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A diffused interface whose chemical potential lies in a Sobolev space

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Abstract. We study a singular perturbation problem arising in the scalar two-phase field model. Given a sequence of functions with a uniform bound on the surface energy, assume the Sobolev norms $W^{1,p}$ of the associated chemical potential fields are bounded uniformly, where $p > \frac{n}{2}$ and n is the dimension of the domain. We show that the limit interface as ε tends to zero is an integral varifold with a sharp integrability condition on the mean curvature.

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1. Introduction

In this paper we study the asymptotic behavior of phase interfaces in the van der Waals-Cahn-Hilliard theory of phase transitions. Let $u : U \subset \mathbb{R}^n \to \mathbb{R}$ $(n \geq 2)$ be a function which, in the context of phase transitions, represents the normalized density distribution of a two-phase fluid and let $W : \mathbb{R} \to \mathbb{R}_+$ be a double well potential with strict minima at ± 1. Here W corresponds to the Helmholtz free energy density [8]. With $\varepsilon \approx 0$ given, define

$$E_\varepsilon(u) = \int_U \frac{\varepsilon}{2}|\nabla u|^2 + \frac{W(u)}{\varepsilon}. \tag{1.1}$$

Here, ε is a parameter giving roughly the order of thickness of phase interface region. It is well known that $\{E_\varepsilon\}_{\varepsilon>0}$ Γ-converge to the interface area as $\varepsilon \to 0$ [11, 18]. That is, the interfaces of the energy minimizers converge to area minimizing surfaces as $\varepsilon \to 0$, and $\frac{1}{2\sigma}E_\varepsilon$ for small ε approximates the area of $U \cap \partial\{u \approx 1\}$, where

$$\sigma = \int_{-1}^{1} \sqrt{W(s)/2}\,ds \tag{1.2}$$

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is the surface energy constant. Even without the energy minimality, but with uniform bounds on E_ε and

$$||-\varepsilon\Delta u+\varepsilon^{-1}W'(u)||_{W^{1,n}(U)},$$

it has been proved [19] that the limit interface has finite area with well-defined mean curvature. Here, $-\varepsilon\Delta u+\varepsilon^{-1}W'(u)$ in the limit is found to be the mean curvature times σ of the limit interface, and in the context of phase transitions, corresponds to the chemical potential field of the two phase fluid. $W^{1,n}(U)$ is the usual Sobolev space with L^n integrable first derivatives.

In this paper, we improve the result of [19] in that one only needs to assume uniform bounds on E_ε and

$$||-\varepsilon\Delta u+\varepsilon^{-1}W'(u)||_{W^{1,p}(U)} \tag{1.3}$$

for some $p>\frac{n}{2}$ to conclude that the limit interface as $\varepsilon\to 0$ has good measure-theoretic properties (see Theorem 2.2 for the precise statements). The result is sharp in the sense that if (1.3) is uniformly bounded only for $p<\frac{n}{2}$ but not for $p\geq\frac{n}{2}$, then limits of interface regions can be diffused over all U, losing the "surface-like" property. If one heuristically regards E_ε as the interface area and $-\varepsilon\Delta u+\varepsilon^{-1}W'(u)$ as the mean curvature field, the $W^{1,p}(U)$ norm control for $p>\frac{n}{2}$ gives $L^{\frac{p(n-1)}{n-p}}$ *trace* norm control of the mean curvature with respect to the surface measure on the interface. Note that $\frac{p(n-1)}{n-p}>n-1$ for $p>\frac{n}{2}$. It is well-known that $(n-1)$-dimensional surface with its mean curvature in L^q, $q>n-1$, behaves well measure-theoretically [1]. The related results for the sharp interface case are discussed by Schätzle [16], and one may also regard the results of this paper as the 'ε-version' of [16].

Instead of assuming (1.3), another interesting assumption is an ε independent bounds on E_ε and

$$\frac{1}{\varepsilon}\int_U\left(-\varepsilon\Delta u+\frac{W'(u)}{\varepsilon}\right)^2. \tag{1.4}$$

Note that (1.4) corresponds to the heuristic uniform L^2 bound of the mean curvature of the interface with respect to the surface measure. The problem is motivated by the Willmore functional and the Allen-Cahn action, and has been studied recently in [2, 5, 12, 10, 13, 15]. In [15], the limit interfaces as $\varepsilon\to 0$ are shown to be integral varifolds with L^2 mean curvature for dimension $n=2,3$. The results obtained there are analogous to those in this paper.

In our analysis, the key point of the proof rests on showing the energy monotonicity formula, which is well known in the context of geometric measure theory. In [19], to control the positive part of $\xi=\frac{\varepsilon}{2}|\nabla u|^2-\frac{W}{\varepsilon}$, which appears as a major obstacle for establishing the monotonicity formula in our setting, we used Aleksandrov-Bakelman-Pucci (ABP) estimate to the differential inequality satisfied by ξ. There, it appears essential to have $W^{1,n}$ norm control of (1.3) to apply ABP estimate. The improvement of the present paper rests on the observation that one can regularize the problem in a "sub ε scale" so the estimate in [19] can be used, where the error terms originating from the nonlinear term W can be controlled. Once the monotonicity formula is established, the rectifiability and integrality of the limit varifold follow, with some modifications, from the argument in [9].

2. Assumptions and main results

2.1. Assumptions and notation

We consider the problem under the following assumptions. The function $W : \mathbb{R} \to [0, \infty)$ is C^3 and $W(\pm 1) = 0$. For some $\gamma \in (-1, 1)$, $W' \leq 0$ on $(\gamma, 1)$ and $W' \geq 0$ on $(-1, \gamma)$. For some $\alpha \in (0, |\gamma|)$ and $\kappa > 0$,

$$W''(s) \geq \kappa \tag{2.1}$$

for all $|s| \geq \alpha$. $U \subset \mathbb{R}^n$ is a bounded domain with Lipschitz boundary ∂U. Assume

$$n > p > \frac{n}{2}. \tag{2.2}$$

For any sequence of $W^{3,p}(U)$ functions $\{u^i\}_{i=1}^{\infty}$ and $\{\varepsilon_i\}_{i=1}^{\infty}$ ($\varepsilon_i > 0$), define

$$f^i = -\varepsilon_i \Delta u^i + \frac{W'(u^i)}{\varepsilon_i}. \tag{2.3}$$

Assume that $\lim_{i\to\infty} \varepsilon_i = 0$ and that there exist constants c_0, λ_0 and E_0 such that

$$\sup_U |u^i| \leq c_0, \tag{2.4}$$

$$||f^i||_{W^{1,p}(U)} = \left(\int_U |f^i|^p + |\nabla f^i|^p \right)^{\frac{1}{p}} \leq \lambda_0, \tag{2.5}$$

$$\int_U \frac{\varepsilon_i}{2} |\nabla u^i|^2 + \frac{W(u^i)}{\varepsilon_i} \leq E_0, \tag{2.6}$$

for all i.

Notation 2.1. We denote

- the open ball in $\mathbb{R}^n$ of radius r and center at x by $B_r(x)$,
- Lebesgue measure by $\mathcal{L}^n$,
- $\omega_n = \mathcal{L}^n(B_1(x))$,
- $(n-1)$-dimensional Hausdorff measure by $\mathcal{H}^{n-1}$.
- We often write B_r for $B_r(x)$ or $B_r(0)$ when no ambiguity arises.

2.2. Immediate consequences

By the Sobolev inequality,

$$||f^i||_{L^{\frac{np}{n-p}}(U)} \leq c_1\lambda_0, \tag{2.7}$$

where c_1 depends only on n, p and U. For $x \in \varepsilon_i^{-1}U$, define $\tilde{u}^i(x) = u^i(\varepsilon_i x)$ and $\tilde{f}^i(x) = f^i(\varepsilon_i x)$. $\tilde{u}^i$ and $\tilde{f}^i$ satisfy

$$\varepsilon_i \tilde{f}^i = -\Delta\tilde{u}^i + W'(\tilde{u}^i) \tag{2.8}$$

on $\varepsilon_i^{-1}U$ and

$$||\varepsilon_i \tilde{f}^i||_{L^{\frac{np}{n-p}}(\varepsilon_i^{-1}U)} + ||\varepsilon_i \nabla\tilde{f}^i||_{L^p(\varepsilon_i^{-1}U)} \leq \varepsilon_i^{2-\frac{n}{p}}(c_1+1)\lambda_0. \tag{2.9}$$

Thus, by the standard elliptic estimates, (2.4), (2.8) and (2.9), there exists a constant c_2 depending only on c_0, n, p, W, U and λ_0 such that, for any $B_1 \subset \varepsilon_i^{-1}U$,

$$||\tilde{u}^i||_{W^{3,p}(B_1)} \leq c_2. \tag{2.10}$$

By the Sobolev inequality and (2.10),

$$||\tilde{u}^i||_{C^{1,2-\frac{n}{p}}(B_1)} \leq c_3. \tag{2.11}$$

Note that $0 < 2 - \frac{n}{p} < 1$ by (2.2). For u^i, (2.11) implies

$$\sup_U |\nabla u^i| \leq c_3\varepsilon_i^{-1}, \tag{2.12}$$

$$\sup_{x,y\in U,\, 0<|x-y|<\varepsilon_i} \frac{|\nabla u^i(x) - \nabla u^i(y)|}{|x-y|^{2-\frac{n}{p}}} \leq c_3\varepsilon_i^{\frac{n}{p}-3}. \tag{2.13}$$

