

## Quaternionic maps and minimal surfaces

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**Abstract.** Let  $(M, J^\alpha, \alpha = 1, 2, 3)$  and  $(N, \mathcal{J}^\alpha, \alpha = 1, 2, 3)$  be hyperkähler manifolds. We study stationary quaternionic maps between  $M$  and  $N$ . We first show that if there are no holomorphic 2-spheres in the target then any sequence of stationary quaternionic maps with bounded energy subconverges to a stationary quaternionic map strongly in  $W^{1,2}(M, N)$ . We then find that certain tangent maps of quaternionic maps give rise to an interesting minimal 2-sphere. At last we construct a stationary quaternionic map with a codimension-3 singular set by using the embedded minimal  $\mathbb{S}^2$  in the hyperkähler surface  $\tilde{M}_2^0$  studied by Atiyah-Hitchin.

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### 1. Introduction

A Riemannian manifold is called hyperkähler if it possesses covariant constant complex structures  $I, J, K$  which satisfy the quaternionic relation

$$I^2 = J^2 = K^2 = IJK = -\text{identity}.$$

Associated to  $I, J, K$  there is a natural family of covariant constant complex structures  $aI + bJ + cK$  where  $(a, b, c)$  is a unit vector in  $\mathbb{R}^3$ . A hyperkähler manifold is Ricci-flat with dimension  $4k$ . Let  $M$  and  $N$  be two hyperkähler manifolds with complex structures  $J^\alpha$  and  $\mathcal{J}^\beta$  respectively for  $\alpha, \beta = 1, 2, 3$  which satisfy the quaternionic identities. A smooth map  $f : M \rightarrow N$  is called a quaternionic map if

$$\sum_{\alpha, \beta=1}^3 A_{\alpha\beta} \mathcal{J}^\beta \circ df \circ J^\alpha = df \quad (1.1)$$

where  $A_{\alpha\beta}$  denote the entries of a constant matrix  $A$  in  $SO(3)$ . Since  $SO(3)$  preserves the quaternionic identities, we can always choose complex structures  $J^\alpha$  for  $M$  and  $\mathcal{J}^\beta$  for  $N$  such that  $A_{\alpha\beta} = \delta_{\alpha\beta}$  in (1.1).

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Quaternionic maps arise from the higher dimensional gauge theory (cf. [C], [DT], [FKS], [MS], [NN], [PG]). More precisely they naturally arise from the adiabatic limit of Hermitian Yang-Mills connections on  $SU(n)$ -bundles on a product of two K3 surfaces. Its linear version in dimension four is the so-called Cauchy-Riemann-Fueter equation (or quaternionic d-bar equations):

$$\partial_{x_1} f - i \partial_{x_2} f - j \partial_{x_3} f - k \partial_{x_4} f = 0$$

for  $f : \mathbb{H} \rightarrow \mathbb{H}$  where  $\mathbb{H}$  is the space of quaternions and  $x_1 + ix_2 + jx_3 + kx_4 \in \mathbb{H}$ .

Assume  $M$  is compact. For any smooth map  $u : M \rightarrow N$ , consider the energy functional

$$E(u) = \frac{1}{2} \int_M |\nabla u|^2$$

and the functional

$$E_T(u) = A_{\alpha\beta} \int_M \langle \omega_{J^\alpha}, u^* \omega_{\mathcal{J}^\beta} \rangle$$

and set

$$I(u) = \frac{1}{2} \int_M |du - A_{\alpha\beta} \mathcal{J}^\beta \circ du \circ J^\alpha|^2.$$

It is clear that  $I(u) = 0$  if and only if  $u$  is a quaternionic map. Since  $u$  pulls back the closed 2-form  $\omega_{\mathcal{J}^\beta}$  to a closed 2-form on  $M$  and  $\omega_{J^\alpha}$  is closed,  $E_T(u)$  is homotopy invariant and depends on  $(A_{\alpha\beta})$ . The following relation holds (cf. [C], [CL1], [FKS])

$$E(u) + E_T(u) = \frac{1}{4} I(u). \tag{1.2}$$

If  $u$  is a quaternionic map, then it minimizes energy in its homotopy class so it is harmonic.

Recall [Sc] that a map in the Sobolev space  $W^{1,2}(M, N)$  is a *stationary harmonic map* if it is a critical point of the energy functional with respect to both of the variations on  $M$  and  $N$  with compact supports. A stationary harmonic map is smooth away from a closed set of zero  $(m - 2)$ -dimensional Hausdorff measure where  $m = \dim M$ . Let  $M$  and  $N$  be two hyperkähler manifolds. A map  $u$  from  $M$  to  $N$  is called a *stationary quaternionic map* if it is a stationary harmonic map and it is a quaternionic map outside its singular set.

It is known that the existence harmonic 2-spheres plays an important role in the study of stationary harmonic maps ([SU], [Lin]).

