Moreover,

$$v(a, z) \leq \frac{\alpha}{\alpha - \beta} d(a), \quad z \in D.$$

Hence, by definition of g_D , one has

$$g_D(a, z) \geq v(a, z) + \frac{\alpha}{\beta - \alpha} d(a),$$

which implies the desired results. □

Remarks. (i) The first inequality in Lemma 2.3 is sharp. To see this, it is enough to consider a bounded hyperconvex domain D and $D_1 = \{w \in D : g_D(a, w) < \alpha\}$, $D_2 = \{w \in D : g_D(a, w) < \beta\}$, where $a \in D$ and $\beta < \alpha < 0$ (cf. [20]).

(ii) It is well known that for a bounded hyperconvex domain $D \subset \mathbb{C}^n$ the pluricomplex Green function g_D is continuous on $D \times \overline{D}$, where $g_D|_{D \times \partial D} := 0$ (see [11]). The same result remains true for an arbitrary hyperconvex open set D . In particular, $e^{g_D(a, \cdot)} - 1$, $a \in D$, is a negative continuous psh exhaustion function of D .

Indeed, to see this, note that the second inequality in Lemma 2.3 applied for a negative psh exhaustion function of D implies that $\lim_{z \rightarrow a, w \rightarrow \partial D} g_D(z, w) = 0$ ³ for

³ Observe that we may choose $D_2 := \mathbb{B}(a, r) \subset\subset D_1 \subset\subset D$ with D_1 hyperconvex.

any $a \in D$ (this lemma also shows that $g_D(z, w) > -\infty$ for $z \neq w$). Moreover, g_D is upper semicontinuous as a function of both variables (see [10]). Assume now that g_D is not continuous at some point $(a, b) \in D \times D$. Then there is $\varepsilon > 0$ and a sequence $(z_j, w_j) \rightarrow (a, b)$ such that $g_D(z_j, w_j) \leq g_D(a, b) - 2\varepsilon$. Let φ be a negative psh exhaustion function of D . We may choose open sets $D_2 \subset\subset D_1 \subset\subset D$ such that $D_2 \times D_1$ contains the above sequence, D_1 is pseudoconvex with smooth boundary,

$$-1 \leq \inf\{g_D(z_j, w) : j \in \mathbb{N}, w \in D \cap \partial D_2\} \leq d(z_j),$$

and

$$\inf_{D \cap \partial D_1} \varphi \geq \frac{\varepsilon}{1 + \varepsilon} \sup_{D \cap \partial D_2} \varphi.$$

Then Lemma 2.3 implies that $g_D(z_j, w_j) \geq g_{D_1}(z_j, w_j) - \varepsilon$. The assumptions for D_1 imply that it is hyperconvex [9]. Hence g_{D_1} is a continuous function and letting $j \rightarrow \infty$ this gives the contradiction

$$g_D(a, b) - 2\varepsilon \geq g_{D_1}(a, b) - \varepsilon \geq g_D(a, b) - \varepsilon.$$

Proof of Proposition 2.2. It is enough to show that

$$\lim_{D \ni w \rightarrow a} g_D(z, w) = 0 \tag{2.1}$$

for any $a \in \partial D$ and any $z \in D$.

First, let $a = \infty$. We shall prove (2.1) under the weaker condition that ∞ is a barrier point. By Proposition 2.1, there exists a strictly psh barrier function φ at ∞ . Choose a smooth function χ such that $\chi = 1$ near z and $\text{supp } \chi \subset\subset D$. Then there is a constant $c > 0$ for which

$$u_z(w) := c\varphi(w) + \chi(w) \log \|w - z\|, \quad w \in D,$$

is a psh function on D with a logarithmic pole at z . Hence, $g_D(z, w) \geq u_z(w)$, $w \in D$, which implies (2.1).

Now let $a \in \partial D \cap \mathbb{C}^n$. Let $r > 0$ be such that $a, z \in \mathbb{B}(0, r)$. If ψ is a psh peak function for D at ∞ , then $\sup_{D \cap \partial \mathbb{B}(0, r)} \psi < 0$. In virtue of the properties of ψ

there is $r' > r$ such that

$$2 \inf_{D \cap \partial \mathbb{B}(0, r')} \psi > \sup_{D \cap \partial \mathbb{B}(0, r)} \psi.$$

Put $\hat{D} := D \cap \mathbb{B}(0, r')$. Then, applying Lemma 2.3, we have

$$g_D(z, w) \geq g_{\hat{D}}(z, w) + \inf_{D \cap \partial \mathbb{B}(0, r)} g_{\hat{D}}(z, \cdot) \geq g_{\hat{D}}(z, w) + \inf_{D \cap \partial \mathbb{B}(0, r)} g_D(z, \cdot).$$

Denote by \tilde{D} the connected component of \hat{D} containing z . Since \tilde{D} is a hyperconvex domain (see [9]), it follows that $\lim_{\tilde{D} \ni w \rightarrow a} g_{\tilde{D}}(z, w) = 0$. On the other hand, $g_{\tilde{D}}(z, w) = 0$ if $w \notin \tilde{D}$. Therefore,

$$\liminf_{w \rightarrow a} g_D(z, w) \geq \inf_{D \cap \partial \mathbb{B}(0,r)} g_D(z, \cdot).$$

Using that r was arbitrary and applying the first case, we are lead to

$$\inf_{D \cap \partial \mathbb{B}(0,r)} g_D(z, \cdot) \xrightarrow{r \rightarrow \infty} 0;$$

hence, we get (2.1). □

Remarks. (i) We do not know whether Proposition 2.2 remains true if ∞ is a (global or local) barrier point.