Let

$$\Phi(s) = \int_0^s \sqrt{W(s)/2}\, ds$$

and define new functions

$$w^i(x) = \Phi(u^i(x))$$

for $i = 1, \cdots$. The Cauchy-Schwarz inequality and (2.6) show

$$\int_U |\nabla w^i| \leq \frac{1}{2}\int_U \frac{\varepsilon_i|\nabla u^i|^2}{2} + \frac{W(u^i)}{\varepsilon_i} \leq \frac{1}{2}E_0. \tag{2.14}$$

The compactness theorem for bounded variation functions, (2.4) and (2.14) show that there exist a converging subsequence which we denote by the same notation as $\{w^i\}$ and the L^1 (and a.e.) limit w^∞. Then define

$$u^\infty(x) = \Phi^{-1}(w^\infty(x)),$$

where Φ^{-1} is the inverse function of Φ. By the a.e. convergence of w^i to w^∞, $u^i \to u^\infty$ in L^1 and by Fatou's lemma and (2.6), $u^\infty = \pm 1$ a.e. on U. Moreover,

$$||\partial\{u^\infty = 1\}||(U) = \frac{1}{2}\int_U |Du^\infty| = \frac{1}{\sigma}\int_U |Dw^\infty| \le \frac{E_0}{2\sigma},$$

where $|Dw^\infty|$ is the total variation of the vector-valued Radon measure Dw^∞, and where $||\partial A||$ is the perimeter of A ([6]).

2.3. The associated varifolds

We associate to each u^i (and w^i) a varifold in a natural way. We refer to [1, 17] for a comprehensive treatment of varifolds.

Let $G(n, n-1)$ be the Grassmann manifold of unoriented $(n-1)$-dimensional planes in $\mathbb{R}^n$. We say that V is an $(n-1)$-dimensional *varifold* in $U \subset \mathbb{R}^n$ if V is a Radon measure on $G_{n-1}(U) = U \times G(n, n-1)$. Let $V_{n-1}(U)$ be the set of all $(n-1)$-dimensional varifolds in U. *Convergence in the varifold sense* means convergence in the usual sense of measure on $G_{n-1}(U)$. For $V \in V_{n-1}(U)$, we let the *weight* $||V||$ be the Radon measure in U defined by

$$||V||(A) = V(\{(x, S) \mid x \in A,\ S \in G(n, n-1)\})$$

for each Borel set $A \subset U$. If M is a $(n-1)$-rectifiable subset ([17]) of U, we define $v(M) \in V_{n-1}(U)$ by

$$v(M)(A) = \mathcal{H}^{n-1}(\{x \in M \mid (x, \mathrm{Tan}^{n-1}(\mathcal{H}^{n-1}\lfloor_M, x)) \in A\})$$

for each Borel set $A \subset G_{n-1}(U)$, where $\mathrm{Tan}^{n-1}(\mathcal{H}^{n-1}\lfloor_M, x)$ is the approximate tangent plane to M at x, which exists for $\mathcal{H}^{n-1}$ a.e. $x \in M$.

We associate to each function u^i a varifold V^i defined naturally as follows. By Sard's theorem, $\{w^i = t\} \subset U$ is a $C^{1,2-\frac{n}{p}}$ hypersurface for $\mathcal{L}^1$ a.e. t. Define $V^i \in V_{n-1}(U)$ by

$$\begin{aligned} V^i(A) &= \int_{-\infty}^{\infty} v(\{w^i = t\})(A)\,dt \\ &= \int_{-\infty}^{\infty} \mathcal{H}^{n-1}(\{w^i = t\} \cap \{x \mid (x, (\nabla w^i(x))^\perp) \in A\})\,dt \end{aligned}$$

for each Borel set $A \subset G_{n-1}(U)$. Here, $(a)^\perp$ denotes the orthogonal hyperplane to the vector a. By the co-area formula [6], we have

$$V^i(A) = \int_{\{x \mid (x, (\nabla w^i(x))^\perp) \in A\}} |\nabla w^i|$$

for each Borel set $A \subset G_{n-1}(U)$. Note that $||V^i||$ is a measure concentrating around the transition region. As a result of the present paper (Theorem 2.2 (1)), we may define an essentially equivalent varifold by

$$\tilde{V}^i(A) = \int_{\{x \in U \mid (x, (\nabla u^i(x))^{\perp}) \in A\}} \frac{\varepsilon_i}{2} |\nabla u^i|^2$$

for $A \subset G_{n-1}(U)$. We prove that $\frac{\varepsilon_i}{2}|\nabla u^i|^2 - |\nabla w^i|$ converges L^1 locally to 0, and thus

$$\lim_{i \to \infty} (V^i - \tilde{V}^i) = 0.$$

The first variation of V^i is given by [14, Sec. 2.1]

$$\delta V^i(g) = \int_U \left(\operatorname{div} g - \sum_{j,k=1}^{n} \frac{w^i_{x_j} w^i_{x_k}}{|\nabla w^i|^2} g^j_{x_k} \right) |\nabla w^i|$$

for each $g \in C^1_c(U; \mathbb{R}^n)$, and

$$\delta \tilde{V}^i(g) = \int_U \left(\operatorname{div} g - \sum_{j,k=1}^{n} \frac{w^i_{x_j} w^i_{x_k}}{|\nabla w^i|^2} g^j_{x_k} \right) \frac{\varepsilon_i}{2} |\nabla u^i|^2.$$

2.4. Main results

With the above assumptions and notation, we show

Theorem 2.2. *Let V^i be the varifold associated with u^i. On passing to a subsequence we can assume that*

$$f^i \to f^\infty \text{ weakly in } W^{1,p}, \quad u^i \to u^\infty \text{ a.e.}, \quad V^i \to V.$$

Then

(1) *For each $\phi \in C_c(U)$,*

$$||V||(\phi) = \lim_{i \to \infty} \int_U \frac{\varepsilon_i}{2} |\nabla u^i|^2 \phi = \lim_{i \to \infty} \int_U \frac{W(u^i)}{\varepsilon_i} \phi$$
$$= \lim_{i \to \infty} \int_U |\nabla w^i| \phi.$$

(2) $\operatorname{supp}||\partial\{u^\infty = 1\}|| \subset \operatorname{supp}||V||$ *and $\{u^i\}$ converges locally uniformly to ± 1 on $U \setminus \operatorname{supp}||V||$.*

(3) *For each $\tilde{U} \subset\subset U$, $0 < b < 1$, $\{|u^i| \le 1 - b\} \cap \tilde{U}$ converges to $\tilde{U} \cap \operatorname{supp}||V||$ in the Hausdorff distance sense.*

(4) *$\sigma^{-1}V$ is an integral varifold. Moreover, the density $\theta(x) = \sigma N(x)$ of V satisfies*

$$N(x) = \begin{cases} odd, & \mathcal{H}^{n-1} a.e.\, x \in M^\infty, \\ even, & \mathcal{H}^{n-1} a.e.\, x \in \mathrm{supp}||V|| \setminus M^\infty, \end{cases}$$

where M^∞ is the reduced boundary of $\{u^\infty = 1\}$.

(5) *The generalized mean curvature vector H of V is given by*

$$H(x) = \begin{cases} \dfrac{f^\infty(x)}{\theta(x)} \nu^\infty(x), & \mathcal{H}^{n-1} a.e.\, x \in M^\infty, \\ 0, & \mathcal{H}^{n-1} a.e.\, x \in \mathrm{supp}||V|| \setminus M^\infty, \end{cases}$$

where ν^∞ is the inward normal for M^∞.

(6) *The generalized mean curvature vector H belongs to $L_{\mathrm{loc}}^{\frac{p(n-1)}{n-p}}$ with respect to $||V||$.*

Note that $\frac{p(n-1)}{n-p} > n-1$ by (2.2). Various results known for integral varifolds with the mean curvature in this class apply [1, 17]. The density function

$$\theta(x) = \lim_{r \to 0} \frac{1}{\omega_{n-1} r^{n-1}} ||V||(B_r(x))$$

exists for all $x \in \mathrm{supp}||V||$ and is upper-semicontinuous. $\sigma^{-1}\theta$ is integer-valued for $\mathcal{H}^{n-1}$ a.e. on $\mathrm{supp}||V||$. There exists an open dense set $\mathcal{O} \subset U$ such that $\mathcal{O} \cap \mathrm{supp}||V||$ is a $C^{1,2-\frac{n}{p}}$ submanifold. A special but interesting case is

Corollary 2.3. *Suppose $\sigma^{-1}\theta = 1$ for $\mathcal{H}^{n-1}$ a.e. on* $\mathrm{supp}||V||$ *(which is equivalent in this case to $\mathcal{H}^{n-1}(\{\sigma^{-1}\theta \geq 2\}) = 0$). Then* $\mathrm{supp}||V||$ *is a $C^{1,2-\frac{n}{p}}$ manifold outside a closed set with $\mathcal{H}^{n-1}$ measure 0. The mean curvature vector of* $\mathrm{supp}||V||$ *is given by $\sigma^{-1} f^\infty \nu^\infty$.*

As is pointed out in [9, Section 5], it is possible that $\sigma^{-1}\theta \geq 2$ has positive measure in general.

