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Varieties with $P_3(X) = 4$ and $q(X) = \dim(X)$

JUNGKAI ALFRED CHEN – CHRISTOPHER D. HACON

Abstract. We classify varieties with $P_3(X) = 4$ and $q(X) = \dim(X)$.

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1. – Introduction

Let X be a smooth complex projective variety. When $\dim(X) \geq 3$ it is very hard to classify such varieties in terms of their birational invariants. Surprisingly, when X has many holomorphic 1-forms, it is sometimes possible to achieve classification results in any dimension. In [Ka], Kawamata showed that: *If X is a smooth complex projective variety with $\kappa(X) = 0$ then the Albanese morphism* $\mathrm{a} : X \longrightarrow \mathrm{A}(X)$ *is surjective. If moreover, $q(X) = \dim(X)$, then X is birational to an abelian variety.* Subsequently, Kollár proved an effective version of this result (cf. [Ko2]): *If X is a smooth complex projective variety with $P_m(X) = 1$ for some $m \geq 4$, then the Albanese morphism* $\mathrm{a} : X \longrightarrow \mathrm{A}(X)$ *is surjective. If moreover, $q(X) = \dim(X)$, then X is birational to an abelian variety.* These results where further refined and expanded as follows:

Theorem 1.1 (cf. [CH1], [CH3], [HP], [Hac2]). *If $P_m(X) = 1$ for some $m \geq 2$ or if $P_3(X) \leq 3$, then the Albanese morphism* $\mathrm{a} : X \longrightarrow \mathrm{A}(X)$ *is surjective. If moreover $q(X) = \dim(X)$, then:*

(1) *If $P_m(X) = 1$ for some $m \geq 2$, then X is birational to an abelian variety.*
(2) *If $P_3(X) = 2$, then $\kappa(X) = 1$ and X is birational to a double cover of its Albanese variety.*
(3) *If $P_3(X) = 3$, then $\kappa(X) = 1$ and X is birational to a bi-double cover of its Albanese variety.*

In this paper we will prove a similar result for varieties with $P_3(X) = 4$ and $q(X) = \dim(X)$. We start by considering the following examples:

Example 1. Let G be a group acting faithfully on a curve C and acting faithfully by translations on an abelian variety $\tilde{K}$, so that $C/G = E$ is an

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elliptic curve and $\dim H^0(C, \omega_C^{\otimes 3})^G = 4$. Let G act diagonally on $\tilde{K} \times C$, then $X := \tilde{K} \times C/G$ is a smooth projective variety with $\kappa(X) = 1$, $P_3(X) = 4$ and $q(X) = \dim(X)$. We illustrate some examples below:

(1) $G = \mathbb{Z}_m$ with $m \geq 3$. Consider an elliptic curve E with a line bundle L of degree 1. Taking the normalization of the m-th root of a divisor $B = (m-a)B_1 + aB_2 \in |mL|$ with $1 \leq a \leq m-1$ and $m \geq 3$, one obtains a smooth curve C and a morphism $g : C \longrightarrow E$ of degree m. One has that
$$g_*\omega_C = \sum_{i=0}^{m-1} L^{(i)}$$
where $L^{(i)} = L^{\otimes i}(-\lfloor \frac{iB}{m} \rfloor)$ for $i = 0, ..., m-1$.

(2) $G = \mathbb{Z}_2$. Let L be a line bundle of degree 2 over an elliptic curve E. Let $C \longrightarrow E$ be the degree 2 cover defined by a reduced divisor $B \in |2L|$.

(3) $G = (\mathbb{Z}_2)^2$. Let L_i for $i = 1, 2$ be line bundles of degree 1 on an elliptic curve E and $C_i \longrightarrow E$ be degree 2 covers defined by disjoint reduced divisors $B_i \in |2L_i|$. Then $C := C_1 \times_E C_2 \longrightarrow E$ is a G cover.

(4) $G = (\mathbb{Z}_2)^3$. For $i = 1, 2, 3, 4$, let P_i be distinct points on an elliptic curve E. For $j = 1, 2, 3$ let L_j be line bundles of degree 1 on E such that $B_1 = P_1 + P_2 \in |2L_1|$, $B_2 = P_1 + P_3 \in |2L_2|$ and $B_3 = P_1 + P_4 \in |2L_3|$. Let $C_j \longrightarrow E$ be degree 2 covers defined by reduced divisors $B_j \in |2L_j|$. Let C be the normalization of $C_1 \times_E C_2 \times_E C_3 \longrightarrow E$, then C is a G cover.

Note that (1) is ramified at 2 points. Following [Be] Section VI.12, one has that $P_2(X) = \dim H^0(C, \omega_C^{\otimes 2})^G = 2$ and $P_3(X) = \dim H^0(C, \omega_C^{\otimes 3})^G = 4$. Similarly (2), (3), (4) are ramified along 4 points and hence $P_2(X) = P_3(X) = 4$.

EXAMPLE 2. Let $q : A \longrightarrow S$ be a surjective morphism with connected fibers from an abelian variety of dimension $n \geq 3$ to an abelian surface. Let L be an ample line bundle on S with $h^0(S, L) = 1$, $P \in \mathrm{Pic}^0(A)$ with $P \notin \mathrm{Pic}^0(S)$ and $P^{\otimes 2} \in \mathrm{Pic}^0(S)$. For D an appropriate reduced divisor in $|L^{\otimes 2} \otimes P^{\otimes 2}|$, there is a degree 2 cover $\mathrm{a} : X \longrightarrow \mathrm{A}$ such that $\mathrm{a}_*(\mathcal{O}_X) = \mathcal{O}_{\mathrm{A}} \oplus (L \otimes P)^\vee$. One sees that $P_i(X) = 1, 4, 4$ for $i = 1, 2, 3$.