On the other hand, if Π is the upper-half plane in \mathbb{C} , then the unbounded domain $D := \Pi \times \mathbb{C}$ is locally hyperconvex at any finite boundary point but D is not hyperconvex.

(ii) Any hyperconvex domain D admits a bounded strictly psh exhaustion function. In particular, its Bergman metric is complete [4].

Indeed, note that Proposition 2.1 implies that there is a strictly psh function ψ on D with $-2 < \psi < -1$. Now let φ be a psh exhaustion function with $-1 < \varphi < 0$. Choose a sequence $\{D_j\}$ of domains such that $D_j \subset\subset D_{j+1}$, $\cup_{j=1}^\infty D_j = D$, and

$$\max_{D_j} \frac{\varphi}{2} \leq b_j := \inf_{D \setminus D_{j+1}} \varphi. \text{ Then}$$

$$\varphi_j := \begin{cases} -b_j \psi & \text{on } D_j \\ \max\{-b_j \psi, \varphi\} & \text{on } D_{j+1} \setminus D_j \\ \varphi & \text{on } D \setminus D_{j+1} \end{cases}$$

is a psh function on D with $-1 < \varphi_j < 0$. Then it is easy to check that $\tilde{\varphi} := \sum_{j=1}^\infty \frac{\varphi_j}{2^j}$

is a strictly psh exhaustion function of D with $-1 < \tilde{\varphi} < 0$.

(iii) The argument at the beginning of the proof of Proposition 2.2 implies that an open set D in \mathbb{C}^n is hyperconvex if and only if any of its boundary points is a barrier point.

More generally, Z. Blocki (see [2], Problem 1) asks whether a bounded domain in \mathbb{C}^n is hyperconvex if any of its boundary points is a local barrier point (here the word ‘‘any’’ is essential – Example 4.1 in Paragraph 4 provides a bounded pseudoconvex domain for which the set of all boundary points that are not local barrier points, is a circle). In the next paragraph we shall see that such a domain is, at least, taut.

3. Hyperbolic, taut and k -complete open sets

Recall that an open set D in \mathbb{C}^n is said to be *taut* if the family $\mathcal{O}(\mathbb{D}, D)$ of all holomorphic mapping from the unit disc $\mathbb{D} := \mathbb{B}_1(0, 1) \subset \mathbb{C}$ into D is *normal*, i.e. any sequence $\{f_j\} \subset \mathcal{O}(\mathbb{D}, D)$ contains a subsequence $\{f_{j_k}\}$ with $f_{j_k} \xrightarrow[k \rightarrow \infty]{} f \in \mathcal{O}(\mathbb{D}, D)$ (convergence in the compact-open topology) or a subsequence that diverges compactly. Observe that this condition is equivalent to the following one: either $\{f_j\}$ is compactly divergent or it allows a subsequence that converges compactly uniformly to an $f \in \mathcal{O}(\mathbb{D}, D)$. It is well-known that any bounded hyperconvex domain is taut and that any taut domain is pseudoconvex and hyperbolic (cf. [11]). Note that the unbounded hyperconvex domains are also taut (cf. [14] or combine Propositions 2.1 and 3.4 below).

We say that $a \in \partial D$ is a *locally taut point* for D if there is a neighborhood U of a such that the set $D \cap U$ is taut. D is said to be *locally taut* if any finite boundary point of D is a locally taut point for D . Obviously, any boundary point of a taut open set is a locally taut point.

Note that D is taut if and only if any of its connected components is taut. Indeed, it is enough to show that if D has infinitely many connected components D_1, D_2, \dots , and $\{f_j\} \subset \mathcal{O}(\mathbb{D}, D)$ such that $\#\{j : f_j(\mathbb{D}) \subset D_k\} < \infty, k \in \mathbb{N}$, then $\{f_j\}$ diverges compactly. To see this, it suffices to observe that any $K \subset\subset D$ intersects finitely many connected components of D by Lebesgue’s covering theorem.

For a domain D in \mathbb{C}^n , denote by l_D its *Lempert function*, i.e.

$$l_D(z, w) := \inf\{|\lambda| : \exists \varphi \in \mathcal{O}(\mathbb{D}, D) \text{ with } \varphi(0) = z \text{ and } \varphi(\lambda) = w\}.$$

Let now D be an open set in \mathbb{C}^n , and $z, w \in D$. If z, w belong to the same connected component \tilde{D} of D , then we set $l_D(z, w) := l_{\tilde{D}}(z, w)$. Otherwise, we define $l_D(z, w) := 1$.

We call $a \in \partial D$ a (global) *t-point* for D if

$$\lim_{z \rightarrow a, w \rightarrow b} l_D(z, w) = 1$$

for any $b \in D$. Note that

$$\begin{aligned} a \text{ is a } t\text{-point if and only if for any } \{f_j\} \subset \mathcal{O}(\mathbb{D}, D) \text{ with } f_j(0) \rightarrow a \\ \text{it follows that } \{f_j\} \text{ diverges compactly.} \end{aligned} \tag{*}$$

We say that $a \in \partial D$ is a *local t-point* for D if there is a neighborhood U of a such that a is a *t-point* for $D \cap U$. Thus, any locally taut point is a local *t-point*.

On the other hand, there are *t-points* which are not locally taut points. For example, for small $\varepsilon > 0$ the domain

$$D_\varepsilon := \{(z, w) \in \mathbb{C}^2 : |z + 1|^2 + |w|^2 \sin |w|^{-1}| < 1, |z| < 2, |w| < \varepsilon\}$$

is non-pseudoconvex at $(0, 0)$ but $|e^z| - 1$ is a barrier function for D_ε at the origin and so $(0, 0)$ is a t -point of D_ε (cf. Proposition 3.4 below).