In this note we investigate the special minimal 2-spheres which arise from the stationary quaternionic maps. We first show that if there are no holomorphic 2-spheres in  $N$  then any sequence of stationary quaternionic maps with bounded energy subconverges to a stationary quaternionic map strongly in  $W^{1,2}(M, N)$ . This result was stated and proved in [CL1] when  $M$  is of dimension four, and the proof

we shall present here is essentially based on that in [CL1]. We then find that certain tangent maps of quaternionic maps give rise to an interesting minimal 2-sphere equation:

$$df J_{\mathbb{S}^2} = - \sum_{k=1}^3 x_k \mathcal{J}^k df$$

where  $f : \mathbb{S}^2 \rightarrow N$ ,  $(x_1, x_2, x_3) \in \mathbb{S}^2$  and  $J_{\mathbb{S}^2}$  is the standard complex structure on  $\mathbb{S}^2$ . We construct a stationary quaternionic map with a codimension-3 singular set by using the embedded minimal  $\mathbb{S}^2$  in the hyperkähler surface  $\tilde{M}_2^0$  studied by Atiyah-Hitchin [AH], where  $\tilde{M}_2^0$  is the double cover of the space  $M_2^0$  of centred 2-monopoles on  $\mathbb{R}^3$  and it is a complete and simply connected hyperkähler surface.

There are interesting results on decomposition of differential forms in quaternionic geometry using representations of special groups (e.g. [Bo], [K], [Sa], [Sw], [W], etc). It is commented in [W] that the quaternionic maps between hyperkähler manifolds can be described by the splitting of  $Sp(1)$ -representations. The authors thank the referee for his bringing this point and the related references in quaternionic geometry to their attention.

## 2. Compactness of stationary quaternionic maps

A sequence of stationary harmonic maps with bounded energies subconverges to a stationary harmonic map strongly in  $W^{1,2}$  topology if there are no harmonic 2-spheres in the target manifold [L]. For stationary quaternionic maps, the absence of holomorphic 2-spheres is sufficient to conclude the strong convergence.

**Theorem 2.1.** *Let  $M$  and  $N$  be compact hyperkähler manifolds with  $\dim M = m$ . Suppose that  $u_k$  is a sequence of stationary quaternionic maps with bounded energies. If  $N$  does not admit holomorphic  $\mathbb{S}^2$ 's with respect to the complex structure  $a_i J^i$  on  $\mathbb{R}^2$  restricted to  $\mathbb{S}^2$  and the complex structure  $a_i \mathcal{J}^i$  on  $N$  for some constants  $a_i$  ( $i = 1, 2, 3$ ) with  $\sum_i a_i^2 = 1$ , then there is a subsequence of  $\{u_k\}$  which converges strongly to a stationary quaternionic map  $u$ .*

*Proof.* We can always assume that  $u_k \rightharpoonup u$  weakly in  $W^{1,2}(M, N)$  and that  $|\nabla u_k|^2 dx \rightharpoonup |\nabla u|^2 dx + \nu$  in the sense of measure as  $k \rightarrow \infty$ . Here  $\nu$  is a non-negative Radon measure on  $M$  with support in  $\Sigma$ , and  $\Sigma$  is the blow-up set of the sequence  $u_k$  which is  $m - 2$  rectifiable [L]. We will prove the Hausdorff measure  $H^{m-2}(\Sigma) = 0$  which implies the strong convergence in  $W^{1,2}(M, N)$ . Assume  $H^{m-2}(\Sigma) \neq 0$ . Then [L] there is a nonconstant harmonic map  $v : \mathbb{R}^m \rightarrow N$  with finite energy and  $\nabla_\Sigma v = 0$ . Here we have identified the tangent space of  $\Sigma$  at  $0 \in \mathbb{R}^m = \mathbb{R}^{m-2} \times \mathbb{R}^2$  with  $\mathbb{R}^{m-2} \times \{0\}$  so  $\nabla_\Sigma$  means the differentiation along  $\mathbb{R}^{m-2} \times \{0\}$ . The rescaling process for constructing  $v$  is taken place around smooth points of  $u_k$  which approach 0, therefore  $v$  is also a smooth quaternionic map (cf. [CT]).

At the point  $0 \in \mathbb{R}^m$ , suppose that  $e$  is in the normal direction of  $\Sigma$ . Let  $K$  be the linear space spanned by  $J^\alpha e$  for  $\alpha = 1, 2, 3$ , so  $K \perp e$ . Since  $\text{rank } dv = 2$ , we have

$dv(e) \neq 0$ . This implies, from the quaternionic map equation,  $\sum_{i=1}^3 \mathcal{J}^i dv(J^i e) \neq 0$  and in turn  $dv(J^i e) \neq 0$  for some  $i$ . Hence  $\dim dv(K) = 1$ . It then follows that there are real constants  $a_1, a_2, a_3$  with  $a_1^2 + a_2^2 + a_3^2 = 1$  such that  $a_i J^i e \in \{0\} \times \mathbb{R}^2$  and  $dv(a_i J^i e) \neq 0$ . Notice that we then have three vectors  $a_i J^j e - a_j J^i e, i \neq j$  which are perpendicular to  $e$  and to  $\sum_{i=1}^3 a_i J^i e$ , so they belong to  $T\Sigma$ . We therefore have  $(a_2 J^1 - a_1 J^2)e \in \text{Ker}(dv), (a_3 J^1 - a_1 J^3)e \in \text{Ker}(dv), (a_2 J^3 - a_3 J^2)e \in \text{Ker}(dv), \mathcal{J}^\alpha dv J^\alpha = dv$ , thus  $dv(\sum_i a_i J^i e)$  can only have components on  $\mathcal{J}^\alpha(dv(e))$ . By a simple calculation, one easily checks that

$$\begin{aligned} dv\left(\sum_{i=1}^3 a_i J^i e\right) &= \sum_{i,j=1}^3 \mathcal{J}^j dv(a_i J^j J^i e) \\ &= -\sum_{i=1}^3 a_i \mathcal{J}^i dv(e) + \mathcal{J}^1 dv(a_2 J^1 J^2 + a_3 J^1 J^3)e \\ &\quad + \mathcal{J}^2 dv(a_1 J^2 J^1 + a_3 J^2 J^3)e + \mathcal{J}^3 dv(a_1 J^3 J^1 + a_2 J^3 J^2)e \\ &= -\sum_{i=1}^3 a_i \mathcal{J}^i dv(e). \end{aligned}$$