EXAMPLE 3. Let $q : \mathrm{A} \longrightarrow E_1 \times E_2$ be a surjective morphism from an abelian variety to the product of two elliptic curves, $p_i : \mathrm{A} \longrightarrow E_i$ the corresponding morphisms, L_i be line bundles of degree 1 on E_i and $P, Q \in \mathrm{Pic}^0(\mathrm{A})$ such that P, Q generate a subgroup of $\mathrm{Pic}^0(\mathrm{A})/\mathrm{Pic}^0(E_1 \times E_2)$ which is isomorphic to $(\mathbb{Z}_2)^2$. Then one has double covers $X_i \longrightarrow \mathrm{A}$ corresponding to divisors $D_1 \in |2(q_1^* L_1 \otimes P)|$, $D_2 \in |2(q_2^* L_2 \otimes Q)|$. The corresponding bi-double cover satisfies
$$\mathrm{a}_*(\omega_X) = \mathcal{O}_{\mathrm{A}} \oplus p_1^* L_1 \otimes P \oplus p_2^* L_2 \otimes Q \oplus p_1^* L_1 \otimes P \otimes p_2^* L_2 \otimes Q$$
One sees that $P_i(X) = 1, 4, 4$ for $i = 1, 2, 3$.

We will prove the following:

THEOREM 1.2. *Let X be a smooth complex projective variety with $P_3(X) = 4$, then the Albanese morphism* $\mathrm{a} : X \longrightarrow \mathrm{A}$ *is surjective* (*in particular* $q(X) \leq \dim(X)$). *If moreover, $q(X) = \dim(X)$, then $\kappa(X) \leq 2$ and we have the following cases:*

(1) *If $\kappa(X) = 2$, then X is birational either to a double cover or to a bi-double cover of* A *as in Examples* 2 *and* 3 *and so $P_2(X) = 4$.*
(2) *If $\kappa(X) = 1$, then X is birational to the quotient $\tilde{K} \times C/G$ where C is a curve, $\tilde{K}$ is an abelian variety, G acts faithfully on C and $\tilde{K}$. One has that either $P_2(X) = 2$ and $C \longrightarrow C/G$ is branched along* 2 *points with inertia group $H \cong \mathbb{Z}_m$ with $m \geq 3$ or $P_2(X) = 4$ and $C \longrightarrow C/G$ is branched along* 4 *points with inertia group $H \cong (\mathbb{Z}_2)^s$ with $s \in \{1, 2, 3\}$. See Example* 1.

NOTATION AND CONVENTIONS. We work over the field of complex numbers. We identify Cartier divisors and line bundles on a smooth variety, and we use the additive and multiplicative notation interchangeably. If X is a smooth projective variety, we let K_X be a canonical divisor, so that $\omega_X = \mathcal{O}_X(K_X)$, and we denote by $\kappa(X)$ the Kodaira dimension, by $q(X) := h^1(\mathcal{O}_X)$ the *irregularity* and by $P_m(X) := h^0(\omega_X^{\otimes m})$ the *m−th plurigenus*. We denote by $\mathrm{a}: X \to \mathrm{A}(X)$ the Albanese map and by $\mathrm{Pic}^0(X)$ the dual abelian variety to $\mathrm{A}(X)$ which parameterizes all topologically trivial line bundles on X. For a $\mathbb{Q}$−divisor D we let $\lfloor D \rfloor$ be the integral part and $\{D\}$ the fractional part. Numerical equivalence is denoted by $\equiv$ and we write $D \prec E$ if $E - D$ is an effective divisor. If $f: X \to Y$ is a morphism, we write $K_{X/Y} := K_X - f^*K_Y$ and we often denote by $F_{X/Y}$ the general fiber of f. A $\mathbb{Q}$-Cartier divisor L on a projective variety X is nef if for all curves $C \subset X$, one has $L.C \geq 0$. For a surjective morphism of projective varieties $f : X \longrightarrow Y$, we will say that a Cartier divisor L on X is Y-big if for an ample line bundle H on Y, there exists a positive integer $m > 0$ such that $h^0(L^{\otimes m} \otimes f^*H^\vee) > 0$. The rest of the notation is standard in algebraic geometry.

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2. – Preliminaries

2.1. – The Albanese map and the Iitaka fibration

Let X be a smooth projective variety. If $\kappa(X) > 0$, then the Iitaka fibration of X is a morphism of projective varieties $f: X' \to Y$, with X' birational to X and Y of dimension $\kappa(X)$, such that the general fiber of f is smooth, irreducible, of Kodaira dimension zero. The Iitaka fibration is determined only up to birational equivalence. Since we are interested in questions of a birational nature, we usually assume that $X = X'$ and that Y is smooth.

X has *maximal Albanese dimension* if $\dim(\mathrm{a}_X(X)) = \dim(X)$. We will need the following facts (cf. [HP], Propositions 2.1, 2.3, 2.12 and Lemma 2.14 respectively).