For convenience, we say that an open set $D \subset \mathbb{C}^n$ is *hyperbolic* if any of its connected components is hyperbolic.

That tautness implies hyperbolicity is also reflected by the following result.

Proposition 3.1. *Any connected component of an unbounded open set D in \mathbb{C}^n is hyperbolic if and only if*

$$\liminf_{z \rightarrow \infty, w \rightarrow b} l_D(z, w) > 0 \tag{3.1}$$

for any $b \in D$.

Proof. Recall that D is hyperbolic if and only if the family $\mathcal{O}(\mathbb{D}, D)$ is *evenly continuous* (cf. [1], [11]), i.e. for any $b \in D$ and any neighborhood U of b there are a neighborhood V of b and $s \in (0, 1)$ such that if $f(0) \in V$, then $f(s\mathbb{D}) \subset U$ for any $f \in \mathcal{O}(\mathbb{D}, D)$.

Using this criterion it is clear that, if D is hyperbolic, then (3.1) holds.

To prove the converse assume that D is not hyperbolic. Then (in virtue of the cited criterion) we may find a point $b \in D$ and sequences $\{s_j\} \subset \mathbb{D}$, $\{f_j\} \subset \mathcal{O}(\mathbb{D}, D)$ with $s_j \rightarrow 0$, $f_j(0) \rightarrow b$, $f_j(s_j) \not\rightarrow b$. If $0 < r := \liminf_{z \rightarrow \infty, w \rightarrow b} l_D(z, w)$, then $f_j(0) \rightarrow b$ implies that $\{f_j\}$ is locally bounded on $r\mathbb{D}$ and therefore $f_j(s_j) \rightarrow b$ by Montel’s theorem giving a contradiction. \square

Proposition 3.2. *Let D be an open set in \mathbb{C}^n . Assume that ∞ is a t -point if D is unbounded. Then the following conditions are equivalent:*

- (i) D is taut;
- (ii) any finite boundary point of D is a locally taut point;
- (iii) any finite boundary point of D is a local t -point;
- (iv) any finite boundary point of D is a t -point.

Proof. It is clear that (iv) \leftrightarrow (i) \rightarrow (ii) \rightarrow (iii). Thus, it suffices to show that (iii) \rightarrow (i). Consider an arbitrary sequence $\{f_j\} \subset \mathcal{O}(\mathbb{D}, D)$. Assume that it is not compactly divergent. Since ∞ is a t -point, it easily follows that this sequence has a subsequence which is locally bounded. Thus, we may assume that $f_j \rightarrow f \in \mathcal{O}(\mathbb{D}, \overline{D})$. It suffices to show that if $a := f(0) \in \partial D$, then $f(\mathbb{D}) \subset \partial D$. Let U be a neighborhood of a such that a is a t -point of $D \cap U$. Observe that there is $r \in (0, 1]$ with $f_j(r\mathbb{D}) \subset U$ for any j large enough. Then, by (*), $f(r\mathbb{D}) \subset \partial D$. Let now $r_0 \in (0, 1]$ be the biggest one with the last property, i.e. $r_0 := \sup\{r \in (0, 1] : f(r\mathbb{D}) \subset \partial D\}$. Assuming $r_0 < 1$, it follows that $f(\overline{r_0\mathbb{D}}) \subset \partial D$. Using that for any s with $|s| = r_0$, $f(s)$ is a local t -point and applying a compactness argument, we conclude (as above) that there is $r_1 \in (r_0, 1]$ with $f(\overline{r_1\mathbb{D}}) \subset \partial D$, giving a contradiction. \square

Remarks. (i) The assumption in Proposition 3.2 that ∞ is a global t -point cannot be replaced by only assuming ∞ to be a local t -point. Indeed, ∞ and 0 are local t -point of \mathbb{C}^* but \mathbb{C}^* is not even hyperbolic.

(ii) Example 4.1 in Paragraph 4 will show that the set of all non-local t -points of a bounded pseudoconvex domain may be very small, e.g. a circle.

(iii) In [18], Theorem 2.5, the following statement may be found:

An unbounded domain $D \subset \mathbb{C}^n$ which is locally taut is taut if and only if D has the Schottky property.

There the Schottky property is defined as follows (see [18], Definition 2.3): D has the Schottky property for $h_{\mathbb{C}^n}$ (here $h_{\mathbb{C}^n}$ is the Euclidean metric of \mathbb{C}^n) if for every point $p \in D$, each relatively compact open set $W \subset \mathbb{C}^n$, and any $r \in (0, 1)$ there exists $S > 0$ such that every $f \in \mathcal{O}(\mathbb{D}, D)$, $f(0) \in W$, satisfies $\|p - f(\lambda)\| \leq S$, $\lambda \in r\overline{\mathbb{D}}$.

The following example may show that with this definition of the Schottky property the above characterization is not true. Namely, take

$$D := \{(z, w) \in \mathbb{C}^2 : 0 < |w|e^{\varphi(z)} < 1\},$$

where φ is a subharmonic function on \mathbb{C} , bounded from below, with $\varphi(z) \xrightarrow{|z| \rightarrow \infty} \infty$ and $\varphi(z) = \varphi(|z|)$, $z \in \mathbb{C}$. The D does not fulfill the above Schottky property. Indeed, take $p := (0, w_0) \in D$ and $W := \mathbb{B}((0, w_0), |w_0|) \subset\subset \mathbb{C}^2$, and $f_{R,\delta}(\lambda) := (R\lambda, \delta)$, $\lambda \in \mathbb{D}$.

On the other side, using Hurwitz’s and Montel’s theorems it is easy to verify that D is nevertheless taut.