At any other point  $(0, x)$  in  $\mathbb{R}^{m-2} \times \mathbb{R}^2$ , the vectors  $e$  and  $\sum_{i=1}^3 a_j \mathcal{J}^j e$  still belong to  $\{0\} \times \mathbb{R}^2$ , and the vectors  $(a_1 \mathcal{J}^2 - a_2 \mathcal{J}^1)e, (a_2 \mathcal{J}^2 - a_3 \mathcal{J}^2)e, (a_1 \mathcal{J}^3 - a_3 \mathcal{J}^1)e$  lie in  $\mathbb{R}^{m-2} \times \{x\}$  hence in the kernel of  $dv$  at  $(0, x)$ , so we can repeat the argument above to conclude  $v$  is holomorphic at  $(x, 0)$  with respect to the same complex structures  $\sum_{i=1}^3 a_i J^i$  and  $\sum_{i=1}^3 a_i \mathcal{J}^i$ . It follows that  $v$  induces a holomorphic map from  $\mathbb{S}^2$  to  $N$ . But no such holomorphic map can exist by assumption. So we must have  $H^{m-2}(\Sigma) = 0$  and in turn  $u_k$  converge strongly to  $u$  in  $W^{1,2}$  norm.  $\square$

**Remark 2.2.** The strong convergence is equivalent to  $H^{m-2}(\Sigma) = 0$  and is equivalent to that the Hausdorff dimension of the singular set  $\text{sing}(u)$  of  $u$  is no bigger than  $m - 3$ . Moreover  $\text{sing}(u)$  is rectifiable since  $N$  real analytic [Si].

### 3. Quaternionic minimal surfaces via quaternionic maps

In this section we study a special class of minimal surfaces which arise from certain tangent maps of the quaternionic maps.

Assume that  $M$  is 4-dimensional hyperkähler manifold and  $N$  is a  $4n$ -dimensional hyperkähler manifold. We can choose a coordinate system around a point  $x$  in  $M$  so that the matrix expressions of the complex structures on  $M$  take the following form:

$$J^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, J^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, J^3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

Note that the three Kähler forms  $\omega_{J_i}, i = 1, 2, 3$  have variable coefficients in these coordinates. For  $f : M \rightarrow N$ , if we denote  $\frac{\partial f}{\partial x_k}$  by  $f_k$  for  $k = 1, 2, 3, 4$  in the coordinate system we have just chosen, the quaternionic map equation (1.1) reads

$$f_1 - a_{\alpha 3} \mathcal{J}^\alpha f_2 + a_{\alpha 2} \mathcal{J}^\alpha f_3 + a_{\alpha 1} \mathcal{J}^\alpha f_4 = 0 \tag{3.1}$$

where we take summation over  $\alpha$ .

Now assume that  $f$  is a homogeneous degree-0 quaternionic map from  $\mathbb{R}^4$  to  $N$  and satisfies  $f(x_1, x_2, x_3, x_4) = f(x_1, x_2, x_3, 0)$ . So  $f$  is singular along the  $x_4$ -axis or it is constant. Note that such an  $f$  is just a tangent map, with a line of singularities, of a quaternionic map from  $M$  to  $N$ .

As a radially independent harmonic map,  $f$  induces a smooth harmonic map from  $\mathbb{S}^2$  to  $N$ :  $\phi(x) = f(x, x_4)$  for  $x \in \mathbb{S}^2 \subset \mathbb{R}^3$ .

**Lemma 3.1.** *With  $f$  and  $\phi$  as above, then*

$$d\phi J_{\mathbb{S}^2} = -a_{\alpha\beta} x_\beta \mathcal{J}^\alpha d\phi. \tag{3.2}$$

*Proof.* Because  $f$  is a homogeneous degree-0 map,

$$\sum_{k=1}^4 x_k f_k = 0$$

and this combined with (3.1) leads to

$$(x_2 + x_1 a_{\alpha 3} \mathcal{J}^\alpha) f_2 + (x_3 - x_1 a_{\alpha 2} \mathcal{J}^\alpha) f_3 = 0.$$

In the spherical coordinates

$$\begin{cases} x_1 = r \sin \alpha \cos \theta \\ x_2 = r \sin \alpha \sin \theta \\ x_3 = r \cos \alpha, \end{cases}$$

it reads

$$(x_2 + x_1 a_{\alpha 3} \mathcal{J}^\alpha) \left( \cos \alpha \sin \theta f_\alpha + \cos \theta \frac{f_\theta}{\sin \alpha} \right) + (x_3 - x_1 a_{\alpha 2} \mathcal{J}^\alpha) (-\sin \alpha f_\alpha) = 0.$$

Multiplying this equation by  $\sin(\alpha)$  yields

$$(x_2 + x_1 a_{\alpha 3} \mathcal{J}^\alpha) \left( x_3 x_2 f_\alpha + x_1 \frac{f_\theta}{\sin \alpha} \right) - (x_3 - x_1 a_{\alpha 2} \mathcal{J}^\alpha) (x_1^2 + x_2^2) f_\alpha = 0$$

i.e.