PROPOSITION 2.1. *Let X be a smooth projective variety of maximal Albanese dimension, and let $f: X \to Y$ be the Iitaka fibration (assume Y smooth). Denote by $f_*: \mathrm{A}(X) \to \mathrm{A}(Y)$ the homomorphism induced by f and consider the commutative diagram:*

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{a}_X} & \mathrm{A}(X) \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f_*} \\ Y & \xrightarrow{\mathrm{a}_Y} & \mathrm{A}(Y). \end{array}$$

Then:

a) *Y has maximal Albanese dimension;*
b) *f_* is surjective and* $\ker f_*$ *is connected of dimension* $\dim(X) - \kappa(X)$*;*
c) *There exists an abelian variety P isogenous to* $\ker f_*$ *such that the general fiber of f is birational to P.*

Let $K := \ker f_*$ and $F = F_{X/Y}$. Define

$$G := \ker\left(\mathrm{Pic}^0(X) \to \mathrm{Pic}^0(F)\right).$$

Then

LEMMA 2.2. *G is the union of finitely many translates of* $\mathrm{Pic}^0(Y)$ *corresponding to the finite group*

$$\overline{G} := G/\mathrm{Pic}^0(Y) \cong \ker\left(\mathrm{Pic}^0(K) \to \mathrm{Pic}^0(F)\right).$$

2.2. – Sheaves on abelian varieties

Recall the following easy corollary of the theory of Fourier-Mukai transforms cf. [M]:

PROPOSITION 2.3. *Let $\psi: \mathcal{F} \hookrightarrow \mathcal{G}$ be an inclusion of coherent sheaves on an abelian variety A inducing isomorphisms $H^i(A, \mathcal{F}\otimes P) \to H^i(A, \mathcal{G}\otimes P)$ for all $i \geq 0$ and all $P \in \mathrm{Pic}^0(A)$. Then ψ is an isomorphism of sheaves.*

Following [M], we will say that a coherent sheaf $\mathcal{F}$ on an abelian variety A is I.T. 0 if $h^i(A, \mathcal{F}\otimes P) = 0$ for all $i > 0$ and for all $P \in \mathrm{Pic}^0(A)$. We will say that an inclusion of coherent sheaves on A, $\psi: \mathcal{F} \hookrightarrow \mathcal{G}$ is an I.T. 0 isomorphism if $\mathcal{F}, \mathcal{G}$ are I.T. 0 and $h^0(\mathcal{G}) = h^0(\mathcal{F})$. From the above proposition, it follows that every I.T. 0 isomorphism $\mathcal{F} \hookrightarrow \mathcal{G}$ is an isomorphism. We will need the following result:

LEMMA 2.4. *Let $f : X \longrightarrow E$ be a morphism from a smooth projective variety to an elliptic curve, such that K_X is E-big. Then, for all $P \in \mathrm{Pic}^0(X)_{tors}$, $\eta \in \mathrm{Pic}^0(E)$ and all $m \geq 2$, $f_*(\omega_X^{\otimes m} \otimes P \otimes f^*\eta)$ is I.T. 0. In particular*

$$\deg(f_*(\omega_X^{\otimes m} \otimes P \otimes f^*\eta)) = h^0(\omega_X^{\otimes m} \otimes P \otimes f^*\eta).$$

The proof of the above lemma is analogous to the proof of Lemma 2.6 of [Hac2]. We just remark that it suffices to show that $f_*(\omega_X^{\otimes m} \otimes P)$ is I.T. 0. The sheaf $f_*(\omega_X^{\otimes m} \otimes P)$ is torsion free and hence locally free on E. By Riemann-Roch

$$h^0(\omega_X^{\otimes m} \otimes P) = h^0(f_*(\omega_X^{\otimes m} \otimes P)) = \chi(f_*(\omega_X^{\otimes m} \otimes P)) = \deg(f_*(\omega_X^{\otimes m} \otimes P)).$$

2.3. – Cohomological support loci

Let $\pi : X \longrightarrow A$ be a morphism from a smooth projective variety to an abelian variety, $T \subset \mathrm{Pic}^0(A)$ the translate of a subtorous and $\mathcal{F}$ a coherent sheaf on X. One can define the cohomological support loci of $\mathcal{F}$ as follows:

$$V^i(X, T, \mathcal{F}) := \{P \in T | h^i(X, \mathcal{F} \otimes \pi^* P) > 0\}.$$

If $T = \mathrm{Pic}^0(X)$ we write $V^i(\mathcal{F})$ or $V^i(X, \mathcal{F})$ instead of $V^i(X, \mathrm{Pic}^0(X), \mathcal{F})$. When $\mathcal{F} = \omega_X$, the geometry of the loci $V^i(\omega_X)$ is governed by the following result of Green and Lazarsfeld (cf. [GL], [EL]):

THEOREM 2.5 (Generic Vanishing Theorem). *Let X be a smooth projective variety. Then:*

a) *$V^i(\omega_X)$ has codimension $\geq i - (\dim(X) - \dim(\mathrm{a}_X(X)))$;*
b) *Every irreducible component of $V^i(X, \omega_X)$ is a translate of a sub-torus of $\mathrm{Pic}^0(X)$ by a torsion point (the same also holds for the irreducible components of $V^i_m(\omega_X) := \{P \in \mathrm{Pic}^0(X) | h^i(X, \omega_X \otimes P) \geq m\}$);*
c) *Let T be an irreducible component of $V^i(\omega_X)$, let $P \in T$ be a point such that $V^i(\omega_X)$ is smooth at P, and let $v \in H^1(X, \mathcal{O}_X) \cong T_P\mathrm{Pic}^0(X)$. If v is not tangent to T, then the sequence*

$$H^{i-1}(X, \omega_X \otimes P) \xrightarrow{\cup v} H^i(X, \omega_X \otimes P) \xrightarrow{\cup v} H^{i+1}(X, \omega_X \otimes P)$$

is exact. Moreover, if P is a general point of T and v is tangent to T then both maps vanish;
d) *If X has maximal Albanese dimension, then there are inclusions:*

$$V^0(\omega_X) \supseteq V^1(\omega_X) \supseteq \cdots \supseteq V^n(\omega_X) = \{\mathcal{O}_X\};$$

e) *Let $f : Y \longrightarrow X$ be a surjective map of projective varieties, Y smooth, then statements analogous to* a), b), c) *for $P \in \mathrm{Pic}^0_{tors}(Y)$ and d) above also hold for the sheaves $R^i f_*\omega_X$. More precisely we refer to* [CH3], [ClH] *and* [Hac5].