With the correct definition of the *Schottky property*, i.e. requiring that the open set W in the above definition is relatively compact in D , the equivalence of (i) and (ii) in Proposition 3.2 is essentially contained in [18].

We do not know any example of a hyperbolic domain which is locally taut but not taut.

In [3] a non trivial example of an unbounded hyperbolic domain is discussed. This domain is also locally taut. As an application of Proposition 3.2 we now show that it is taut.

Let X be any closed, 1-dimensional complex subvariety of \mathbb{C}^2 . In [3] (see also [7]), a domain $\Omega \subset \mathbb{C}^2$ containing X with $\overline{\Omega} \neq \mathbb{C}^2$ and a biholomorphic map Φ from Ω onto \mathbb{C}^2 are constructed such that $D := \mathbb{C}^2 \setminus \Phi(X)$ is a hyperbolic domain. Using the ideas from [3] and Proposition 3.2, unfortunately it turns out that one has

Example 3.3. The domain D from [3] is taut.

Proof. Since $\Phi(X)$ is an analytic subset of \mathbb{C}^2 , it follows from Montel’s and Hurwitz’s theorems that any point of $\Phi(X)$ is a local t -point of D .

To see that ∞ is a t -point for D , we use that Φ is the limit of biholomorphic maps $\Phi_l : \mathbb{C}^2 \rightarrow \mathbb{C}^2, l \in \mathbb{N}$, which have the following property:

$$\text{if } \varphi \in \mathcal{O}(\mathbb{D}, \mathbb{B}(0, l + 2) \setminus \Phi(X)), \varphi(0) \in \mathbb{B}(0, l/2)$$

$$\text{and } \text{dist}(\varphi(0), \Phi_l(X)) \geq 1/l, \text{ then } \varphi((1 - 2^{-l})\mathbb{D}) \subset \mathbb{B}(0, l + 1).$$

Let now $\{\varphi_j\} \subset \mathcal{O}(\mathbb{D}, D)$ be such that $\varphi_j(0) \rightarrow b \in D$ as $j \rightarrow \infty$. Fixing $r \in (0, 1)$ we may find $m \in \mathbb{N}$ with $\prod_{k=m}^\infty (1 - 2^{-k}) \geq r$ and $\text{dist}(\varphi_j(0), \Phi_l(X)) \geq 1/l$ for any $j, l \geq m$. For given $j \geq m$, let $l \geq m$ be such that $\varphi_j(r\mathbb{D}) \subset \mathbb{B}(0, l + 2)$. Then $\varphi_j(r(1 - 2^{-l})\mathbb{D}) \subset \mathbb{B}(0, l + 1)$ and, by induction, $\varphi_j(rs\mathbb{D}) \subset \mathbb{B}(0, m + 1)$, where $s := \prod_{k=m}^l (1 - 2^{-k})$. Hence $\varphi_j(r^2\mathbb{D}) \subset \mathbb{B}(0, m + 1)$ for any $j \geq m$ and thus

$$\liminf_{z \rightarrow \infty, w \rightarrow b} l_D(z, w) \geq r^2.$$

Letting $r \rightarrow 1$ shows that ∞ is a t -point for D and, therefore, D is taut. □

In connection with the above example, it would be interesting to know whether there is a domain $\mathbb{C}^2 \setminus X$ with X an analytic subset of \mathbb{C}^2 such that D is hyperbolic but not taut.

On the other hand, there is the following sufficient criterion for a t -point.

Proposition 3.4. *Any barrier point of a hyperbolic open set D in \mathbb{C}^n is a t -point.*

Proof. Let $a \in \partial D$ and let φ be a barrier function at a . Assume that a is not a t -point. Then there exist $\{f_j\} \subset \mathcal{O}(\mathbb{D}, D)$ and $\{t_j\} \subset \mathbb{D}$ with $f_j(0) \rightarrow a, t_j \rightarrow t_0 \in \mathbb{D}$, and $f_j(t_j) \rightarrow b \in D$. Since D is hyperbolic, it follows that for any neighborhood V of b there is a neighborhood U of t_0 such that $f_j(U) \subset V$ for any large j . We may assume that $V \subset\subset D, U \subset r\mathbb{D}$ for some $r \in (0, 1)$, and $f_j(U) \subset V$ for any j . Then we find $\varepsilon < 0$ with $\int_{r\mathbb{D}} \varphi \circ f_j < \varepsilon$ for any j . On the other hand, the last integral is larger or equal $\pi r^2 \varphi(f_j(0))$. Using $\varphi(f_j(0)) \rightarrow 0$, we get a contradiction. □

Combining Propositions 3.2 and 3.4 leads to the following result.

Corollary 3.5. *If any finite boundary point of an unbounded open set D in \mathbb{C}^n is a locally taut point and ∞ is a barrier point, then D is taut.*

In particular, if any finite boundary point is a local barrier point and ∞ is a barrier point, then D is taut.

Remark. The first part of Corollary 3.5 is contained as Proposition 2 in [8] (see also Corollary 2.10 in [18]).

Recall now that the Kobayashi pseudodistance k_D of a domain D in \mathbb{C}^n is the largest pseudodistance which does not exceed $\tanh^{-1} l_D$. The Kobayashi pseudo-metric κ_D at a point $(z, X) \in D \times \mathbb{C}^n$ is defined as

$$\kappa_D(z; X) := \inf\{\alpha > 0 : \exists \varphi \in \mathcal{O}(\mathbb{D}, D) \text{ with } \varphi(0) = z \text{ and } \alpha\varphi'(0) = X\}.$$

There is a tight relationship between the Kobayashi pseudodistance and metric, namely (cf. [11]),

$$k_D(z, w) = \inf \left\{ \int_0^1 \kappa_D(\gamma(t); \gamma'(t)) dt \right\},$$

where the infimum is taken over all piecewise C^1 -curves $\gamma : [0, 1] \rightarrow D$ connecting z and w .