3. Monotonicity formula

In this section, we denote u^i, f^i, ε_i by u, f, ε and assume all the assumptions set in 2.1 are satisfied. We assume $\tilde{U}$ is open and $\tilde{U} \subset\subset U$. For $x \in U$ and $0 < r < \mathrm{dist}(x, \partial U)$, define

$$E(r, x) = \frac{1}{r^{n-1}} \int_{B_r(x)} \frac{\varepsilon}{2} |\nabla u|^2 + \frac{W(u)}{\varepsilon}.$$

The key point of this section is show that E is an almost monotone increasing function of r (Proposition 3.7).

We denote $W - \varepsilon f u$ by $\tilde{W}$. First we need the following Lemma. The statement and the proof are identical to [19, Lemma 3.1], so we simply cite the result.

Lemma 3.1. *For $B_r(x) \subset U$, we have*

$$\frac{d}{dr}\left\{\frac{1}{r^{n-1}}\int_{B_r(x)}\left(\frac{\varepsilon}{2}|\nabla u|^2 + \frac{\tilde{W}}{\varepsilon}\right)\right\}$$
$$= \frac{1}{r^n}\int_{B_r(x)}\left(\frac{\tilde{W}}{\varepsilon} - \frac{\varepsilon}{2}|\nabla u|^2\right) + \frac{\varepsilon}{r^{n+1}}\int_{\partial B_r(x)}\big((y-x)\cdot\nabla u\big)^2 \tag{3.1}$$
$$- \frac{1}{r^n}\int_{B_r(x)}\big((y-x)\cdot\nabla f\big)u.$$

Next we need the following lemma, which we prove later in this section.

Lemma 3.2. *There exist constants $0 < \beta_1 < 1$ and $\varepsilon_1 > 0$ which depend only on c_0, λ_0, W, n, p and $\operatorname{dist}(\tilde{U}, \partial U)$ such that, if $\varepsilon < \varepsilon_1$,*

$$\sup_{\tilde{U}}\left(\frac{\varepsilon}{2}|\nabla u|^2 - \frac{W(u)}{\varepsilon}\right) \le \varepsilon^{-\beta_1}. \tag{3.2}$$

Lemma 3.1 and 3.2 give a lower energy density ratio bound for $r < O(\varepsilon^{\beta_1})$.

Lemma 3.3. *There exist constants $0 < \varepsilon_2, c_4, c_5 < 1$ which depend only on c_0, λ_0, W, n, p and $\operatorname{dist}(\tilde{U}, \partial U)$ such that, if $B_{\varepsilon^{\beta_1}}(x) \subset \tilde{U}$, $|u(x)| \le \alpha$ and $\varepsilon < \varepsilon_2$, then*

$$E(r, x) \ge c_4 \qquad \text{for } \varepsilon \le r \le c_5\varepsilon^{\beta_1}. \tag{3.3}$$

Proof. By integrating (3.1) over $[\varepsilon, r]$ and dropping the positive terms, we have

$$E(r,x) - E(\varepsilon,x) - \frac{1}{r^{n-1}}\int_{B_r} uf + \frac{1}{\varepsilon^{n-1}}\int_{B_\varepsilon} uf \tag{3.4}$$
$$\ge -\int_\varepsilon^r \frac{d\tau}{\tau^n}\int_{B_\tau}\left\{\left(\frac{\varepsilon}{2}|\nabla u|^2 - \frac{W}{\varepsilon}\right)_+ + uf + ((y-x)\cdot\nabla f)u\right\}.$$

By Hölder's inequality, (2.4) and (2.7),

$$\left|\frac{1}{r^{n-1}}\int_{B_r} uf\right| \le c_0c_1\lambda_0 r^{2-\frac{n}{p}}, \quad \left|\frac{1}{\varepsilon^{n-1}}\int_{B_\varepsilon} uf\right| \le c_0c_1\lambda_0\varepsilon^{2-\frac{n}{p}}. \tag{3.5}$$

Similarly

$$\left|\int_\varepsilon^r \frac{d\tau}{\tau^n}\int_{B_\tau} uf\right| \le c_0c_1\lambda_0\int_\varepsilon^r \tau^{1-\frac{n}{p}}\,d\tau \le \frac{c_0c_1\lambda_0}{2-\frac{n}{p}} r^{2-\frac{n}{p}}, \tag{3.6}$$

$$\left|\int_\varepsilon^r \frac{d\tau}{\tau^n}\int_{B_\tau}((y-x)\cdot\nabla f)u\right| \le c_0\lambda_0\int_\varepsilon^r \tau^{1-\frac{n}{p}}\,d\tau \le \frac{c_0\lambda_0}{2-\frac{n}{p}}r^{2-\frac{n}{p}}. \tag{3.7}$$

By (3.2)

$$\int_\varepsilon^r \frac{d\tau}{\tau^n}\int_{B_\tau}\left(\frac{\varepsilon}{2}|\nabla u|^2-\frac{W}{\varepsilon}\right)_+ \le \omega_n\varepsilon^{-\beta_1}r. \tag{3.8}$$

Since $|u(x)|\le\alpha$, and by (2.12), $|u(y)|\le\frac{\alpha+1}{2}$ for all $y\in B_{\frac{\varepsilon}{c_3(1-\alpha)}}(x)$. Here, we assume $c_3(1-\alpha)\ge 1$ (by choosing c_3 large if necessary) without loss of generality. Let

$$c_4=\frac{\omega_n}{2(c_3(1-\alpha))^n}\min_{|t|\le\frac{1+\alpha}{2}}W(t).$$

Note that $c_4>0$. Since $W(u(y))\ge\min_{|t|\le\frac{1+\alpha}{2}}W(t)$ on $B_{\frac{\varepsilon}{c_3(1-\alpha)}}(x)$,

$$\begin{aligned}E(\varepsilon,x)=\frac{1}{\varepsilon^{n-1}}\int_{B_\varepsilon}\frac{\varepsilon}{2}|\nabla u|^2+\frac{W}{\varepsilon}&\ge\frac{1}{\varepsilon^{n-1}}\int_{B_{\frac{\varepsilon}{c_3(1-\alpha)}}}\frac{W}{\varepsilon}\\&\ge\frac{\omega_n}{(c_3(1-\alpha))^n}\min_{|t|\le\frac{1+\alpha}{2}}W(t)=2c_4.\end{aligned} \tag{3.9}$$

We now restrict r so that terms in (3.5)-(3.8) remain smaller than c_4, i.e., we choose c_5 small so that $r\le c_5\varepsilon^{\beta_1}$ implies $\omega_n\varepsilon^{-\beta_1}r\le\frac{c_4}{2}$ (so $c_5=\frac{c_4}{2\omega_n}$). Then by restricting ε depending on $c_4, c_5, c_0, c_1, \lambda_0, n$ and p, we have $(3.5)+(3.6)+(3.7)\le\frac{c_4}{2}$. Then we have the desired inequality (3.3) from (3.4). ☐

Proposition 3.4. *There exist constants* $0<\beta_2<1$, $0<c_6$ *and* $0<\varepsilon_3$ *which depend only on* $c_0,\ \lambda_0,\ W,\ n,\ p$ *and* $\operatorname{dist}(\tilde U,\partial U)$ *such that if* $B_r(x)\subset\tilde U$, $c_5\varepsilon^{\beta_1}\le r\le 1$ *and* $\varepsilon\le\varepsilon_3$, *then*

$$\frac{1}{r^n}\int_{B_r(x)}\left(\frac{\varepsilon}{2}|\nabla u|^2-\frac{W}{\varepsilon}\right)_+\le\frac{c_6}{r^{1-\beta_2}}(E(r,x)+1). \tag{3.10}$$

Proof. First set $\beta_2=\frac{1-\beta_1}{2\beta_1}$ and $\beta_3=\frac{1+\beta_1}{2}$. β_2 and β_3 are chosen so that

$$\beta_1\beta_2=\beta_3-\beta_1, \tag{3.11}$$

$$0<\beta_2<1,\qquad 0<\beta_1<\beta_3<1. \tag{3.12}$$

Here, we re-define β_1 such that $\beta_1>\frac{1}{3}$, if necessary, so that $\beta_2<1$ is satisfied. We estimate the integral of (3.10) by separating $B_r(x)$ into three disjoint sets. Set

$$\mathcal{A}=\{x\in B_r\setminus B_{r-\varepsilon^{\beta_3}}\},$$

$$\mathcal{B}=\{x\in B_{r-\varepsilon^{\beta_3}}\,|\operatorname{dist}(\{|u|\le\alpha\},x)<\varepsilon^{\beta_3}\},$$

$$\mathcal{C}=\{x\in B_{r-\varepsilon^{\beta_3}}\,|\operatorname{dist}(\{|u|\le\alpha\},x)\ge\varepsilon^{\beta_3}\}.$$

Note that $r\ge c_5\varepsilon^{\beta_1}>\varepsilon^{\beta_3}$ for all small ε by (3.12).