$$-x_1 (x_2 + x_1 a_{\alpha 3} \mathcal{J}^\alpha) \frac{f_\theta}{\sin \alpha} = \left( x_2 x_3 (x_2 + x_1 a_{\alpha 3} \mathcal{J}^\alpha) - (x_3 - x_1 a_{\alpha 2} \mathcal{J}^\alpha) (x_1^2 + x_2^2) \right) f_\alpha.$$

Multiplying  $x_2 - x_1 a_{\alpha 3} J^\alpha$  from left on both sides of the equation above, we obtain,

$$-x_1(x_1^2 + x_2^2) \frac{f_\theta}{\sin \alpha} = x_1(x_1^2 + x_2^2) (x_1 a_{\alpha 1} J^\alpha + x_2 a_{\alpha 2} J^\alpha + x_3 a_{\alpha 3} J^\alpha) f_\alpha$$

here we have used  $a_{\alpha 3} J^\alpha \cdot a_{\beta 2} J^\beta = a_{\gamma 1} J^\gamma$  with the summation convention over repeated indices applied. So we see  $\phi$  satisfies the equation:

$$d\phi J_{\mathbb{S}^2} = -a_{\alpha\beta} x_\beta \mathcal{J}^\alpha d\phi.$$

This finishes the proof. □

Note that  $a_{\alpha\beta} x_\beta \mathcal{J}^\alpha$  is only defined along the image surface  $f(\mathbb{S}^2)$  and  $f$  cannot be holomorphic with respect to any complex structure in the 2-sphere family of complex structures on  $N$ .

Let  $\Sigma$  be a Riemann surface,  $N^{4n}$  a hyperkähler manifold with the complex structures  $\mathcal{J}^1, \mathcal{J}^2, \mathcal{J}^3$  which satisfy the quaternion relation  $\mathcal{J}^1 \mathcal{J}^2 = \mathcal{J}^3$ . Let  $\vec{a} = (a_1, a_2, a_3)$  be smooth functions  $\Sigma \rightarrow \mathbb{S}^2$ .

**Definition 3.2.** Let  $f : \Sigma \rightarrow N^{4n}$  be a smooth immersion which satisfies

$$df J_\Sigma = - \sum_{k=1}^3 a_k \mathcal{J}^k df, \tag{3.3}$$

where  $\vec{a} = (a_1, a_2, a_3) : \Sigma \rightarrow \mathbb{S}^2$ . We say  $f$  is a quaternionic surface in  $N^{4n}$ . If in addition  $f$  is harmonic, we say  $f$  is a quaternionic minimal surface.

Condition (3.3) requires the image of  $df$  lying in the span of  $\mathcal{J}^1 df, \mathcal{J}^2 df, \mathcal{J}^3 df$ . In the twistor space approach to minimal surfaces and harmonic maps, this condition is called "inclusive" (see [AM], [ES], [R], [Sa] and the references therein).

It is not difficult to see that if  $f$  satisfies (3.3) then  $f$  is conformal. Furthermore, any conformal immersion from  $(\Sigma, J_\Sigma)$  to a 4-dimensional hyperkähler manifold satisfies the equation (3.3). In fact, suppose that  $e_1, e_2$  is an orthonormal frame of  $\Sigma$ . Because  $f$  is conformal and  $df(e_1) \perp df(e_2)$ , we have

$$df(e_1) = c_i J^i df(e_2) \text{ and } df(e_2) = d_i J^i df(e_1)$$

with  $\sum_i c_i^2 = 1$  and  $\sum_i d_i^2 = 1$ . It is clear that

$$c_i |df(e_2)|^2 = \langle df(e_1), J^i df(e_2) \rangle = -\langle J^i df(e_1), df(e_2) \rangle = -d_i |df(e_1)|^2.$$

Since  $|df(e_2)|^2 = |df(e_1)|^2 = 1/2 |df|^2$ , we have  $c_i = -d_i$  hence (3.3) holds.

**Lemma 3.3.** *Let  $u : \Sigma_1 \rightarrow \Sigma_2$  be a holomorphic map between two Riemann surfaces with complex structures  $J_{\Sigma_1}$  and  $J_{\Sigma_2}$  respectively. Then for any smooth map  $f : \Sigma_2 \rightarrow N$  which satisfies (3.3) with  $\vec{a} : \Sigma_1 \rightarrow \mathbb{S}^2$ ,  $f \circ u : \Sigma_1 \rightarrow N$  satisfies (3.3) with  $\vec{a} \circ u : \Sigma_1 \rightarrow \mathbb{S}^2$ . If  $f(\Sigma_2)$  is a quaternionic minimal surface, then  $f \circ u(\Sigma_1)$  is also a quaternionic minimal surface.*

*Proof.* Then for any  $x \in \Sigma_1$

$$\begin{aligned} d(f \circ u)_x J_{\Sigma_1}(x) &= df_{u(x)} \circ du_x J_{\Sigma_1}(x) \\ &= df_{u(x)} \circ J_{\Sigma_2}(u(x)) du_x \\ &= -a_i(u(x)) \mathcal{J}_{u(x)}^i df_{u(x)} \circ du_x \\ &= -a_i(u(x)) \mathcal{J}_{u(x)}^i d(f \circ u)_x. \end{aligned}$$