It is well-know that a bounded domain D in \mathbb{C}^n is k -complete (i.e. complete w.r.t. to the Kobayashi pseudodistance) if and only if D is locally k -complete (cf. [6], [11]).

Similarly to the Lempert function, we may define the Kobayashi pseudodistance for an open set D in \mathbb{C}^n . If z, w belong to the same connected component \tilde{D} of D , then we set $k_D(z, w) := k_{\tilde{D}}(z, w)$. Otherwise, we put $k_D(z, w) := \infty$.

Then it is clear that D is k -complete if and only if any of its connected component is k -complete.

Now we introduce the notions of (local) k -points and k' -points for an open set $D \subset \mathbb{C}^n$.

We say that $a \in \partial D$ is a k -point for D if

$$\lim_{z \rightarrow a} k_D(z, b) = \infty$$

for any $b \in D$ (equivalently, for one $b \in D$ from any connected component of D). For example, it is essentially shown in [8] (see also [12]) that any *local holomorphic peak point* (possible ∞) is a k -point.

We call $a \in \partial D$ a k' -point for D if there is no k_D -Cauchy sequence converging to a . It is clear that any k -point is a k' -point. The notions of *local k -points* and *local k' -points* may be defined in an obvious way. Then we have

Proposition 3.6. *Let D be an open set in \mathbb{C}^n . Assume that ∞ is a k' -point if D is unbounded. The following conditions are equivalent:*

- (i) D is k -complete;
- (ii) any finite boundary point of D admits a neighborhood U such that $D \cap U$ is k -complete;
- (iii) any finite boundary point of D is a local k' -point;
- (iv) any finite boundary point of D is a k' -point;
- (v) any boundary point of D is a local k -point;
- (vi) any boundary point of D is a k -point.

Proof. It is trivial that (i) \rightarrow (ii) \rightarrow (iii), and (vi) \rightarrow (v) \rightarrow (iii). We also have (i) \rightarrow (vi) (cf. Theorem 7.3.2 in [11]).

Observe that the assumption that ∞ is a k' -point implies

$$\liminf_{w \rightarrow \infty} k_D(z, w) > 0, \quad z \in D$$

(if this is not the case, use the triangle inequality to get a contradiction). Hence, D is hyperbolic by Proposition 4 and thus any k_D -Cauchy sequence has at most one accumulation point in D . Since any boundary point is a k' -point, it follows that such a sequence has no accumulation points on ∂D . So (iv) \rightarrow (i).

Therefore, it is enough to show that (iii) \rightarrow (iv).

Assume the contrary. Then there exists a k_D -Cauchy sequence $\{z_j\} \subset D$, $z_j \rightarrow a \in \partial D \setminus \{\infty\}$.

Claim. There is $r > 0$ such that

$$3\varepsilon := \inf\{k_D(z_j, w) : j \in \mathbb{N}, w \in D \setminus \mathbb{B}(0, r)\} > 0.$$

Otherwise, for any l there are $j_l \in \mathbb{N}$ and $w_l \in D$ with $\|w_l\| > l$ and $k_D(z_{j_l}, w_l) < \frac{1}{l}$. Then

$$k_D(w_l, w_m) < \frac{1}{l} + \frac{1}{m} + k_D(z_{j_l}, z_{j_m}), \quad l, m \in \mathbb{N}.$$

If the sequence $\{j_l\}$ consists of only a finite number of elements of \mathbb{N} , then, by considering a subsequence if necessary, we may assume that $j_l = j_m$, for any l and m and, hence, $\{w_l\}$ is a k_D -Cauchy sequence. Otherwise, we may assume that $j_l \rightarrow \infty$. Then $\{w_l\}$ is again a k_D -Cauchy sequence, since z_{j_l} is one. This is a contradiction, since ∞ is a k' -point. So the claim is proved.

Now we may assume that $k_D(z_l, z_m) < \varepsilon$ for any l, m and consider a piecewise C^1 -curve $\gamma_{l,m} : [0, 1] \rightarrow D$ connecting z_l and z_m such that

$$k_D(z_l, z_m) \leq \int_0^1 \kappa_D(\gamma_{l,m}(t); \gamma'_{l,m}(t)) dt < 2k_D(z_l, z_m) < 2\varepsilon.$$

Since $k_D(z_l, \gamma_{l,m}(t)) < 2\varepsilon$, it follows that $\gamma_{l,m}([0, 1]) \subset \tilde{D} := D \cap \mathbb{B}(0, r)$ and

$$\tanh^{-1} l_D(\gamma_{l,m}(t), w) \geq k_D(\gamma_{l,m}(t), w) > \varepsilon, \quad t \in [0, 1], \quad w \in D \setminus \mathbb{B}(0, r).$$

Then the inequality (cf. the proof of Proposition 7.2.9 in [11])

$$\kappa_{\tilde{D}}(z; X) \inf_{w \in D \setminus \mathbb{B}(0, r)} l_D(z, w) \leq \kappa_D(z; X), \quad z \in \tilde{D},$$

implies that

$$k_{\tilde{D}}(z_l, z_m) < 2(\coth \varepsilon)k_D(z_l, z_m).$$

Hence $\{z_j\}$ is a $k_{\tilde{D}}$ -Cauchy sequence.