Estimate on $\mathcal{A}$

Since $\mathcal{L}^n(\mathcal{A}) \leq n\omega_n r^{n-1}\varepsilon^{\beta_3}$, with (3.2),

$$\frac{1}{r^n}\int_{\mathcal{A}}\left(\frac{\varepsilon}{2}|\nabla u|^2 - \frac{W}{\varepsilon}\right)_+ \leq \frac{\varepsilon^{-\beta_1}}{r^n}\mathcal{L}^n(\mathcal{A}) \leq \frac{n\omega_n}{r}\varepsilon^{\beta_3-\beta_1}. \tag{3.13}$$

Since $r \geq c_5\varepsilon^{\beta_1}$ and by (3.11),

$$\frac{1}{r^{\beta_2}} \leq \frac{1}{c_5^{\beta_2}\varepsilon^{\beta_1\beta_2}} = \frac{1}{c_5^{\beta_2}\varepsilon^{\beta_3-\beta_1}}. \tag{3.14}$$

Thus (3.13) and (3.14) show

$$\frac{1}{r^n}\int_{\mathcal{A}}\left(\frac{\varepsilon}{2}|\nabla u|^2 - \frac{W}{\varepsilon}\right)_+ \leq \frac{n\omega_n}{c_5^{\beta_2}r^{1-\beta_2}}. \tag{3.15}$$

Estimate on $\mathcal{B}$

We estimate $\mathcal{L}^n(\mathcal{B})$ first. Apply the Vitali covering lemma [6] to the family of balls $\{B_{\varepsilon^{\beta_3}}(x)\}_{x\in\{|u|\leq\alpha\}\cap\mathcal{B}}$ so that $\{B_{\varepsilon^{\beta_3}}(x_i)\}_{i=1}^N$ is a disjoint family of balls and so that $\mathcal{B} \subset \cup_{i=1}^N B_{5\varepsilon^{\beta_3}}(x_i)$. Then

$$\mathcal{L}^n(\mathcal{B}) \leq \omega_n 5^n \varepsilon^{n\beta_3} N. \tag{3.16}$$

Since $x_i \in \{|u| \leq \alpha\}$ and $\varepsilon \leq \varepsilon^{\beta_3} \leq c_5\varepsilon^{\beta_1}$, (3.3) gives $E(\varepsilon^{\beta_3}, x_i) \geq c_4$ for $i = 1, \cdots, N$, that is,

$$c_4\varepsilon^{(n-1)\beta_3} \leq \int_{B_{\varepsilon^{\beta_3}}(x_i)}\left(\frac{\varepsilon}{2}|\nabla u|^2 + \frac{W}{\varepsilon}\right). \tag{3.17}$$

Since they are disjoint balls, by summing (3.17) over i,

$$\begin{aligned} Nc_4\varepsilon^{(n-1)\beta_3} &\leq \int_{\cup_{i=1}^N B_{\varepsilon^{\beta_3}}(x_i)}\left(\frac{\varepsilon}{2}|\nabla u|^2 + \frac{W}{\varepsilon}\right) \\ &\leq \int_{B_r}\left(\frac{\varepsilon}{2}|\nabla u|^2 + \frac{W}{\varepsilon}\right) = r^{n-1}E(r,x). \end{aligned} \tag{3.18}$$

Using (3.16) and (3.18), we obtain

$$\mathcal{L}^n(\mathcal{B}) \leq \frac{\omega_n 5^n \varepsilon^{\beta_3} r^{n-1}}{c_4}E(r,x). \tag{3.19}$$

We now estimate the integral on $\mathcal{B}$ using (3.2), (3.14) and (3.19),

$$\begin{aligned}\frac{1}{r^n}\int_{\mathcal{B}}\left(\frac{\varepsilon}{2}|\nabla u|^2-\frac{W}{\varepsilon}\right)_+ &\le \frac{\varepsilon^{-\beta_1}}{r^n}\mathcal{L}^n(\mathcal{B}) \le \frac{\omega_n 5^n}{c_4}\frac{\varepsilon^{\beta_3-\beta_1}}{r}E(r,x)\\ &\le \frac{\omega_n 5^n}{c_4 c_5^{\beta_2} r^{1-\beta_2}}E(r,x). \qquad (3.20)\end{aligned}$$

Estimate on $\mathcal{C}$

Define a Lipschitz function ϕ as follows:

$$\phi(x)=\min\{1,\varepsilon^{-\beta_3}\mathrm{dist}(\{|y|\ge r\}\cup\{|u|\le\alpha\},x)\}.$$

ϕ is 0 on the set $\{|u|\le\alpha\}\cup\{|y|\ge r\}$, 1 on $\mathcal{C}$ and $|\nabla\phi|\le\varepsilon^{-\beta_3}$. Using this ϕ, we estimate $\frac{\varepsilon}{2}|\nabla u|^2$ which is larger than $\frac{\varepsilon}{2}|\nabla u|^2-\frac{W}{\varepsilon}$. Differentiate (2.3) with respect to x_j, multiply it by $u_{x_j}\phi^2$ and sum over j. Then

$$\int\sum_{j=1}^n \varepsilon u_{x_j}\Delta u_{x_j}\phi^2=\int\frac{W''}{\varepsilon}|\nabla u|^2\phi^2-\nabla f\cdot\nabla u\phi^2. \qquad (3.21)$$

Integrate by parts the left-hand side of (3.21) as well as the second term of the right-hand side to obtain

$$\begin{aligned}&\int \varepsilon|\nabla^2 u|^2\phi^2+\frac{W''}{\varepsilon}|\nabla u|^2\phi^2\\ &=\int-\sum_{i,j=1}^n 2\varepsilon u_{x_j}u_{x_ix_j}\phi\phi_{x_i}-f(\Delta u\phi^2+2\phi\nabla u\cdot\nabla\phi). \qquad (3.22)\end{aligned}$$

We estimate the right-hand side of (3.22) by the Cauchy-Schwarz inequality,

$$\le\frac{1}{2}\int\varepsilon|\nabla^2u|^2\phi^2+c_7(n)\int(\varepsilon|\nabla u|^2|\nabla\phi|^2+f^2\phi^2\varepsilon^{-1}).$$

Since $|u|\ge\alpha$ on the support of ϕ, $W''\ge\kappa$ by (2.1). Thus

$$\int\frac{\varepsilon}{2}|\nabla^2u|^2\phi^2+\frac{\kappa}{\varepsilon}|\nabla u|^2\phi^2\le c_7\int(\varepsilon|\nabla u|^2|\nabla\phi|^2+f^2\phi^2\varepsilon^{-1}). \qquad (3.23)$$

Since $|\nabla\phi|\le\varepsilon^{-\beta_3}$, and using (2.7) and Hölder's inequality,

$$\begin{aligned}\int\frac{\kappa}{\varepsilon}|\nabla u|^2\phi^2 &\le c_7\left(\varepsilon^{-2\beta_3}\int_{B_r}\varepsilon|\nabla u|^2+\varepsilon^{-1}\|f\|^2_{L^{\frac{np}{n-p}}}(\mathcal{L}^n(B_r))^{\frac{np-2(n-p)}{np}}\right)\\ &\le c_8\left(\varepsilon^{-2\beta_3}\int_{B_r}\varepsilon|\nabla u|^2+\varepsilon^{-1}c_1^2\lambda_0^2 r^{n-\frac{2(n-p)}{p}}\right). \qquad (3.24)\end{aligned}$$

Since $\phi = 1$ on $\mathcal{C}$, multiplying (3.24) by $\frac{\varepsilon^2}{\kappa r^n}$,

$$\frac{1}{r^n}\int_{\mathcal{C}} \frac{\varepsilon}{2}|\nabla u|^2 \le \frac{c_8}{\kappa}\left(\frac{\varepsilon^{2-2\beta_3}}{r}E(r,x) + \varepsilon c_1^2\lambda_0^2 r^{2-\frac{2n}{p}}\right). \tag{3.25}$$

Using the definition of $\beta_1,\ \beta_2,\ \beta_3$ and $r \ge c_5\varepsilon^{\beta_1}$, one can check that

$$\frac{\varepsilon^{2-2\beta_3}}{r} \le \frac{\varepsilon^{\beta_1\beta_2}}{r^{1-\beta_2}c_5^{\beta_2}}, \tag{3.26}$$

and using $\varepsilon \le r$,

$$\varepsilon r^{2-\frac{2n}{p}} \le \frac{1}{r^{\frac{2n}{p}-3}}, \tag{3.27}$$

where $\frac{2n}{p} - 3 < 1$ by (2.2). Thus (3.25), (3.26) and (3.27) show

$$\frac{1}{r^n}\int_{\mathcal{C}} \frac{\varepsilon}{2}|\nabla u|^2 \le \frac{c_8}{\kappa}\left(\frac{\varepsilon^{\beta_1\beta_2}}{c_5^{\beta_2}r^{1-\beta_2}}E(r,x) + \frac{c_1^2\lambda_0^2}{r^{\frac{2n}{p}-3}}\right). \tag{3.28}$$

Finally, re-defining $\beta_2 = \min\{\beta_2, 4 - \frac{2n}{p}\}$ and (3.15), (3.20) and (3.28), we obtain (3.10) with an appropriate choice of c_6. ☐