When X is of maximal Albanese dimension, its geometry is very closely connected to the properties of the loci $V^i(\omega_X)$. We recall the following two results from [CH2]:

THEOREM 2.6. *Let X be a variety of maximal Albanese dimension. The translates through the origin of the irreducible components of $V^0(\omega_X)$ generate a subvariety of $\mathrm{Pic}^0(X)$ of dimension $\kappa(X) - \dim(X) + q(X)$. In particular, if X is of general type then $V^0(X, \omega_X)$ generates $\mathrm{Pic}^0(X)$.*

PROPOSITION 2.7. *Let X be a variety of maximal Albanese dimension and G, Y defined as in Proposition 2.1. Then*

a) $V^0(X, \mathrm{Pic}^0(X), \omega_X) \subset G$;
b) *For every $P \in G$, the loci $V^0(X, \mathrm{Pic}^0(X), \omega_X) \cap \big(P + \mathrm{Pic}^0(Y)\big)$ are non-empty;*
c) *If P is an isolated point of $V^0(X, \mathrm{Pic}^0(X), \omega_X)$, then $P = \mathcal{O}_X$.*

The following result governs the geometry of $V^0(\omega_X^{\otimes m})$ for all $m \geq 2$:

PROPOSITION 2.8. *Let X be a smooth projective variety of maximal Albanese dimension, $f: X \to Y$ the Iitaka fibration (assume Y smooth) and G defined as in Proposition 2.1. If $m \geq 2$, then $V^0(\omega_X^{\otimes m}) = G$. Moreover, for any fixed $Q \in V^0(\omega_X^{\otimes m})$, and all $P \in \mathrm{Pic}^0(Y)$ one has $h^0(\omega_X^{\otimes m} \otimes Q \otimes P) = h^0(\omega_X^{\otimes m} \otimes Q)$.*

We will also need the following lemma proved in [CH2] Section 3.

LEMMA 2.9. *Let X be a smooth projective variety and D an effective a_X-exceptional divisor on X. If $\mathcal{O}_X(D) \otimes P$ is effective for some $P \in \mathrm{Pic}^0(X)$, then $P = \mathcal{O}_X$.*

The following result is due to Ein and Lazarsfeld (see [HP], Lemma 2.13):

LEMMA 2.10. *Let X be a variety such that $\chi(\omega_X) = 0$ and such that $\mathrm{a}_X : X \longrightarrow \mathrm{A}(X)$ is surjective and generically finite. Let T be an irreducible component of $V^0(\omega_X)$, and let $\pi_B : X \longrightarrow B := \mathrm{Pic}^0(T)$ be the morphism induced by the map $\mathrm{A}(X) \longrightarrow \mathrm{Pic}^0(\mathrm{Pic}^0(X)) \longrightarrow B$ corresponding to the inclusion $T \hookrightarrow \mathrm{Pic}^0(X)$.*

Then there exists a divisor $D_T \prec R := \mathrm{Ram}(\mathrm{a}_X) = K_X$, vertical with respect to π_B (i.e. $\pi_B(D_T) \neq B$), such that for general $P \in T$, $G_T := R - D_T$ is a fixed divisor of each of the linear series $|K_X + P|$.

We have the following useful Corollary:

COROLLARY 2.11. *In the notation of Lemma 2.10, if $\dim(T) = 1$, then for any $P \in T$, there exists a line bundle of degree 1 on B such that $\pi_B^* L_P \prec K_X + P$.*

PROOF. By [HP] Step 8 of the proof of Theorem 6.1, for general $Q \in T$, there exists a line bundle of degree 1 on B such that $\pi_B^* L_Q \prec K_X + Q$. Write $P = Q + \pi_B^* \eta$ where $\eta \in \mathrm{Pic}^0(B)$. Then, since

$$h^0(\omega_X \otimes P \otimes \pi_B^*(L_Q \otimes \eta)^\vee) = h^0(\pi_{B,*}(\omega_X \otimes Q) \otimes L_Q^\vee) \neq 0,$$

one sees that there is an inclusion $\pi_B^*(L_Q \otimes \eta) \longrightarrow \omega_X \otimes P$. □

Recall the following result (cf. [Hac2], Lemma 2.17):

LEMMA 2.12. *Let X be a smooth projective variety, let L and M be line bundles on X, and let $T \subset \mathrm{Pic}^0(X)$ be an irreducible subvariety of dimension t. If for all $P \in T$, $\dim|L+P| \geq a$ and $\dim|M-P| \geq b$, then $\dim|L+M| \geq a+b+t$.*

LEMMA 2.13. *Let T be a 1-dimensional component of $V^0(\omega_X)$, $E := T^\vee$ and $\pi : X \longrightarrow E$ the induced morphism. Then $P|_F \cong \mathcal{O}_F$ for all $P \in T$.*

PROOF. Let G_T, D_T be as in Lemma 2.10, then for $P \in T$ we have $|K_X + P| = G_T + |D_T + P|$ and hence the divisor $D_T + P$ is effective. It follows that $(D_T + P)|_F$ is also effective. However D_T is vertical with respect to π and hence $D_T|_F \cong \mathcal{O}_F$. By Lemma 2.9, one sees that $P|_F \cong \mathcal{O}_F$. □

3. – Kodaira dimension of Varieties with $P_3(X) = 4$, $q(X) = \dim(X)$

The purpose of this section is to study the Albanese map and Iitaka fibration of varieties with $P_3 = 4$ and $q = \dim(X)$. We will show that: 1) the Albanese map is surjective, 2) the image of the Iitaka fibration is an abelian variety (and hence the Iitaka fibration factors through the Albanese map), 3) we have that $\kappa(X) \leq 2$.