Now choose $r' > 0$ such that a is a k' -point for $\hat{D} := D \cap \mathbb{B}(a, r')$. Since \tilde{D} is bounded one has

$$\inf\{k_{\tilde{D}}(z_j, w) : j \gg 1, w \in \tilde{D} \setminus \mathbb{B}(a, r')\} > 0$$

and then repeating the argument of the previous paragraph, we see that the $k_{\tilde{D}}$ -Cauchy sequence $\{z_j\}$ is also a $k_{\hat{D}}$ -Cauchy sequence, giving a contradiction. So (iii) implies (iv). \square

Remark. This proof shows that any local k' -point of a bounded domain is a global one. The same is not true for k -points. Indeed, let Π be the upper half-plane in \mathbb{C} , $H := \Pi \times \mathbb{C}$, $\mathbb{B}_2 := \mathbb{B}_2(0, 1)$, and $D := 2\mathbb{B}_2 \setminus \overline{\mathbb{B}_2 \cap H}$. Note that 0 is a k -point for $D \cap \mathbb{B}_2$, since this domain is convex, but it is not even a t -point for D .

Note that if ∞ is a local peak point one has

Proposition 3.7. *If ∞ is a psh peak point and a local k' -point of an unbounded open set D in \mathbb{C}^n , then ∞ is a k' -point.*

Proof. Let $r > 0$ be such that ∞ is a k' -point for $\tilde{D} := D \setminus \overline{\mathbb{B}(0, r)}$. Then one has

$$\kappa_{\tilde{D}}(z; X) = \inf_{w \in D \cap \overline{\mathbb{B}(0, r)}} l_D(z, w) \leq \kappa_D(z; X), \quad z \in \tilde{D},$$

On the other hand, if $r_2 > r_1 > r$, then

$$\begin{aligned} & \inf\{\tanh^{-1} l_D(z, w) : z \in D \setminus \mathbb{B}(0, r_1), w \in D \cap \overline{\mathbb{B}(0, r)}\} \\ & \geq \inf\{k_D(z, w) : z \in D \cap \partial\mathbb{B}(0, r_1), w \in D \cap \partial\mathbb{B}(0, r)\} \\ & \geq \min\{k_{D \cup \mathbb{B}(0, r_2)}(z, w) : z \in \partial\mathbb{B}(0, r_1), w \in \partial\mathbb{B}(0, r)\} =: \varepsilon. \end{aligned}$$

Since the notion of psh peak point is of a local character, one has that ∞ is a psh peak point for $D \cup \mathbb{B}(0, r_2)$. Thus, by Proposition 1, any connected component of this set is hyperbolic and hence $\varepsilon > 0$. So

$$\kappa_{\tilde{D}}(z; X) \leq (\coth \varepsilon) \kappa_D(z; X), \quad z \in D \setminus \mathbb{B}(0, r_1).$$

Assume now that there exists a k_D -Cauchy sequence converging to ∞ . Repeating the argument of the proof of Proposition 3.6, it follows that all the members of this sequence belonging to \tilde{D} form a $k_{\tilde{D}}$ -Cauchy sequence, which is a contradiction. \square

4. Hartogs domains

Let G be a domain in \mathbb{C}^n . Recall that a domain $D \subset G \times \mathbb{C}^m$ is called a *Hartogs domain over G with m -dimensional (balanced) fibers* if

$$D = \{(z, w) \in G \times \mathbb{C}^m : h(z, w) < 1\},$$

where $h : G \times \mathbb{C}^m \rightarrow [0, \infty)$ is an upper semicontinuous function with $h(z, \lambda w) = |\lambda| h(z, w)$, $z \in G$, $\lambda \in \mathbb{D}$, $w \in \mathbb{C}^m$ (cf. [12]). It clear that D is bounded if and only if G is bounded and $\inf_{G \times \mathbb{S}} h > 0$, where \mathbb{S} is the unit sphere in \mathbb{C}^m . Note also that D is pseudoconvex if and only if G is pseudoconvex and $\log h$ is psh.

In the following example, which is inspired by [9], different kinds of boundary points appear.

If

$$\theta(z_1) := \sum_{j=1}^{\infty} 2^{-j} \log |1 - 2^{2j-1} z_1|,$$

$$\varphi(z_1) := \max\{|z_1|, 1 + \theta(z_1)\},$$

then

$$D := \{(z_1, z_2) \in \mathbb{D} \times \mathbb{C} : |z_2| e^{\varphi(z_1)} < 1\}$$

is a bounded pseudoconvex Hartogs domain over \mathbb{D} with one-dimensional fibers. Let E_0 be the set of all boundary points (a_1, a_2) of D with $a_1 \neq 0$. Set

$$E_1 := \{(0, a_2) \in \mathbb{C}^2 : |a_2| = 1\},$$

$$E_2 := \{(0, a_2) \in \mathbb{C}^2 : e^{-1} < |a_2| < 1\},$$

$$E_3 := \{(0, a_2) \in \mathbb{C}^2 : |a_2| = e^{-1}\}.$$

Observe that $\partial D = \cup_{j=1}^4 E_j$. One has

Example 4.1.

- (i) Any point of $E_0 \cup E_1$ is a barrier point and, therefore, by Proposition 3.4, a t -point;
- (ii) any point of E_2 is a local barrier point but not a global t -point and hence, by Proposition 3.4, a local t -point which is not a global barrier point; in particular, by Proposition 3.2, D is not a taut domain;
- (iii) any point of E_3 is not a local t -point and thus not a local barrier point.

In particular, $\partial D \setminus E_3$ is the set of local t -points. Then, applying Proposition 3.2 for a neighborhood of any of these points, it follows that $\partial D \setminus E_3$ is also the set of all locally taut points.

Proof. (i) Note that φ is a continuous function on \mathbb{C}^* . Hence the points of E_0 are barrier points. Moreover, $0 > \log |z_2| \rightarrow 0$ as $D \ni (z_1, z_2) \rightarrow E_1$. Thus the points of E_1 are also barrier points.