If  $f$  is harmonic and  $u$  is holomorphic,  $f \circ u$  is harmonic. □

**Proposition 3.4.** *A quaternionic surface in  $N^{4n}$  is a minimal surface if and only if  $\vec{a}$  is holomorphic with respect to the complex structure on  $\Sigma$  which makes the metric  $g$  Hermitian and the standard complex structure on  $\mathbb{S}^2$ .  $\vec{a}$  is constant if and only if the quaternionic surface is a holomorphic curve.*

*Proof.* Since  $f$  is conformal, a quaternionic surface in  $N^{4n}$  is a minimal surface if and only if  $f$  is a harmonic map from  $\Sigma$  to  $N$ . Let  $e_1, e_2$  be an orthonormal frame on  $\Sigma$  which satisfies  $Ie_1 = e_2, Ie_2 = -e_1$ . Note that, by the definition,

$$f_1 := df(e_1) = \sum_{i=1}^3 a_i J^i f_2, \quad f_2 := df(e_2) = -\sum_{i=1}^3 a_i J^i f_1.$$

Taking the normal coordinates centred at  $x$  and  $f(x)$ , we have

$$\begin{aligned} \Delta f &= -\nabla_2 \left( \sum_{i=1}^3 a_i J^i \right) f_1 + \nabla_1 \left( \sum_{i=1}^3 a_i J^i \right) f_2 \\ &= \left( -\sum_{i=1}^3 \nabla_2 a_i J^i - \left( \sum_{i=1}^3 \nabla_1 a_i J^i \right) \left( \sum_{i=1}^3 a_i J^i \right) \right) f_1 \\ &= (-\nabla_2 a_1 - a_3 \nabla_1 a_2 + a_2 \nabla_1 a_3) J^1 f_1 \\ &\quad + (-\nabla_2 a_2 - a_1 \nabla_1 a_3 + a_3 \nabla_1 a_1) J^2 f_1 \\ &\quad + (-\nabla_2 a_3 - a_2 \nabla_1 a_1 + a_1 \nabla_1 a_2) J^3 f_1. \end{aligned} \tag{3.4}$$

Since  $f$  is harmonic, it follows that

$$\begin{cases} \nabla_2 a_1 + a_3 \nabla_1 a_2 - a_2 \nabla_1 a_3 = 0 \\ \nabla_2 a_2 + a_1 \nabla_1 a_3 - a_3 \nabla_1 a_1 = 0 \\ \nabla_2 a_3 + a_2 \nabla_1 a_1 - a_1 \nabla_1 a_2 = 0. \end{cases} \tag{3.5}$$

Solving (3.5) and using  $a_1 \nabla_2 a_1 + a_2 \nabla_2 a_2 + a_3 \nabla_2 a_3 = 0$ , one gets

$$\begin{cases} \nabla_1 a_1 + a_2 \nabla_2 a_3 - a_3 \nabla_2 a_2 = 0 \\ \nabla_1 a_2 + a_3 \nabla_2 a_1 - a_1 \nabla_2 a_3 = 0 \\ \nabla_1 a_3 + a_1 \nabla_2 a_2 - a_2 \nabla_2 a_1 = 0. \end{cases} \tag{3.6}$$

We can rewrite (3.5) as

$$\nabla_2 \vec{a} = \vec{a} \times \nabla_1 \vec{a},$$

and rewrite (3.6) as

$$\nabla_1 \vec{a} = -\vec{a} \times \nabla_2 \vec{a}.$$

Noting that the standard complex structure on  $\mathbb{S}^2$  at  $\vec{a}$  is  $\vec{a} \times$ , we can see that  $\vec{a}$  satisfies the equations (3.5) and (3.6) if and only if it is a holomorphic map with respect to the complex structure on  $\Sigma$  which makes the metric  $g$  Hermitian and the standard complex structure on  $\mathbb{S}^2$ .  $\square$

Remark that if we write the equation in  $b_i = -a_i$  then  $\vec{b}$  is anti-holomorphic and if  $N$  is 4-dimensional the above result was obtained in [ES] and by S.S. Chern if  $N = \mathbb{R}^4$ .

In particular, when a quaternionic surface is minimal, the mapping  $\vec{a}$  satisfies the harmonic map equation to the standard sphere:

$$\Delta \vec{a} + |\nabla \vec{a}|^2 \vec{a} = 0. \tag{3.7}$$

The following theorem is known to be true for minimal surface in a Kähler-Einstein manifold of real dimension 4 (cf. [CW]) by noticing that  $a_k = \cos \alpha_k$  where  $\alpha_k$  is the Kähler angle of the surface  $f(\Sigma)$  with respect to the Kähler form  $\omega_{\mathcal{J}^k}$  in  $N$ .

**Theorem 3.5.** *If a quaternionic surface in  $N^{4n}$  is a minimal surface with  $\vec{a} = (a_1, a_2, a_3) : \Sigma \rightarrow \mathbb{S}^2$ , then*

$$\Delta a_k + 2 \frac{|\nabla a_k|^2 a_k}{1 - a_k^2} = 0.$$

*Proof.* We only need to prove the result for  $a_1$ . First we compute the Laplacian of  $a_1$  as follows. Again we take the normal coordinates centred at  $x \in M$  and at  $f(x) \in N$ . Differentiating in  $\nabla_2$  of

$$\nabla_2 a_1 = a_2 \nabla_1 a_3 - a_3 \nabla_1 a_2$$

yields

$$\nabla_{22}^2 a_1 = \nabla_2 a_2 \nabla_1 a_3 + a_2 \nabla_{12}^2 a_3 - \nabla_2 a_3 \nabla_1 a_2 - a_3 \nabla_{12}^2 a_2.$$