We begin by fixing some notation. We write

$$V_0(X, \omega_X) = \cup_{i \in I} S_i$$

where S_i are irreducible components. Let T_i denote the translate of S_i passing through the origin and $\delta_i := \dim(S_i)$. For any $i, j \in I$, let $\delta_{i,j} := \dim(T_i \cap T_j)$.

Recall that $V_0(X, \omega_X) \subset G \to \bar{G} := G/\mathrm{Pic}^0(Y)$. For any $\eta \in \bar{G}$, we fix once and for all S_η a maximal dimensional component which maps to η. In particular, T_0 denotes the translate through the origin of a maximal dimensional component $S_0 \subset V^0(X, \omega_X) \cap \mathrm{Pic}^0(Y)$. If X is of maximal Albanese dimension with $q(X) = \dim(X)$, then its Iitaka fibration image Y is of maximal Albanese dimension with $q(Y) = \dim(Y) = \kappa(X)$. Moreover, by Proposition 2.7, one has $\delta_i \geq 1, \forall i \neq 0$.

We denote by $P_{m,\alpha} := h^0(X, \omega_X^{\otimes m} \otimes \alpha)$ for $\alpha \in \mathrm{Pic}^0(X)$. Now let Q_i (Q_η resp.) be a general element in S_i (S_η resp.), we denote by $P_{m,i} := h^0(X, \omega_X^{\otimes m} \otimes Q_i)$ ($P_{m,\eta}$ resp.). We remark that it is convenient to choose Q_i (Q_η resp.) to be torsion so that the results of Kollár on higher direct images of dualizing sheaves will also apply to the sheaf $\omega_X \otimes Q_i$. Proposition 2.8 can be rephrased as

$$(1) \qquad P_{m,\alpha} = P_{m,\alpha+\beta} \quad \forall \alpha \in \mathrm{Pic}^0(X),\ \beta \in \mathrm{Pic}^0(Y),\ m \geq 2.$$

Notice that if $\alpha \notin G$ then also $\alpha + \beta \notin G$ and so both numbers are equal to 0.

By Lemma 2.12 one has, for any $\eta, \zeta \in \bar{G}$,

$$\left\{ \begin{array}{l} P_{2,\eta+\zeta} \geq P_{1,\eta} + P_{1,\zeta} + \delta_{\eta,\zeta} - 1, \\ P_{2,2\eta} \geq 2P_{1,\eta} + \delta_\eta - 1, \\ P_{3,\eta+\zeta} \geq P_{1,\eta} + P_{2,\zeta} + \delta_\eta - 1. \end{array} \right. \tag{2}$$

Here $\delta_\eta = \delta_i$ and $\delta_{\eta,\zeta} = \delta_{i,j}$ if T_η, T_ζ are represented by T_i, T_j respectively.
The following lemma is very useful when $\kappa \geq 2$.

LEMMA 3.1. *Let X be a variety of maximal Albanese dimension with $\kappa(X) \geq 2$. Suppose that there is a surjective morphism $\pi : X \to E$ to an elliptic curve E, and suppose that there is an inclusion $\varphi : \pi^*L \to \omega_X^{\otimes m} \otimes P$ for some $m \geq 2$, $P|_F = \mathcal{O}_F$ where F is a general fiber of π and L is an ample line bundle on E. Then the induced map $L \to \pi_*(\omega_X^{\otimes m} \otimes P)$ is not an isomorphism,* $\operatorname{rank}(\pi_*(\omega_X^{\otimes m} \otimes P)) \geq 2$ *and $h^0(X, \omega_X^{\otimes m} \otimes P) > h^0(E, L)$.*

PROOF. By the easy addition theorem, $\kappa(F) \geq 1$. Hence by Theorem 1.1, $P_m(F) \geq 2$ for $m \geq 2$. The sheaf $\pi_*(\omega_X^{\otimes m} \otimes P)$ has rank equal to $h^0(F, \omega_X^{\otimes m} \otimes P|_F) = h^0(F, \omega_F^{\otimes m}) \geq 2$. Therefore, $L \to \pi_*(\omega_X^{\otimes m} \otimes P)$ is not an isomorphism. Since they are non-isomorphic I.T.0 sheaves, it follows that $h^0(\pi_*(\omega_X^{\otimes m} \otimes P)) > h^0(L)$. □

COROLLARY 3.2. *Keep the notation as in Lemma* 3.1. *If there is a morphism $\pi' : X \to E'$ and an inclusion $\pi'^*L' \hookrightarrow \omega_X \otimes P^\vee$ for some ample line bundle L' on E' and $P \in \operatorname{Pic}^0(X)$ with $P|_{F'} = \mathcal{O}_{F'}$, then for all $m \geq 2$*

$$P_{m+1}(X) \geq 2 + h^0(X, \omega_X^{\otimes m} \otimes P) > 2 + h^0(E', L').$$

PROOF. The inclusion $\pi'^*L' \hookrightarrow \omega_X \otimes P^\vee$ induces an inclusion

$$\pi'^*L' \otimes \omega_X^{\otimes m} \otimes P \hookrightarrow \omega_X^{\otimes m+1}.$$