(ii) Let $a := (0, a_2) \in E_2$. We may assume that $a_2 > 0$. Choose $c \in (a_2, 1)$ and observe that the discs $f_j(t) := (2^{1-2j}, ct)$ belong to $\mathcal{O}(\mathbb{D}, D)$ if $2^{1-2j} \leq -\log c$. Since $f_j(t) \rightarrow f(t) := (0, ct)$ locally uniformly, $f(0) \in D$ and $f(a_2/c) = a$, it follows that a is not a t -point.

To see that a is a local barrier point, observe first that if F is the union of disjoint discs $\mathbb{D}(2^{1-2j}, 2^{-2j})$, $j = 1, 2, \dots$, then

$$\lim_{F \ni z_1 \rightarrow 0} \theta(z_1) = 0.$$

Indeed, $\log |1 - 2^{2j-1} z_1| \geq -\log 2$ for any $z_1 \notin F$ and any $j \geq 1$, hence

$$\theta(z_1) \geq \sum_{j=1}^k 2^{-j} \log |1 - 2^{2j-1} z_1| - \log 2 \sum_{j=k+1}^{\infty} 2^{-j}, \quad k \geq 1.$$

Letting $F \not\ni z_1 \rightarrow 0, k \rightarrow \infty$, and recalling that θ is upper semicontinuous, one obtains the desired equality.

Note now that if C is the complement of the closed disc $r\overline{\mathbb{D}}$ with $e^{-1} < r < 1$ and $z \in D \cap (\mathbb{D} \times C)$, then

$$\theta(z_1) < -\log(re) < 0.$$

From above it follows that there exists $\varepsilon > 0$ such that if z_1 satisfies this inequality and $|z_1| < \varepsilon$, then $z_1 \in F$. Hence

$$D \cap (\varepsilon\mathbb{D} \times C) \subset F \times C.$$

Defining $p(z) := -j^{-1}$ on $\mathbb{D}(2^{1-2j}, 2^{-2j}) \times C$, it follows that p is a barrier function for $D \cap (\varepsilon\mathbb{D} \times C)$ at any a with $a_1 = 0$ and $r < |a_2| < 1$. Since $r \in (e^{-1}, 1)$ was arbitrary chosen, we are done.

(iii) It follows from (i), (ii) and Proposition 3.2 that the set of boundary points which are not local t -points is non-empty and it is contained in E_3 . Since E_3 is an orbit w.r.t. z_2 -rotations, this implies the statement. \square

Remark. It is also possible to prove (iii) directly by constructing suitable discs as in the proof of (ii).

In [17] (see also [12]) it is shown that if D is hyperbolic, then

$$G \text{ is hyperbolic and } \inf_{K \times \mathbb{S}} h > 0 \text{ for any } K \subset\subset G. \tag{**}$$

Indeed, since $G \times \{0\} \subset D$, then G is hyperbolic. Assume now that $\inf_{K \times \mathbb{S}} h = 0$ for some $K \subset\subset G$. Then there is a sequence $D \supset \{a_j\}$ and a point $a := (a', a'') \in G \times \mathbb{C}^m, a'' \neq 0$, such that $a_j \rightarrow a$ and $h(a_j) \rightarrow 0$ as $j \rightarrow \infty$. Note that the discs $f_j(t) := (a'_j, ta''_j(h(a_j))^{-1})$ belong to $\mathcal{O}(\mathbb{D}, D), f_j(0) \rightarrow \tilde{a} := (a', 0) \in D$, and $f_j(t) \rightarrow \infty$ for any $t \in \mathbb{D} \setminus \{0\}$. Hence $\liminf_{z \rightarrow \infty, w \rightarrow \tilde{a}} l_D(z, w) = 0$, which contradicts Proposition 4.

For the special case of Hartogs domains over \mathbb{D} with 1-dimensional fibers the converse to (**) is shown in [17]. In [12], the converse in general is posed as an open question. Here we obtain a positive answer as a consequence of Proposition 3.1.

Proposition 4.2. *If (**) holds, then D is hyperbolic.*

Proof. Assume the contrary. It follows from Proposition 3.1 that we may find sequences $\{t_j\} \subset \mathbb{D}$ and $\{f_j\} \subset \mathcal{O}(\mathbb{D}, D)$ with $t_j \rightarrow 0$, $f_j(t_j) \rightarrow \infty$ and $f_j(0) \rightarrow a \in D$. Let $a = (a', a'')$ with $a' \in G$ and $a'' \in \mathbb{C}^m$. Then $f'_j(t_j) \rightarrow a'$ by the hyperbolicity of G . Hence $f'_j(t_j) \in K$ for some $K \subset\subset D$. Then $f''_j(t_j) \rightarrow \infty$ and one has

$$1 > h(f(t_j)) = \|f''(t_j)\| h(f'(t_j), \frac{f''(t_j)}{\|f''(t_j)\|}) \geq \|f''(t_j)\| \inf_{K \times \mathbb{S}} h \rightarrow \infty,$$

which is a contradiction. □

The next example shows that the limit in Proposition 3.1 can be positive but less than 1.

Example 4.3. There is a hyperbolic unbounded domain in \mathbb{C}^3 for which ∞ is not a t -point.

Proof. Note that the function

$$\Phi(z_1, z_2) := \begin{cases} \max\{|z_1|, 1 - |ez_2|\}, & |z_2| < e^{-1} \\ |z_1|, & |z_2| \geq e^{-1} \end{cases}$$

is positive and continuous on the domain D from Example 4.1 above. Hence, by Proposition 4.2,

$$\tilde{D} := \{(z, w) \in D \times \mathbb{C} : |w|\Phi(z) < 1\}$$

is a hyperbolic Hartogs domain.