Proposition 3.5. *There exist constants* $0 < c_9,\ 0 < r_0 \le 1$ *depending only on* $c_0,\ \lambda_0,\ W,\ n,\ p$ *and* $\operatorname{dist}(\tilde{U}, \partial U)$ *such that, if* $\varepsilon \le r \le r_0$, $\varepsilon \le \varepsilon_3$, $B_r(x) \subset \tilde{U}$ *and* $|u(x)| \le \alpha$, *then*

$$E(r,x) \ge c_9. \tag{3.29}$$

Proof. The idea of the proof is to use (3.1), (3.3) and (3.10), and show that $E(r,x)$ can not decrease much as r increases from $r \ge c_5\varepsilon^{\beta_1}$. Note (3.29) is already proved in Lemma 3.3 with $c_9 = c_4$ and $\varepsilon \le r \le c_5\varepsilon^{\beta_1}$, so we assume $c_5\varepsilon^{\beta_1} \le r \le 1$. First note that the terms coming from f, such as the second term of

$$\text{LHS of (3.1)} = \frac{d}{dr}E(r,x) - \frac{d}{dr}\left(\frac{1}{r^{n-1}}\int_{B_r} uf\right), \tag{3.30}$$

can be estimated by (2.4), Hölder's inequality and (2.7) as

$$\begin{aligned}\left|\frac{d}{dr}\left(\frac{1}{r^{n-1}}\int_{B_r} uf\right)\right| &\le \frac{n-1}{r^n}c_0\int_{B_r}|f| + \frac{c_0}{r^{n-1}}\int_{\partial B_r}|f| \\ &\le c_{10}r^{1-\frac{n}{p}} + \frac{c_0}{r_{n-1}}\int_{\partial B_r}|f|. \end{aligned}\tag{3.31}$$

Integrating the second term of (3.31) over $[r_1, r_2]$ (which we need to do later),

$$\int_{r_1}^{r_2} \frac{c_0 dr}{r^{n-1}} \int_{\partial B_r} |f| \leq \frac{c_0}{r^{n-1}} \int_{B_r} |f| \Big|_{r=r_1}^{r_2} + \int_{r_1}^{r_2} \frac{c_0(n-1)dr}{r^n} \int_{B_r} |f|$$

$$\leq c_{11} r_2^{2-\frac{n}{p}}, \tag{3.32}$$

where c_{11} depends only on $c_0,\ c_1,\ \lambda_0,\ n$ and p. Similarly, as in (3.5) and (3.7),

$$\left| \frac{1}{r^n} \int_{B_r} uf \right| \leq c_0 c_1 \lambda_0 r^{1-\frac{n}{p}}, \quad \left| \frac{1}{r^n} \int_{B_r} ((y-x)\cdot \nabla f) u \right| \leq c_0 \lambda_0 r^{1-\frac{n}{p}}. \tag{3.33}$$

Combining (3.1), (3.30), (3.31), (3.33) as well as (3.10), we obtain

$$\frac{d}{dr} E(r,x) \geq -c_{12} r^{1-\frac{n}{p}} - \frac{c_0}{r^{n-1}} \int_{\partial B_r} |f| + \frac{1}{r^n} \int_{B_r} \left(\frac{W}{\varepsilon} - \frac{\varepsilon}{2} |\nabla u|^2 \right)_+$$

$$- \frac{c_6}{r^{1-\beta_2}} (E(r,x)+1). \tag{3.34}$$

Define $r_0 \leq \min\{1, \operatorname{dist}(x, \partial \tilde{U})\}$ as the supremum such that

$$E(r,x) \geq \frac{c_4}{2} \qquad \text{for } r \in [c_5 \varepsilon^{\beta_1}, r_0] \tag{3.35}$$

holds. By (3.10), we know $r_0 > c_5 \varepsilon^{\beta_1}$. Dividing both sides of (3.34) by $E(r,x)$ for $r \in [c_5 \varepsilon^{\beta_1}, r_0]$, and using (3.35), we have

$$\frac{d}{dr} \ln E(r,x) \geq -\frac{2}{c_4} \left(c_{12} r^{1-\frac{n}{p}} + \frac{c_0}{r^{n-1}} \int_{\partial B_r} |f| + \frac{c_6 (1+\frac{c_4}{2})}{r^{1-\beta_2}} \right). \tag{3.36}$$

Integrating (3.36) over $[c_5 \varepsilon^{\beta_1}, r_0]$ gives, with (3.32),

$$\ln \left(\frac{E(r_0,x)}{E(c_5 \varepsilon^{\beta_1}, x)} \right) \geq -c_{13} \left(r_0^{2-\frac{n}{p}} + r_0^{\beta_2} \right), \tag{3.37}$$

where c_{13} depends only on $c_0,\ c_1,\ \lambda_0,\ n,\ p$ and $\operatorname{dist}(\tilde{U}, \partial U)$.
If $r_0 = \min\{1, \operatorname{dist}(x, \partial \tilde{U})\}$ then we are done. If not, we have $E(r_0, x) = \frac{c_4}{2}$, while $E(c_5 \varepsilon^{\beta_1}, x) \geq c_4$ by (3.3). Thus, (3.37) shows

$$-c_{13} \left(r_0^{2-\frac{n}{p}} + r_0^{\beta_2} \right) \leq \ln \left(\frac{1}{2} \right),$$

or

$$r_0^{2-\frac{n}{p}} + r_0^{\beta_2} \geq \frac{\ln 2}{c_{13}}. \tag{3.38}$$

(3.38) gives a lower bound for r_0 independent of ε, and we have proved (3.29) with $c_9 = \frac{c_4}{2}$ and a small r_0 chosen so that a reverse inequality holds in (3.38), for example $r_0^{2-\frac{n}{p}} + r_0^{\beta_2} = \frac{\ln 2}{2c_{13}}$. □

Proposition 3.6. *There exist constants* c_{14} *depending only on* c_0, c_1, λ_0, W, n, p *and* $\operatorname{dist}(\tilde{U}, \partial U)$ *such that, if* $\varepsilon \le \varepsilon_3$, $B_r(x) \subset \tilde{U}$, $|u(x)| \le \alpha$ *and* $\varepsilon \le r \le r_0$, *then*

$$E(r,x) \le c_{14}(E_0 + 1). \tag{3.39}$$

Proof. Integrate (3.36) for $[r, r_0]$, $r \ge c_5 \varepsilon^{\beta_1}$. Then we obtain the same inequality

$$\ln\left(\frac{E(r_0,x)}{E(r,x)}\right) \ge -c_{13}\left(r_0^{2-\frac{n}{p}} + r_0^{\beta_2}\right). \tag{3.40}$$

Since $B_{r_0}(x) \subset \tilde{U}$, by (2.6),

$$E(r_0,x) = \frac{1}{r_0^{n-1}} \int_{B_{r_0}(x)} \left(\frac{\varepsilon}{2}|\nabla u|^2 + \frac{W}{\varepsilon}\right) \le \frac{E_0}{r_0^{n-1}}. \tag{3.41}$$

Then (3.40) and (3.41) show

$$E(r,x) \le E_0 r_0^{1-n} \exp\left(-c_{13}\left(r_0^{2-\frac{n}{p}} + r_0^{\beta_2}\right)\right). \tag{3.42}$$

For $\varepsilon \le r \le c_5\varepsilon^{\beta_1}$, in (3.4), replace r there by $c_5\varepsilon^{\beta_1}$ and ε by r. One checks using similar estimates to (3.5), (3.6) and (3.8) that

$$E(c_5\varepsilon^{\beta_1}, x) \ge E(r,x) - c_4. \tag{3.43}$$

Then (3.42) and (3.43) with an appropriate choice of c_{14} show (3.39). ☐

Proposition 3.7. *There exists a constant* c_{15} *depending only on* c_0, c_1, λ_0, W, n, p *and* $\operatorname{dist}(\tilde{U}, \partial U)$ *such that, if* $\varepsilon \le \varepsilon_3$, $B_r(x) \subset \tilde{U}$, $|u(x)| \le \alpha$ *and* $c_5\varepsilon^{\beta_1} \le s \le r \le r_0$, *then*

$$\begin{aligned} &E(r,x) - E(s,x) \\ &\ge -c_{15}\left(r^{2-\frac{n}{p}} + r^{\beta_2} + E_0 r^{\beta_2}\right) + \int_s^r \frac{d\tau}{\tau^n} \int_{B_\tau(x)} \left(\frac{W}{\varepsilon} - \frac{\varepsilon}{2}|\nabla u|^2\right)_+. \end{aligned} \tag{3.44}$$

Proof. In the range $c_5\varepsilon^{\beta_1} \le r \le r_0$, (3.34) is valid. Thus, using (3.32) and (3.39) and integrating (3.34) over $[s, r]$, we immediately obtain (3.44) with an appropriate choice of c_{15}. ☐

For the rest of this section, we prove Lemma 3.2. First, it is not hard to show the following, as in [19, Proposition 3.7]. The proof is omitted.

Lemma 3.8. *There exist constants* ε_4 *and* $\eta > 0$ *depending only on* λ_0, c_0, n, p, W *and* $\operatorname{dist}(\tilde{U}, \partial U)$ *such that*

$$\sup_{\tilde{U}} |u| \le 1 + \varepsilon^\eta \tag{3.45}$$

for $\varepsilon \le \varepsilon_4$.

It is convenient to rescale the problem by $x \mapsto \frac{x}{\varepsilon}$. We define $\tilde{u}(x) = u(\varepsilon x)$, $\tilde{f}(x) = f(\varepsilon x)$, and subsequently drop $\tilde{\cdot}$ for simplicity. We have

$$-\Delta u + W'(u) = \varepsilon f$$

on $\varepsilon^{-1}U$ and we need to prove

$$\sup_{\varepsilon^{-1}U} \left(\frac{1}{2}|\nabla u|^2 - W(u)\right) \leq \varepsilon^{1-\beta_1} \tag{3.46}$$

for some $0 < \beta_1 < 1$ for all sufficiently small ε. To do so, we need the following lemma [19, Lemma 3.9]. The statement is changed slightly for the purpose of application here.