By Riemann-Roch, one has

$$P_{m+1}(X) \geq h^0(E', L' \otimes \pi'_*(\omega_X^{\otimes m} \otimes P)) \geq h^0(E', \pi'_*(\omega_X^{\otimes m} \otimes P)) + \operatorname{rank}(\pi'_*(\omega_X^{\otimes m} \otimes P)).$$

By Proposition 2.7, there exists $\alpha \in \operatorname{Pic}^0(Y)$ such that $h^0(\omega_X^{\otimes m-1} \otimes P^{\otimes 2} \otimes \alpha) \neq 0$ and hence there is an inclusion

$$\pi'^*L' \hookrightarrow \omega_X^{\otimes m} \otimes P \otimes \alpha.$$

By Proposition 2.8 and Lemma 3.1,

$$h^0(X, \omega_X^{\otimes m} \otimes P) = h^0(X, \omega_X^{\otimes m} \otimes P \otimes \alpha) > h^0(E', L'). \qquad \square$$

REMARK 3.3. Let X be a variety with $\kappa(X) \geq 2$. Suppose that there is a 1-dimensional component $S_i \subset V^0(\omega_X)$. We often consider the induced map $\pi : X \to E := T_i^\vee$. It is easy to see that π factors through the Iitaka fibration. By Corollary 2.11 and Lemma 2.13, there is an inclusion $\varphi : \pi^*L \to \omega_X \otimes P$ for some $P \in \operatorname{Pic}^0(X)$ with $P|_F = \mathcal{O}_F$ and some ample line bundle L on E. In what follows, we will often apply Lemma 3.1 and Corollary 3.2 to this situation.

LEMMA 3.4. *Let X be a variety of maximal Albanese dimension with $\kappa(X) \geq 2$ and $P_3(X) = 4$. Then for any $\zeta \neq 0 \in \bar{G}$, one has $P_{2,\zeta} \leq 2$.*

PROOF. If $P_{2,\zeta} \geq 3$, then by (2) and Proposition 2.7, one sees that $\delta_{-\zeta} = 1$. Let $\pi : X \longrightarrow E := T^{\vee}_{-\zeta}$ be the induced morphism. Then there is an ample line bundle L on the elliptic curve E and an inclusion $L \longrightarrow \pi_*(\omega_X \otimes Q_{-\zeta})$. By Corollary 3.2, $P_3(X) \geq 2 + P_{2,\zeta} \geq 5$ which is impossible. □

THEOREM 3.5. *Let X be a smooth projective variety with $P_3(X) = 4$, then the Albanese morphism* $\mathrm{a} : X \longrightarrow \mathrm{A}$ *is surjective.*

PROOF. We follow the proof of Theorem 5.1 of [HP]. Assume that $\mathrm{a} : X \longrightarrow \mathrm{A}$ is not surjective, then we may assume that there is a morphism $f : X \longrightarrow Z$ where Z is a smooth variety of general type, of dimension at least 1, such that its Albanese map $\mathrm{a}_Z : Z \longrightarrow S$ is birational onto its image. By the proof of Theorem 5.1 of [HP], it suffices to consider the cases in which $P_1(Z) \leq 3$ and hence $\dim(Z) \leq 2$. If $\dim(Z) = 2$, then $q(Z) = \dim(S) \geq 3$ and since $\chi(\omega_Z) > 0$, one sees that $V^0(\omega_Z) = \mathrm{Pic}^0(S)$. By the proof of Theorem 5.1 of [HP], one has that for generic $P \in \mathrm{Pic}^0(S)$,

$$P_3(X) \geq h^0(\omega_Z \otimes P) + h^0(\omega_X^{\otimes 3} \otimes f^* \omega_Z^{\vee} \otimes P) + \dim(S) - 1 \geq 1 + 2 + 3 - 1 \geq 5.$$

This is a contradiction, so we may assume that $\dim(Z) = 1$. It follows that $g(Z) = q(Z) = P_1(Z) \geq 2$ and one may write $\omega_Z = L^{\otimes 2}$ for some ample line bundle L on Z. Therefore, for general $P \in \mathrm{Pic}^0(Z)$, one has that $h^0(\omega_Z \otimes L \otimes P) \geq 2$ and proceeding as in the proof of Theorem 5.1 of [HP], that $h^0(\omega_X^{\otimes 3} \otimes f^*(\omega_Z \otimes L)^{\vee} \otimes P) \geq 2$. It follows as above that

$$P_3(X) \geq h^0(\omega_Z \otimes L \otimes P) + h^0(\omega_X^{\otimes 3} \otimes f^*(\omega_Z \otimes L)^{\vee} \otimes P) + \dim(S) - 1 \geq 2+2+2-1 \geq 5.$$

This is a contradiction and so $\mathrm{a} : X \longrightarrow \mathrm{A}$ is surjective. □

PROPOSITION 3.6. *Let X be a smooth projective variety with $P_3(X) = 4$, $q(X) = \dim(X)$, then*

1. *X is not of general type and*
2. *if $\kappa(X) \geq 2$, then*

$$V^0(\omega_X) \cap f^* \mathrm{Pic}^0(Y) = \{\mathcal{O}_X\}.$$

PROOF. If $\kappa(X) = 1$, then clearly X is not of general type as otherwise X is a curve with $P_3(X) = 5g - 5 > 4$. We thus assume that $\kappa(X) \geq 2$. It suffices to prove (2) as then (1) will follow from Theorem 2.6.

If all points of $V^0(\omega_X) \cap f^* \mathrm{Pic}^0(Y)$ are isolated, then the above statement follows from Proposition 2.7. Therefore, it suffices to prove that $\delta_0 = 0$. (Recall that δ_0 is the maximal dimension of a component in $\mathrm{Pic}^0(Y)$.)