Multiplying  $a_3, a_2, a_1$  accordingly to the following three equations

$$a_3 \nabla_1 a_1 = \nabla_2 a_2 + a_1 \nabla_1 a_3$$

$$a_2 \nabla_1 a_1 = a_1 \nabla_1 a_2 - \nabla_2 a_3$$

$$a_1 \nabla_1 a_1 = -a_2 \nabla_1 a_2 - a_3 \nabla_1 a_3$$

then summing them up leads to

$$\nabla_1 a_1 = a_3 \nabla_2 a_2 - a_2 \nabla_2 a_3.$$



Differentiating in  $\nabla_1$  gives

$$\nabla_{11}^2 a_1 = \nabla_1 a_3 \nabla_2 a_2 + a_3 \nabla_{21}^2 a_2 - \nabla_1 a_2 \nabla_2 a_3 - a_2 \nabla_{21}^2 a_3.$$

Now we conclude

$$\Delta a_1 = 2(\nabla_1 a_3 \nabla_2 a_2 - \nabla_1 a_2 \nabla_2 a_3)$$

and we may write the right hand side in terms which only involve  $\nabla_1$  as follows:

$$\begin{aligned} \nabla_1 a_3 \nabla_2 a_2 - \nabla_1 a_2 \nabla_2 a_3 &= \nabla_1 a_3 (a_3 \nabla_1 a_1 - a_1 \nabla_1 a_3) \\ &\quad - \nabla_1 a_2 (a_1 \nabla_1 a_2 - a_2 \nabla_1 a_1) \\ &= a_3 \nabla_1 a_1 \nabla_1 a_3 - a_1 |\nabla_1 a_3|^2 \\ &\quad - a_1 |\nabla_1 a_2|^2 + a_2 \nabla_1 a_1 \nabla_1 a_2 \\ &= -a_1 (|\nabla_1 a_1|^2 + |\nabla_1 a_2|^2 + |\nabla_1 a_3|^2). \end{aligned}$$

So we have just shown

$$\Delta a_1 = -2a_1 (|\nabla_1 a_1|^2 + |\nabla_1 a_2|^2 + |\nabla_1 a_3|^2). \tag{3.8}$$

On the other hand, we have

$$\begin{aligned} |\nabla a_1|^2 &= |\nabla_1 a_1|^2 + |\nabla_2 a_1|^2 \\ &= |\nabla_1 a_1|^2 + (a_2 \nabla_1 a_3 - a_3 \nabla_1 a_2)^2 \\ &= |\nabla_1 a_1|^2 + a_2^2 |\nabla_1 a_3|^2 + a_3^2 |\nabla_1 a_2|^2 - 2a_2 a_3 \nabla_1 a_2 \nabla_1 a_3. \end{aligned}$$

However,

$$\begin{aligned} &(1 - a_1^2)(|\nabla_1 a_1|^2 + |\nabla_1 a_2|^2 + |\nabla_1 a_3|^2) - |\nabla a_1|^2 \\ &= -a_1^2 |\nabla_1 a_1|^2 + (1 - a_1^2 - a_3^2) |\nabla_1 a_2|^2 + (1 - a_1^2 - a_2^2) |\nabla_1 a_3|^2 + 2a_2 a_3 \nabla_1 a_2 \nabla_1 a_3 \\ &= -a_1^2 |\nabla_1 a_1|^2 + a_2^2 |\nabla_1 a_2|^2 + a_3^2 |\nabla_1 a_3|^2 + 2a_2 a_3 \nabla_1 a_2 \nabla_1 a_3 \\ &= 0 \end{aligned} \tag{3.9}$$

by recalling  $a_1 \nabla_1 a_1 = a_2 \nabla_1 a_2 + a_3 \nabla_1 a_3$ .

Putting (3.8) and (3.9) together, we have

$$\Delta a_1 = -2 \frac{|\nabla a_1|^2 a_1}{1 - a_1^2},$$

which completes the proof. □

**Theorem 3.6.** *Suppose that  $f$  is a minimal quaternionic surface in  $N^4$ . Then either  $f$  is constant or the Euler characteristic number  $\frac{1}{2\pi} \chi(Nf(\Sigma))$  of the normal bundle of  $f(\Sigma)$  is  $2g - 2 - 2 \deg \vec{a}$ . In particular, if  $f \in C^2(\mathbb{S}^2, N^4)$  satisfies the equation*

$$df J_{\mathbb{S}^2} = - \sum_{i=1}^3 x_i \mathcal{J}^i df, \tag{3.10}$$

where  $x \in \mathbb{S}^2 \subset \mathbb{R}^3$ , then either  $f$  is constant or the Euler characteristic number of the normal bundle of  $f(\mathbb{S}^2)$  is  $-4$ .

*Proof.* Let  $\Sigma_0 = f(\Sigma)$ .  $\Sigma_0$  is a minimal surface in  $N$  because  $f$  is harmonic and conformal. Proposition 4.2 and Proposition 4.3 in [CT] assert, for a compact minimal surface in a Kähler-Einstein surface  $N$ , that the generalized adjunction formula

$$\begin{aligned} \chi(T\Sigma_0) + \chi(N\Sigma_0) &= \int_{\Sigma} \Omega_{12} + \Omega_{34} - \frac{1}{2} \int_{\Sigma} |\nabla J_{\Sigma_0}|^2 \\ &= 2\pi \int_{\Sigma} \alpha c_1(N) - \frac{1}{2} \int_{\Sigma} |\nabla J_{\Sigma}|^2 \end{aligned}$$

holds for some function  $\alpha$  on  $\Sigma_0$ , where  $\Omega_{12}, \Omega_{34}$  are the curvature tensors of  $N$  along the tangential and normal directions of  $\Sigma_0$  respectively. The term  $|\nabla J_{\Sigma_0}|^2$  is equal to  $2|h_{12}^4 - h_{11}^3|^2 + 2|h_{22}^4 - h_{12}^3|^2$  where  $h_{ij}^k$  are the second fundamental forms of  $\Sigma_0$  in  $N$ .