Lemma 3.9. *Suppose* $0 < \eta, \beta_4 < 1$, $\eta \leq \beta_4$, c_{16} *are given. Then there exist* $\varepsilon_5 > 0$, $c_{17} > 0$ *depending only on* η, β_4, c_{16}, n *and* W *with the following properties: Suppose* $v \in C^3(B_{\varepsilon^{-\beta_4}})$, $g \in C^1(B_{\varepsilon^{-\beta_4}})$, $\varepsilon \leq \varepsilon_5$,

$$-\Delta v + W'(v) = \varepsilon g$$

on $B_{\varepsilon^{-\beta_4}}$ *and*

$$\sup_{B_{\varepsilon^{-\beta_4}}} |v| \leq 1 + \varepsilon^{\eta}, \qquad \sup_{B_{\varepsilon^{-\beta_4}}} \left(\frac{1}{2}|\nabla v|^2 - W(v)\right) \leq c_{16}.$$

Then

$$\sup_{B_{\frac{\varepsilon^{-\beta_4}}{2}}} \left(\frac{1}{2}|\nabla v|^2 - W(v)\right) \leq c_{17}\left(\varepsilon^{1-\beta_4}||g||_{W^{1,n}(B_{\varepsilon^{-\beta_4}})} + \varepsilon^{\eta}\right). \tag{3.47}$$

Now we prove (3.46). The idea is to regularize u so that we have suitable control of $W^{1,n}$ norm for the regularized problem. Let $\phi \in C^{\infty}(\mathbb{R}^n)$ be a non-negative, radially symmetric function with support in $B_1(0)$ and $\int_{B_1(0)} \phi = 1$. Define $\phi_s(x) = \frac{1}{s^n}\phi\left(\frac{x}{s}\right)$ for $s > 0$, so that $\lim_{s \to 0} \phi_s$ is the delta function. For $1 > \beta_5 > 0$ to be chosen depending only on n and p later, define for $x \in \varepsilon^{-1}\tilde{U}$

$$v(x) = (u * \phi_{\varepsilon^{\beta_5}})(x) = \int u(x-y)\phi_{\varepsilon^{\beta_5}}(y)\,dy. \tag{3.48}$$

Using (2.11), we see from the definition (3.48) that

$$\sup_{\varepsilon^{-1}\tilde{U}} |v - u| \leq c_3 \varepsilon^{\beta_5}, \tag{3.49}$$

$$\sup_{\varepsilon^{-1}\tilde{U}} |\nabla v - \nabla u| \leq c_3 \varepsilon^{\beta_5(2-\frac{n}{p})}. \tag{3.50}$$

Next we define g to be

$$g = f * \phi_{\varepsilon^{\beta_5}} + \varepsilon^{-1}\left\{W'(v) - (W'(u)) * \phi_{\varepsilon^{\beta_5}}\right\}. \tag{3.51}$$

Note that v and g satisfy

$$-\Delta v + W'(v) = \varepsilon g. \tag{3.52}$$

We next estimate the $W^{1,n}$ norm of g on $B_{\varepsilon^{-\beta_4}} \subset \varepsilon^{-1}\tilde{U}$. Here, β_4 is not fixed, but we will choose $0 < \beta_4 < 1$ later depending only on n and p. The first term of (3.51) can be estimated as

$$||f * \phi_{\varepsilon^{\beta_5}}||_{W^{1,n}(B_{\varepsilon^{-\beta_4}})} \le (1 + \varepsilon^{-\beta_5} c_{18})||f||_{L^n(B_{2\varepsilon^{-\beta_4}})}, \tag{3.53}$$

where c_{18} depends only on ϕ and n, which are fixed. By Hölder's inequality and (2.7) (note the scaling is different),

$$\begin{aligned} ||f||_{L^n(B_{2\varepsilon^{-\beta_4}})} &\le ||f||_{L^{\frac{np}{n-p}}(B_{2\varepsilon^{-\beta_4}})} \left\{\omega_n (2\varepsilon^{-\beta_4})^n\right\}^{\frac{np-n+p}{np}} \\ &\le c_1 \lambda_0 \varepsilon^{1-\frac{n}{p}-\beta_4\frac{np-n+p}{p}} (2^n \omega_n)^{\frac{np-n+p}{np}}. \end{aligned} \tag{3.54}$$

(3.35) and (3.54) show

$$||f * \phi_{\varepsilon^{\beta_5}}||_{W^{1,n}(B_{\varepsilon^{-\beta_4}})} \le c_{19} \varepsilon^{1-\frac{n}{p}-\beta_4\frac{np-n+p}{p}-\beta_5}, \tag{3.55}$$

where c_{19} depends only on n, c_1, λ_0 and ϕ. To estimate the second term of (3.51), use

$$W'(v) - (W'(u)) * \phi_{\varepsilon^{\beta_5}} = (W'(v) - W'(u)) + \{W'(u) - (W'(u)) * \phi_{\varepsilon^{\beta_5}}\}$$

and note using (3.49) and (3.50) that

$$\sup|W'(v) - W'(u)| \le \sup|W''| \cdot \sup|u - v| \le c_{20}\varepsilon^{\beta_5}, \tag{3.56}$$

$$\begin{aligned} \sup|\nabla(W'(v) - W'(u))| &\le \sup|W''| \cdot \sup|\nabla v - \nabla u| \\ &\quad + \sup|\nabla u| \cdot \sup|W'''| \cdot \sup|u - v| \\ &\le c_{21}\varepsilon^{\beta_5(2-\frac{n}{p})}, \end{aligned} \tag{3.57}$$

$$\sup|W'(u) - (W'(u)) * \phi_{\varepsilon^{\beta_5}}| \le c_{22}\varepsilon^{\beta_5}, \tag{3.58}$$

$$\sup|\nabla\{W'(u) - (W'(u)) * \phi_{\varepsilon^{\beta_5}}\}| \le c_{23}\varepsilon^{\beta_5(2-\frac{n}{p})}, \tag{3.59}$$

where c_{20}-c_{23} depend only on c_3 and W. Thus, (3.56)-(3.59) show

$$||\varepsilon^{-1}\{W'(v) - (W'(u)) * \phi_{\varepsilon^{\beta_5}}\}||_{W^{1,n}(B_{\varepsilon^{-\beta_4}})} \le c_{24}\varepsilon^{\beta_5(2-\frac{n}{p})-1-\beta_4}. \tag{3.60}$$

Thus (3.51), (3.55) and (3.60) show that

$$||g||_{W^{1,n}(B_{\varepsilon^{-\beta_4}})} \leq c_{19}\varepsilon^{1-\frac{n}{p}-\beta_4\frac{np-n+p}{p}-\beta_5} + c_{24}\varepsilon^{\beta_5(2-\frac{n}{p})-1-\beta_4}. \tag{3.61}$$

Here we apply Lemma 3.9 to v and g. With $\beta_4 \geq \eta$ and $c_{16} = \sup\left(\frac{1}{2}|\nabla v|^2 - W(v)\right)$ which is bounded independent of ε, we obtain with (3.47) and (3.61) that

$$\sup_{B_{\frac{\varepsilon^{-\beta_4}}{2}}} \left(\frac{1}{2}|\nabla v|^2 - W(v)\right)$$

$$\leq c_{17}\left(c_{19}\varepsilon^{2-\frac{n}{p}-\beta_4\frac{np-n+2p}{p}-\beta_5} + c_{24}\varepsilon^{\beta_5(2-\frac{n}{p})-2\beta_4} + \varepsilon^{\eta}\right). \tag{3.62}$$

Choose $0 < \beta_4,\ \beta_5 < 1$ small so that (note $2 - \frac{n}{p} > 0$)

$$2 - \frac{n}{p} - \beta_4\frac{np-n+2p}{p} - \beta_5 > 0, \qquad \beta_5(2 - \frac{n}{p}) - 2\beta_4 > 0. \tag{3.63}$$

Basically, choose β_4 and β_5 small so the first inequality holds, and then choose β_4 even smaller so the second inequality holds. Such choices of β_4 and β_5 can be made depending only on n and p. Now the right-hand side of (3.62) is bounded by $\varepsilon^{1-\beta_1}$ by further restricting ε if necessary and choosing appropriate β_1 close to 1. Finally, note that the difference between $|\nabla u|$ and $|\nabla v|$ as well as $W(u)$ and $W(v)$ are $O(\varepsilon^{\beta_5(2-\frac{n}{p})})$. Thus we obtain the estimate (3.46) for $\frac{1}{2}|\nabla u|^2 - W(u)$.

Remark 3.10. In the last part of the proof, the requirement for f is that $||f||_{L^q}$ is controlled suitably for some $q > n$, as in (3.54). On the other hand, the gradient bound $||\nabla f||_{L^p}$ is essential in the proof for the mean curvature of the limit varifold as in Section 4.

4. Rectifiability and integrality of the limit varifold

Define

$$\mu = \lim_{i\to\infty}\left(\frac{\varepsilon_i}{2}|\nabla u^i|^2 + \frac{W(u^i)}{\varepsilon_i}\right)dx.$$

Proposition 4.1. *There exist constants* $0 < D_1 \leq D_2 < \infty$ *which depend only on* c_0, λ_0, n, p, E_0, $\operatorname{dist}(\tilde{U}, \partial U)$ *and* W *such that*

$$D_1 r^{n-1} \leq \mu(B_r(x)) \leq D_2 r^{n-1} \tag{4.1}$$

for all $0 < r < r_0$, $x \in \operatorname{supp}\mu \cap \tilde{U}$ *and* $B_r(x) \subset \tilde{U}$.