Suppose that $\delta_0 \geq 2$. Then by (2) and Proposition 2.8, one has

$$P_2 \geq 1 + 1 + \delta_0 - 1 \geq 3, \qquad P_3 \geq 3 + 1 + \delta_0 - 1 \geq 5$$

which is impossible.

Suppose now that $\delta_0 = 1$, i.e. there is a 1-dimensional component $S_0 \subset V^0(\omega_X) \cap f^* \mathrm{Pic}^0(Y)$. Let $\pi : X \longrightarrow E := T_0^{\vee}$ be the induced morphism. By

Corollary 2.11, for some general $P \in S_0$, there exists a line bundle of degree 1 on E and an inclusion $\pi^*L \longrightarrow \omega_X \otimes P$. By Lemma 2.13, $P|_{F_{X/E}} \cong \mathcal{O}_{F_{X/E}}$.

We consider the inclusion $\varphi : L^{\otimes 2} \longrightarrow \pi_*(\omega_X^{\otimes 2} \otimes P^{\otimes 2})$. By Lemma 3.1, one sees that $h^0(\omega_X^{\otimes 2} \otimes P^{\otimes 2}) \geq 3$, and $\operatorname{rank}(\pi_*(\omega_X^{\otimes 2} \otimes P^{\otimes 2})) \geq 2$. So

$$\begin{aligned} P_3(X) &= h^0(\omega_X^{\otimes 3} \otimes P^{\otimes 3}) \geq h^0(\omega_X^{\otimes 2} \otimes P^{\otimes 2} \otimes \pi^* L) \\ &= h^0(\pi_*(\omega_X^{\otimes 2} \otimes P^{\otimes 2}) \otimes L) \geq \deg(\pi_*(\omega_X^{\otimes 2} \otimes P^{\otimes 2})) + \operatorname{rank}(\pi_*(\omega_X^{\otimes 2} \otimes P^{\otimes 2})) \\ &\geq 3 + 2 \end{aligned}$$

and this is the required contradiction. □

PROPOSITION 3.7. *Let X be a smooth projective variety with $P_3(X) = 4$, $q(X) = \dim(X)$, and $f : X \longrightarrow Y$ be a birational model of its Iitaka fibration. Then Y is birational to an abelian variety.*

PROOF. Since X, Y are of maximal Albanese dimension, $K_{X/Y}$ is effective. If $h^0(\omega_Y \otimes P) > 0$, it follows that $h^0(\omega_X \otimes f^*P) > 0$ and so by Proposition 3.6, $f^*P = \mathcal{O}_X$. By Proposition 2.1, the map $f^* : \operatorname{Pic}^0(Y) \longrightarrow \operatorname{Pic}^0(X)$ is injective and hence $P = \mathcal{O}_Y$. Therefore $V^0(\omega_Y) = \{\mathcal{O}_Y\}$ and by Theorem 2.6, one has $\kappa(Y) = 0$ and hence Y is birational to an abelian variety. □

We are now ready to describe the cohomological support loci of varieties with $\kappa(X) \geq 2$ explicitly. Recall that by Proposition 2.7, for all $\eta \neq 0 \in \bar{G}$, $\delta_\eta \geq 1$.

THEOREM 3.8. *Let X be a smooth projective variety with $P_3(X) = 4, q(X) = \dim(X)$ and $\kappa(X) \geq 2$. Then $\kappa(X) = 2$ and $\bar{G} \cong (\mathbb{Z}_2)^s$ for some $s \geq 1$.*

PROOF. The proof consists of following claims:

CLAIM 3.9. If $\kappa(X) \geq 2$ and $T \subset V^0(\omega_X)$ is a positive dimensional component, then $T + T \subset \operatorname{Pic}^0(Y)$, i.e. $\bar{G} \cong (\mathbb{Z}_2)^s$.

PROOF OF CLAIM 3.9. It suffices to prove that $2\eta = 0$ for $0 \neq \eta \in \bar{G}$. Suppose that $2\eta \neq 0$, we will find a contradiction.

We first consider the case that $\delta_\eta \geq 2$ and $\delta_{-2\eta} \geq 2$. Then by (2), $P_{2,2\eta} \geq 1 + 1 + \delta_\eta - 1 \geq 3$, and $P_3 \geq 3 + 1 + \delta_{-2\eta} - 1 \geq 5$ which is impossible.

We then consider the case that $\delta_\eta \geq 2$ and $\delta_{-2\eta} = 1$. Again we have $P_{2,2\eta} \geq 3$. We consider the induced map $\pi : X \to E := T^\vee_{-2\eta}$ and the inclusion $\varphi : \pi^*L \to \omega_X \otimes Q_{-2\eta}$ where E is an elliptic curve and L is an ample line bundle on E. It follows that there is an inclusion

$$\pi^*L \otimes (\omega_X \otimes Q_\eta)^{\otimes 2} \to \omega_X^{\otimes 3} \otimes Q_\eta^{\otimes 2} \otimes Q_{-2\eta}.$$

By Lemma 3.1, one has that $\operatorname{rank}(\pi_*(\omega_X \otimes Q_\eta)^{\otimes 2}) \geq 2$. By Proposition 2.8, Riemann-Roch and Lemma 2.4

$$\begin{aligned} P_3(X) &= h^0(\omega_X^{\otimes 3} \otimes Q_\eta^{\otimes 2} \otimes Q_{-2\eta}) \geq h^0(\pi^*L \otimes (\omega_X \otimes Q_\eta)^{\otimes 2}) \\ &= h^0((\omega_X \otimes Q_\eta)^{\otimes 2}) + \operatorname{rank}(\pi_*(\omega_X \otimes Q_\eta)^{\otimes 2}) \geq P_{2,2\eta} + 2 \geq 5, \end{aligned}$$

which is impossible.