Since  $c_1(N) = 0$ , we have

$$\chi(T\Sigma_0) + \chi(N\Sigma_0) = -\frac{1}{2} \int_{\Sigma_0} |\nabla J_{\Sigma_0}|^2. \tag{3.11}$$

In particular, an embedded holomorphic  $\mathbb{S}^2$  has self-intersection number  $-2$  in  $M$  with  $C_1(M) = 0$ .

On the other hand, for any solution of (3.5), by Proposition 3.4 and Theorem 3.5 and Proposition 3.2 in [CL2] (specializing the general formula for cosine of the Kähler angle along the mean curvature flow to minimal surface) and (3.7), we always have

$$|\nabla J_{\Sigma_0}|^2 = |\nabla \vec{a}|^2 = \frac{2|\nabla a_i|^2}{1 - a_i^2} \tag{3.12}$$

for  $i = 1, 2, 3$ . One then has

$$\begin{aligned} \frac{1}{2\pi} \chi(N\Sigma) &= -\frac{1}{4\pi} \int_{\Sigma_g} |\nabla \vec{a}|^2 + 2g - 2 \\ &= 2g - 2 - 2\text{deg } \vec{a}. \end{aligned}$$

Here we recall for holomorphic  $\vec{a}$  to  $\mathbb{S}^2$ ,

$$\text{deg } \vec{a} = \frac{1}{\text{vol}(\mathbb{S}^2)} \int_{\Sigma_g} \text{Jac}(\vec{a}) = \frac{1}{4\pi} \int_{\Sigma_g} |\partial \vec{a}|^2 = \frac{1}{8\pi} \int_{\Sigma_g} |\nabla \vec{a}|^2.$$

Now if  $\Sigma = \mathbb{S}^2$  and  $\vec{a}(x) = (x_1, x_2, x_3)$ ,  $f : \mathbb{S}^2 \rightarrow N$  is harmonic because  $\vec{a} : \mathbb{S}^2 \rightarrow \mathbb{S}^2$  is the identity map. We conclude

$$\frac{1}{2\pi} \chi(N\Sigma) = -2 - \frac{1}{4\pi} \int_{\mathbb{S}^2} |\nabla x|^2 = -4.$$

This completes the proof. □

Based on the results we obtained so far, we next construct an example of stationary quaternionic map from  $\mathbb{R}^4$  with a line of singularities. For any smooth map  $\phi : \mathbb{S}^2 \rightarrow N$ , we have an extension  $f(x, x_4) := \phi(x/|x|)$  for any  $x \in \mathbb{R}^3 \setminus \{0\}$ . Moreover, the proof of Lemma 3.1 can be reversed to produce a quaternionic map with the  $x_4$ -axis as its singular set from a map  $\phi$  which satisfies (3.2).

In the monograph [AH], Atiyha and Hitchin considered the space  $M_2^0$  of centred 2-monopoles on  $\mathbb{R}^3$  with finite action. It is a complete hyperkähler manifold of dimension 4.  $SO(3)$  acts on  $M_2^0$  isometrically and this action lifts to a double (also Riemannian universal) covering  $\tilde{M}_2^0$ . The space of axisymmetric monopoles, which constitute a special class of solutions to the monopole equations, defines an embedded minimal  $\mathbb{R}P^2$  in  $M_2^0$ . This  $\mathbb{R}P^2$  lifts to an embedded minimal  $\mathbb{S}^2$  in the hyperkähler manifold  $\tilde{M}_2^0$ .

**Corollary 3.7.** *There does exist a nontrivial minimal quaternionic sphere  $\phi$  in the hyperkähler manifold  $\tilde{M}_2^0$  with  $\vec{a} = (x_1, x_2, x_3)$ . The extended map  $f$  from  $\phi$  is a stationary quaternionic map from  $\mathbb{R}^4$  to  $\tilde{M}_2^0$  with the entire  $x_4$ -axis as singular set.*

*Proof.* We take the nontrivial embedded minimal  $\mathbb{S}^2$  in  $\tilde{M}_2^0$  discussed above. The Euler characteristic number of the normal bundle of this minimal 2-sphere is  $-4$  as shown in [AH].

By Theorem 3.6, we know that the minimal 2-sphere is a minimal quaternionic sphere  $\phi_0$  with a function  $\vec{a}_0$  in its definition, and  $\deg \vec{a}_0 = 1$ . Since  $\vec{a}_0 : \mathbb{S}^2 \rightarrow \mathbb{S}^2$  is holomorphic and of degree 1, it is diffeomorphic because the sum of orders of the zeros of  $|\partial \vec{a}_0|$  is  $-\deg(\vec{a}_0)(2 \cdot 0 - 2) + (2 \cdot 0 - 2) = 0$ ,  $|\partial \vec{a}_0|$  has no zeros, and therefore the inverse  $\vec{a}_0^{-1}$  of  $\vec{a}_0$  exists and is holomorphic. So,  $\phi := \phi_0 \circ \vec{a}_0^{-1}$  is a nontrivial minimal quaternionic sphere with  $\vec{a} = (x_1, x_2, x_3)$  by Lemma 3.3.