Proof. For any $x \in \operatorname{supp}\mu \cap \tilde{U}$, if we show that there exists a subsequence $\{x_{i_j}\}_{j=1}^{\infty}$ such that $|u^{i_j}(x_{i_j})| \leq \alpha$ and $\lim_{j\to\infty} x_{i_j} = x$, then (3.29) and (3.39) prove (4.1) immediately, since

$$c_9 \leq E(r, x_{i_j}) = \frac{1}{r^{n-1}} \int_{B_r(x_{i_j})} \left(\frac{\varepsilon_{i_j}}{2} |\nabla u^{i_j}|^2 + \frac{W}{\varepsilon_{i_j}} \right) \leq c_{14}(E_0 + 1)$$

and

$$\lim_{j\to\infty} \frac{1}{r^{n-1}} \int_{B_r(x_{i_j})} \left(\frac{\varepsilon_{i_j}}{2} |\nabla u^{i_j}|^2 + \frac{W}{\varepsilon_{i_j}} \right) = \frac{1}{r^{n-1}} \mu(B_r(x))$$

for $0 < r < r_0$ a.e. $\mathcal{L}^1$. To show the above claim, assume the contrary. This means that there exists some $r > 0$ such that $|u^i| \geq \alpha$ on $B_r(x)$ for all large i. Without loss of generality, assume $u^i \geq \alpha$ on $B_r(x)$. Then one can repeat the argument leading to (3.23) with ϕ there replaced by $C_c^1(B_r(x))$. Then one can show that $\lim_{i\to\infty} \int \frac{\varepsilon_i}{2} |\nabla u^i|^2 \phi^2 = 0$. Multiplying $u^i - 1$ to the equation (2.3) and using $W'(u^i)(u^i - 1) \geq \frac{\kappa}{2}(u^i - 1)^2$, one can derive that $0 = \lim_{i\to\infty} \int \frac{(u^i-1)^2}{\varepsilon_i} \phi^2 = \lim_{i\to\infty} \int \frac{W}{\varepsilon_i} \phi^2$. This contradicts $x \in \operatorname{supp}\mu$. Thus we proved the claim. $\square$

The immediate consequence of Proposition 4.1 and its proof is

Proposition 4.2. *Either* $u^i \to +1$ *or* -1 *uniformly on each compact subset of* $U \setminus \operatorname{supp}\|V\|$. *In particular,* $\operatorname{supp}\|\partial\{u^\infty = 1\}\| \subset \operatorname{supp}\|V\|$.

The vanishing of the so-called discrepancy

$$\xi^i = \frac{\varepsilon_i}{2} |\nabla u^i|^2 - \frac{W(u^i)}{\varepsilon_i}$$

follows by the same proof as in [9, Proposition 4.3].

Proposition 4.3. $\xi^i \to 0$ *in* $L^1_{\text{loc}}(U)$. *Moreover, both* $\frac{\varepsilon_i}{2}|\nabla u^i|^2 - |\nabla w^i|$ *and* $\frac{W(u^i)}{\varepsilon_i} - |\nabla w^i|$ *also converge to zero in* $L^1_{\text{loc}}(U)$.

The information on the mean curvature of the limit varifold is obtained similarly.

Proposition 4.4. *The limit varifold satisfies* $\|V\| = \frac{1}{2}\mu$ *and is rectifiable. The first variation of* V *is given by*

$$\delta V(g) = \frac{1}{2} \int_U u^\infty \operatorname{div}(f^\infty g) = -\int_{M^\infty} f^\infty g \cdot \nu^\infty \, d\mathcal{H}^{n-1}$$

for any $g \in C_c^1(U; \mathbb{R}^n)$, *where* $M^\infty \subset \operatorname{supp}\|V\|$ *is the reduced boundary of* $\{u^\infty = 1\}$ *and* f^∞ *on* M^∞ *is the trace of* $f^\infty \in W^{1,p}(U)$ *which is well-defined.*

The generalized mean curvature vector H is given by

$$H(x) = \begin{cases} \dfrac{f^\infty(x)}{\theta(x)}\nu^\infty(x), & \mathcal{H}^{n-1} a.e.\, x \in M^\infty, \\ 0, & \mathcal{H}^{n-1} a.e.\, x \in \mathrm{supp}\|V\| \setminus M^\infty, \end{cases}$$

where θ is the density function for $\|V\|$ which exist everywhere on $\mathrm{supp}\|V\|$. *Moreover,*

$$f^\infty\lfloor_{M^\infty} \in L_{\mathrm{loc}}^{\frac{p(n-1)}{n-p}}(U, \mathcal{H}^{n-1}).$$

Proof. The proof is identical to [19, Proposition 4.4], except for the part that $f^\infty\lfloor_{M^\infty} \in L_{\mathrm{loc}}^{\frac{p(n-1)}{n-p}}(U, \mathcal{H}^{n-1})$. This is due to $f^\infty \in W^{1,p}(U)$ instead of $W^{1,n}(U)$, and the modification is straightforward. ☐

The proof of integrality requires some non-trivial modifications from [9, Section 5] for one of the three propositions. Since it may be more confusing to sketch the proof, we more or less write out the details on this point. It is the first proposition which states that the energy is uniformly small in ε in the region $\{|u| \approx 1\}$. Since integrality is a local question, for simplicity we assume that $U = B_3$ and $\tilde{U} = B_1$ in the following.

Proposition 4.5. *Given $s > 0$, there exist positive constants $b < 1$ and ε_6 depending only on λ_0, c_0, E_0, W and s such that*

$$\int_{B_1\cap\{|u|\geq 1-b\}} \left(\frac{\varepsilon}{2}|\nabla u|^2 + \frac{W}{\varepsilon}\right) \leq s$$

whenever $\varepsilon \leq \varepsilon_6$.

To prove this, we use three lemmata. Define

$$Z_\alpha = \{x \in B_3 \mid u(x) \in [-\alpha, \alpha]\}.$$

Lemma 4.6. *Given $0 < \beta_6 < 2 - \frac{n}{p}$, there exist positive constants c_{25} and ε_7 which depend only on λ_0, W and β_6 such that, if $x \in B_1$ and $|u(x)| < 1 - 2\varepsilon^\beta$ for some β with*

$$\frac{1}{c_{25}|\ln\varepsilon|} < \beta < \min\{\beta_6, \frac{1}{c_{25}\varepsilon|\ln\varepsilon|}\},$$

then

$$\mathrm{dist}(x, Z_\alpha) \leq c_{25}\beta\varepsilon|\ln\varepsilon|,$$

provided $0 < \varepsilon < \varepsilon_7$.

Proof. Without loss of generality, assume $x = 0$. Rescale x by ε and write $\tilde{u}(x) = u(\varepsilon x)$. We use a radially symmetric function which solves

$$\begin{cases} \Delta\psi = \dfrac{\kappa}{4}\psi \text{ on } \mathbb{R}^n, \\ \psi(0) = 1, \end{cases} \tag{4.2}$$

which exists uniquely and also satisfies $\psi \geq 1$ on $\mathbb{R}^n$. The function ψ grows exponentially as $|x| \to \infty$, so there exists a constant c_{25} depending only on κ and n such that

$$\psi(x) > \exp(|x|/c_{25}) \qquad \text{for } |x| \geq 1. \tag{4.3}$$

Let $r = c_{25}\beta|\ln\varepsilon|$. We choose r so that $1 - \varepsilon^\beta \exp(r/c_{25}) = 0$, and by the assumption imposed on β, $1 \leq r \leq \varepsilon^{-1}$. We are given the assumption that $|\tilde{u}(0)| < 1 - 2\varepsilon^\beta$ and without loss, assume $\tilde{u}(0) < 1 - 2\varepsilon^\beta$. Suppose for a contradiction that

$$\inf_{B_r} \tilde{u} > \alpha. \tag{4.4}$$

Define $\phi(x) = 1 - \varepsilon^\beta\psi(x)$. Then ϕ satisfies $\Delta\phi = \frac{\kappa}{4}(\phi - 1)$ by (4.2). By (4.3) and (4.4), $\phi(x) < 1 - \varepsilon^\beta \exp(r/c_{25}) < \alpha < \inf_{B_r} \tilde{u}$ on $|x| = r$. Hence

$$\phi - \tilde{u} < 0 \qquad \text{on } |x| = r. \tag{4.5}$$

Since $\tilde{u}(0) < 1 - 2\varepsilon^\beta$ and $\phi(0) = 1 - \varepsilon^\beta$,

$$\sup_{B_r}(\phi - \tilde{u}) \geq \varepsilon^\beta. \tag{4.6}$$

On B_r, we apply the Aleksandrov-Bakelman-Pucci estimate [7, Lemma 9.3] to $(\phi - \tilde{u})_+$. By (4.2) and (2.8),

$$\Delta(\phi - \tilde{u}) = -\frac{\kappa}{4}(1 - \phi) - W'(\tilde{u}) + \varepsilon\tilde{f}. \tag{4.7}$$

Since we use (4.7) only on $\{\phi \geq \tilde{u}\}$ and $\phi \leq 1$, by (4.4), we have $\alpha \leq \tilde{u} \leq 1$ on $\{\phi \geq \tilde{u}\}$. Thus by (2.1)

$$-W'(\tilde{u}) \geq \kappa(1 - \tilde{u}) \geq \kappa(1 - \phi) \tag{4.8}$$

on $\{\phi \geq \tilde{u}\}$. (4.8) and (4.7) show that on $\{\phi \geq \tilde{u}\}$

$$\Delta(\phi - \tilde{u}) \geq \frac{3}{4}\kappa(1 - \phi) + \varepsilon\tilde{f} \geq \varepsilon\tilde{f}. \tag{4.9}$$