Now fix $c \in (e^{-1}, 1)$. Observe that the disc $f_j(t) := (2^{1-2j}, ct)$ belongs to $\mathcal{O}(\mathbb{D}, D)$ for any j large enough. If $g_j(t) := (cet)^j$, then $|g_j(t)| < 1 \leq \min\{2^{2j-1}, (1 - ce|t|)^{-1}\}$ when $ce|t| < 1$, and $|g_j(t)| < 2^{2j-1}$ for $t \in \mathbb{D}$ and $j \geq 2$, i.e. $|g_j(t)|\Phi(f_j(t)) < 1$ on \mathbb{D} . Hence $h_j := (f_j, g_j) \in \mathcal{O}(\mathbb{D}, \tilde{D})$ with $h_j(0) \rightarrow 0 \in \tilde{D}$. Moreover, for any t with $ce|t| > 1$, we have $h_j(t) \rightarrow \infty$. Thus

$$\liminf_{u \rightarrow \infty, v \rightarrow 0} l_{\tilde{D}}(u, v) \leq (ce)^{-1}.$$

For $c \rightarrow 1$ one has

$$\liminf_{u \rightarrow \infty, v \rightarrow 0} l_{\tilde{D}}(u, v) \leq e^{-1}.$$

Thus ∞ is not a t -point for D . □

Remarks. (i) In fact, the last limit is equal to e^{-1} . Indeed, if $\tilde{D} \supset ((z^j, w^j))_j \rightarrow \infty$, then it is easy to see that $z^j_1 \rightarrow 0$ and $\liminf_{j \rightarrow \infty} |z^j_2| \geq e^{-1}$. Since $D \subset \mathbb{D}^2$, it follows that if $((\hat{z}^j, \hat{w}^j))_j \rightarrow 0$, then

$$\begin{aligned} \liminf_{j \rightarrow \infty} l_{\tilde{D}}((z^j, w^j), (\hat{z}^j, \hat{w}^j)) &\geq \liminf_{j \rightarrow \infty} l_D(z^j, \hat{z}^j) \\ &\geq \liminf_{j \rightarrow \infty} l_{\mathbb{D}^2}(z^j, \hat{z}^j) \\ &= \liminf_{j \rightarrow \infty} |z^j_2| \geq e^{-1}. \end{aligned}$$

(ii) Observe that if we replace in Example 13 e^{-1} by any $r \in (e^{-1}, 1)$, then we have the same conclusions with r instead of e^{-1} . In particular, since ∞ is not a t -point, it is also not a barrier point. On the other hand, using the proof of Example 4.1 (ii), it is easy to see that ∞ is a local barrier point for \tilde{D} (with r instead of e^{-1}).

Concerning tautness and hyperconvexity of Hartogs domains one has the following

Proposition 4.4. *A Hartogs domain D is taut (hyperconvex) if and only if G is taut (hyperconvex) and $\log h$ is a continuous psh function with $\log h^{-1}(-\infty) = G \times \{0\}$.*

Proof. We shall follow the proof of the analogous result in the case of bounded Hartogs domains (see [13]).

First we shall prove the implication \Leftarrow . Assume that h has the above properties.

Suppose that G is hyperconvex, i.e. it admits a negative psh exhaustion function φ . Denoting by $\tilde{\varphi}(z, w) := \varphi(z)$ the trivial extension of φ to D , it is easy to see that $\max\{\tilde{\varphi}, \log h\}$ is a negative psh exhaustion function of D , implying that D is hyperconvex.

Now assume that G is taut and there exists a sequence $\{f_j\} \subset \mathcal{O}(\mathbb{D}, D)$ that does not diverge compactly. Then we may assume that $f'_j \rightarrow f' \in \mathcal{O}(\mathbb{D}, G)$. In particular, for any $r \in (0, 1)$ we have $\cup_{j=1}^\infty f'_j(r\mathbb{D}) =: K_r \subset\subset D$. Since $\inf_{K_r \times \mathbb{S}} h > 0$, it follows that $(f''_j)_j$ is uniformly bounded on $r\mathbb{D}$. By Montel's theorem one may assume that $f''_j \rightarrow f'' \in \mathcal{O}(\mathbb{D}, \mathbb{C}^m)$. Set $f := (f', f'')$. Using that $f'(\mathbb{D}) \subset G$ and the continuity of h , we get $h(f(t)) \leq 1$ for any $t \in \mathbb{D}$. Then the maximum principle implies that either $h \circ f < 1$ or $h \circ f \equiv 1$, i.e. either $f(\mathbb{D}) \in D$ or $f(\mathbb{D}) \in \partial D$.

To prove the converse implication \Rightarrow , note that if D is pseudoconvex and hyperbolic, then $\log h$ is a psh function and $h > 0$ on $G \times \mathbb{S}$. Moreover, it is clear that tautness (hyperconvexity) of D implies tautness (hyperconvexity) of G . Assume now that h is not continuous at some point $a \in D$. Then we may find a sequence $\{a_j\} \subset D$ and $r > 1$ such that $a_j \rightarrow a$ for $j \rightarrow \infty$ and $(h(a))^{-1} < r \leq (h(a_j))^{-1}$ for any j . Note that the discs $f_j(t) := (a'_j, ra''_j t)$ belong to $\mathcal{O}(\mathbb{D}, D)$. On the other hand, for the limit disc $f(t) := (a', ra'' t)$ we have $f(0) \in D$ and $f((h(a)r)^{-1}) \in \partial D$. This contradicts tautness (hyperconvexity) of D . □

Remark. For Hartogs domains with one-dimensional fibers, the part of the statement regarding tautness is obtained in [16] (see also [14] for a more general situation).

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