Recall that action of the complex structure  $J_{\mathbb{S}^2}$  at  $x \in \mathbb{S}^2$  is given by the standard cross product  $x \times$ . Write  $\vec{a}_0 = (a_{01}, a_{02}, a_{03})$ . Then

$$a_{0i}(x) = -\frac{\langle d\phi_0(x \times e), \mathcal{J}^i d\phi_0(e) \rangle_x}{|d\phi_{0x}(e)|^2}$$

and  $d\phi_0$  at  $x$  is the same as  $d\phi$  at  $-x$  because  $\phi_0$  is the lift from  $\mathbb{R}P^2$ . We then conclude

$$\vec{a}_0(-x) = -\vec{a}_0(x), \quad \vec{a}_0^{-1}(-x) = -\vec{a}_0^{-1}(x).$$

The chain rule implies

$$\begin{aligned} |\nabla\phi(-x)|^2 &= |\nabla\phi_0(\vec{a}_0^{-1}(-x))|^2 |\nabla\vec{a}_0^{-1}(-x)|^2 \\ &= |\nabla\phi_0(-\vec{a}_0^{-1}(x))|^2 |\nabla\vec{a}_0^{-1}(x)|^2 \\ &= |\nabla\phi_0(\vec{a}_0^{-1}(x))|^2 |\nabla\vec{a}_0^{-1}(x)|^2 = |\nabla\phi(x)|^2 \end{aligned}$$

because  $\phi_0$  is the lift from  $\mathbb{R}P^2$ . Therefore for  $i = 1, 2, 3$ ,

$$\int_{\mathbb{S}^2} x_i |\nabla \phi|^2 = 0.$$

The fact that the extended map  $f$  is stationary follows from the lemma below.  $\square$

The lemma below is known to experts. For the sake of completeness, we present a proof of it.

**Lemma 3.8.** *Let  $\phi$  be a smooth harmonic map from  $\mathbb{S}^2$  to a Riemannian manifold  $N$ . Then the extended map  $f$  of  $\phi$ , which is defined by  $f(x, x') = \phi(x/|x|)$  for  $x = (x_1, x_2, x_3) \neq (0, 0, 0)$ ,  $x' \in \{0\} \times \mathbb{R}^{m-3} \subset \mathbb{R}^m$ , is a stationary harmonic map if and only if  $\phi$  satisfies*

$$\int_{\mathbb{S}^2} x_i |\nabla \phi|^2 = 0, \quad i = 1, 2, 3, \quad (x_1, x_2, x_3) \in \mathbb{S}^2.$$

*Proof.* In fact, we have

$$\nabla_{x'} f = 0, \quad \frac{\partial f}{\partial r} = 0, \quad r = \sqrt{x_1^2 + x_2^2 + x_3^2}.$$

Define a cut-off function by

$$\eta_\epsilon(r, \alpha, \beta, x') = \begin{cases} 1 & r \geq \epsilon \\ \frac{2}{\epsilon} \left( r - \frac{\epsilon}{2} \right) & \epsilon/2 < r < \epsilon \\ 0 & r \leq \epsilon/2 \end{cases}$$

where  $x_1 = r \sin \alpha \cos \beta$ ,  $x_2 = r \sin \alpha \sin \beta$ ,  $x_3 = r \cos \beta$ .

For any smooth vector field  $X = (X_1, \dots, X_m)$  in  $\mathbb{R}^m$  with compact support, because  $f$  is smooth away from  $\{0\} \times \mathbb{R}^{m-3}$ , we have

$$\begin{aligned} 0 &= \int_{\mathbb{R}^m} (|\nabla f|^2 \delta_{ij} - 2\nabla_i f \nabla_j f) \nabla_j (\eta_\epsilon X_i) \\ &= \int_{\mathbb{R}^m} (|\nabla f|^2 \delta_{ij} - 2\nabla_i f \nabla_j f) \nabla_j \eta_\epsilon X_i \\ &\quad + \int_{\mathbb{R}^m} (|\nabla f|^2 \delta_{ij} - 2\nabla_i f \nabla_j f) \eta_\epsilon \nabla_j X_i. \end{aligned}$$

It then follows

$$\begin{aligned} \int_{\mathbb{R}^m} (|\nabla f|^2 \delta_{ij} - 2\nabla_i f \nabla_j f) \nabla_j X_i &= \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^m} (|\nabla f|^2 \delta_{ij} - 2\nabla_i f \nabla_j f) \eta_\epsilon \nabla_j X_i \\ &= - \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^m} (|\nabla f|^2 \delta_{ij} - 2\nabla_i f \nabla_j f) \nabla_j \eta_\epsilon X_i \end{aligned}$$

Therefore,  $f$  is stationary if and only if

$$\lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^m} (|\nabla f|^2 \delta_{ij} - 2\nabla_i f \nabla_j f) \nabla_j \eta_\epsilon X_i = 0.$$

Direct computation shows that the above condition is equivalent to

$$\int_{\mathbb{R}^{m-3}} \int_{\mathbb{S}^2} |\nabla \phi|^2 \sum_{i=1}^3 x_i X_i(0, x') d\sigma dx' = 0.$$

Since  $X$  is arbitrary, we see the desired statement holds.  $\square$

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