Thus

$$\sup_{B_r}(\phi - \tilde{u})_+ \leq c_{26}(n) r ||\varepsilon\tilde{f}||_{L^n(B_r)}. \tag{4.10}$$

By (2.7) we have

$$||\tilde{f}||_{L^{\frac{np}{n-p}}(B_r)} \le \varepsilon^{1-\frac{n}{p}} c_1 \lambda_0 \tag{4.11}$$

and Hölder's inequality applied to (4.10) and (4.11) shows

$$\begin{aligned}\sup_{B_r}(\phi - \tilde{u})_+ &\le c_{26}\varepsilon r(\omega_n r^n)^{\frac{2p-n}{np}} ||\tilde{f}||_{L^{\frac{np}{n-p}}(B_r)} \\ &\le c_{27} r^{3-\frac{n}{p}} \varepsilon^{2-\frac{n}{p}},\end{aligned} \tag{4.12}$$

where c_{27} depends only on n, p, c_1 and λ_0. By (4.6) and (4.12) and the definition of r,

$$\varepsilon^{\beta} \le c_{27}(c_{25}\beta|\ln\varepsilon|)^{3-\frac{n}{p}} \varepsilon^{2-\frac{n}{p}}. \tag{4.13}$$

Since $\beta \le \beta_6 < 2 - \frac{n}{p}$, (4.13) leads to

$$1 \le c_{27}(c_{25}\beta_6|\ln\varepsilon|)^{3-\frac{n}{p}} \varepsilon^{2-\frac{n}{p}-\beta_6}. \tag{4.14}$$

This is impossible if ε is restricted small enough depending only on c_{25}, c_{27}, β_6, n and p. Thus we derive a contradiction to (4.4), and we only need to re-scale back to conclude the proof. □

Lemma 4.7. *There exist positive constants c_{28} and ε_8 depending only on λ_0, c_0, E_0, W, n and p such that if $\varepsilon \le r \le 1$, then*

$$\mathcal{L}^n(\{x \in B_2 \mid \operatorname{dist}(x, Z_\alpha) < r\}) \le c_{28} r,$$

provided that $\varepsilon \le \varepsilon_8$.

The proof of Lemma 4.7 is exactly the same as [9, Lemma 5.3], so we omit the proof.

Lemma 4.8. *Given $0 < \beta < 1$ and $0 < s$, there exists a positive constant ε_9 depending only on c_0, λ_0, E_0, W, β, s, n and p such that*

$$\int_{B_1 \cap \{|u| \ge 1-\varepsilon^{\beta}\}} \left(\frac{\varepsilon}{2}|\nabla u|^2 + \frac{W}{\varepsilon}\right) \le s,$$

provided that $\varepsilon \le \varepsilon_9$.

Proof. Since $\{|u| \ge 1-\varepsilon^{\beta}\} \subset \{|u| \ge 1-\varepsilon^{\beta'}\}$ for $\beta > \beta'$, without loss of generality we may choose smaller $\beta > 0$ so that $1 - \beta - \beta_1 > 0$ if necessary. We estimate the integral on three disjoint sets. Define

$$\mathcal{A} = \{x \in B_1 \mid \operatorname{dist}(x, Z_\alpha) < \varepsilon^{1-\beta},\ 1 \ge |u(x)| \ge 1 - \varepsilon^{\beta}\},$$

$$\mathcal{B} = \{x \in B_1 \mid \operatorname{dist}(x, Z_\alpha) \ge \varepsilon^{1-\beta},\ 1 \ge |u(x)| \ge 1 - \varepsilon^{\beta}\},$$

$$\mathcal{C} = \{x \in B_1 \mid |u(x)| \ge 1\}.$$

By Lemma 4.7, for $\varepsilon \le \varepsilon_8$

$$\int_{\mathcal{A}} \frac{W(u)}{\varepsilon} \le c_{28}\varepsilon^{1-\beta} \sup_{\mathcal{A}} \frac{W(u)}{\varepsilon}. \tag{4.15}$$

Since $W(u) \le c(1-|u|)^2$ for a suitable $c > 0$ depending only on W, $W(u) \le c\varepsilon^{2\beta}$ on $\mathcal{A}$. (4.15) then shows

$$\int_{\mathcal{A}} \frac{W}{\varepsilon} \le c_{29}\varepsilon^{\beta}. \tag{4.16}$$

By (3.2),

$$\int_{\mathcal{A}} \frac{\varepsilon}{2}|\nabla u|^2 \le \int_{\mathcal{A}} \varepsilon^{-\beta_1} + \frac{W}{\varepsilon} \le c_{28}\varepsilon^{1-\beta-\beta_1} + c_{29}\varepsilon^{\beta}. \tag{4.17}$$

(4.16) and (4.17) show that the contribution from the integral on $\mathcal{A}$ can be made small for ε small. Next, define the Lipschitz function ϕ by

$$\phi(x) = \min\{1, \varepsilon^{-1+\beta}\operatorname{dist}(x, Z_\alpha \cup \big(\mathbb{R}^n \setminus B_{1+\varepsilon^{1-\beta}}))\}.$$

Then $\phi = 1$ on $\mathcal{B}$, $\phi = 0$ on Z_α, and $|\nabla\phi| \le \varepsilon^{-1+\beta}$. In particular, $|u| \ge \alpha$ on the support of ϕ. Then, by (3.24) where we use the ϕ above,

$$\int \frac{\kappa}{\varepsilon}|\nabla u|^2\phi^2 \le c_8\left(\varepsilon^{-2+2\beta}\int_{B_2} \varepsilon|\nabla u|^2 + \varepsilon^{-1}c_{30}\right).$$

Since $\phi = 1$ on $\mathcal{B}$,

$$\int_{\mathcal{B}} \varepsilon|\nabla u|^2 \le c_8\kappa^{-1}(2\varepsilon^{2\beta}E_0 + \varepsilon c_{30}). \tag{4.18}$$

By Proposition 4.3 and (4.18), for ε sufficiently small

$$\begin{aligned}\int_{\mathcal{B}} \frac{W}{\varepsilon} &\le \int_{B_1}\left|\frac{W}{\varepsilon} - \frac{\varepsilon}{2}|\nabla u|^2\right| + \int_{\mathcal{B}} \frac{\varepsilon}{2}|\nabla u|^2 \\ &\le \frac{s}{4} + c_8\kappa^{-1}(2\varepsilon^{2\beta}E_0 + \varepsilon c_{30}).\end{aligned} \tag{4.19}$$

(4.18) and (4.19) show that the integral on $\mathcal{B}$ can be made smaller than $\frac{s}{2}$ for all small ε. Finally, for the integral on $\mathcal{C}$, multiply $(u-1)_+\phi^2$ to (2.3) and integrate by parts, where $\phi \in C_c^1(B_2)$, $|\nabla\phi| \le 2$ and $\phi = 1$ on B_1. Then, using Lemma 3.8, one can easily check that the integral on $\mathcal{C}$ can be made small. This concludes the proof. □

Now we are ready to prove Proposition 4.5. The only difference from the proof of [9, Proposition 5.1] is that one cannot use Lemma 4.6 for β close to 1 (such as $\frac{2}{3}$ used there) because $\beta < \beta_6 < 2 - \frac{n}{p}$ and β_6 can be small. To overcome this difficulty, we have Lemma 4.8 to show the uniform smallness of the energy. Thus a simple modification of [9, Proposition 5.1] and Proposition 4.3 show Proposition 4.5. The remaining proof of the integrality can be accomplished by modifying the proof in [9], where one shows that the error term coming from f can be handled as a small term using Hölder's inequality and (2.5). Since it is carried out in [19], we omit the proof.

5. Remarks

(1) Though it is unclear what can be said about the case of $p = \frac{n}{2}$, it is interesting to know if anything can be said about the limit varifold in general.

(2) As in [19, Section 5.2], the result of this paper shows the following: For the solutions of the Cahn-Hilliard equation

$$\begin{cases} u_t = \Delta f & \text{on } U \times (0,\infty), \\ f = -\varepsilon\Delta u + \dfrac{W'(u)}{\varepsilon}, & \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial f}{\partial \nu} = 0 & \text{on } \partial U \times (0,\infty), \\ u(x,0) = u_0(x) & \text{on } U \end{cases}$$

Chen [4, Lemma 3.4] showed that

$$\int_0^t ||f(\cdot,t)||^2_{W^{1,2}(U)}\,dt \le C$$

where C does not depend on ε. Moreover, with a suitable growth condition on W, one can also show that $\sup|u|$ is bounded uniformly in ε. Thus, given a sequence of $\{\varepsilon_i\}$ with the finite energy initial data $\{u_0^i\}$, the assumptions in this paper are satisfied for the solution of the equation for $n = 2, 3$ for a.e. t, after choosing a (time-dependent) subsequence. Since we can not conclude any continuity properties of limit varifolds in the time direction, the result obtained via our result is not satisfactory.

(3) One can extend our results to corresponding time-dependent problems such as the Allen-Cahn equation with inhomogeneous forcing term. We would like to resolve these problems in the future.

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