Lastly, we consider the case that $\delta_\eta = 1$. There is an induced map $\pi : X \to E := T_\eta^\vee$ and an inclusion $\pi^*L \to \omega_X \otimes Q_\eta$. Hence there is an inclusion $\varphi : \pi^*L^{\otimes 2} \to (\omega_X \otimes Q_\eta)^{\otimes 2}$. By Lemma 3.1, we have $P_{2,2\eta} \geq 3$. We now proceed as in the previous cases.

Therefore, any element $\eta \in \bar{G}$ is of order 2 and hence $\bar{G} \cong (\mathbb{Z}_2)^s$. □

CLAIM 3.10. If there is a surjective map with connected fibers to an elliptic curve $\pi : X \longrightarrow E$ and an inclusion $\pi^*L \longrightarrow \omega_X \otimes P$ for an ample line bundle L on E and $P \in \mathrm{Pic}^0(X)$ (in particular if $\delta_i = 1$ for some $i \neq 0$ cf. Corollary 2.11). Then $\kappa(X) = 2$.

PROOF OF CLAIM 3.10. Since K_X is effective, there is also an inclusion $L \to \pi_*(\omega_X^{\otimes 2} \otimes P)$. By Lemma 3.1, one has $\mathrm{rank}(\pi_*(\omega_X^{\otimes 2} \otimes P)) \geq 2$, $h^0(\pi_*(\omega_X^{\otimes 2} \otimes P)) \geq 2$. Consider the inclusion

$$\pi_*(\omega_X^{\otimes 2} \otimes P) \otimes L \longrightarrow \pi_*(\omega_X^{\otimes 3} \otimes P^{\otimes 2}).$$

Since

$$\begin{aligned} P_3(X) = h^0(\pi_*(\omega_X^{\otimes 3} \otimes P^{\otimes 2})) &\geq h^0(\pi_*(\omega_X^{\otimes 2} \otimes P) \otimes L) \\ &\geq \deg(\pi_*(\omega_X^{\otimes 2} \otimes P)) + \mathrm{rank}(\pi_*(\omega_X^{\otimes 2} \otimes P)), \end{aligned}$$

it follows that

$$\deg(\pi_*(\omega_X^{\otimes 2} \otimes P)) = \mathrm{rank}(\pi_*(\omega_X^{\otimes 2} \otimes P)) = 2$$

and the above homomorphism of sheaves induces an isomorphism on global sections and hence is an isomorphism of sheaves (cf. Proposition 2.3). Therefore,

$$P_3(F) = h^0(\omega_F^{\otimes 3} \otimes P^{\otimes 2}) = 2.$$

By Theorem 1.1, it follows that $\kappa(F) = 1$ and by easy addition, one has that

$$\kappa(X) \leq \kappa(F) + \dim(E) = 2.$$ □

CLAIM 3.11. For all $i \neq 0$, $P_{1,i} = 1$.

PROOF OF CLAIM 3.11. If $P_{1,i} \geq 2$, then by (2),

$$4 \geq P_2 \geq 2P_{1,i} + \delta_i - 1.$$

It follows that $\delta_i = 1$. Let $E = T^\vee$ and $\pi : X \longrightarrow E$ be the induced morphism. We follow Lemma 2.10 and let $L := \pi_*(\mathcal{O}_X(D_T) \otimes Q_i)$. The sheaf L is torsion free and hence locally free. Since D_T is vertical, L is of rank 1, i.e. a line bundle. There is an inclusion $\pi^*L \longrightarrow \omega_X \otimes Q_i$ and one has $h^0(E, L) = h^0(\omega_X \otimes Q_i) \geq 2$. Consider the inclusion $\pi^*L^{\otimes 2} \longrightarrow \omega_X^{\otimes 2} \otimes Q_i^{\otimes 2}$. By Lemma 3.1, one sees that

$$P_3 \geq P_{2,2i} = h^0(\omega_X^{\otimes 2} \otimes Q_i^{\otimes 2}) > h^0(E, L^{\otimes 2}) \geq 4,$$

which is impossible. □

CLAIM 3.12. If $\kappa(X) = \dim(S)$ for some component S of $V^0(\omega_X)$, then $\kappa(X) = 2$.

PROOF OF CLAIM 3.12. Let Q be a general point in S, and T be the translate of S through the origin. By Proposition 3.7, one sees that the induced map $X \to T^\vee$ is isomorphic to the Iitaka fibration. We therefore identify Y with $T^\vee$. We assume that $\dim(S) \geq 3$ and derive a contradiction. First of all, by (2)

$$P_3(X) = h^0(\omega_X^{\otimes 3} \otimes Q^{\otimes 2}) \geq h^0(\omega_X^{\otimes 2} \otimes Q) + \dim(S)$$

and so $h^0(\omega_X^{\otimes 2} \otimes Q) = 1$ and $\dim(S) = 3$.

Let H be an ample line bundle on Y and for m a sufficiently big and divisible integer, fix a divisor $B \in |mK_X - f^*H|$. After replacing X by an appropriate birational model, we may assume that B has simple normal crossings support. Let $L = \omega_X \otimes \mathcal{O}_X(-\lfloor B/m \rfloor)$, then $L \equiv f^*(H/m) + \{B/m\}$ i.e. L is numerically equivalent to the sum of the pull back of an ample divisor and a k.l.t. divisor and so one has

$$h^i(Y, f_*(\omega_X \otimes L \otimes Q) \otimes \alpha) = 0 \quad \text{for all} \quad i > 0 \quad \text{and } \alpha \in \mathrm{Pic}^0(Y).$$

Comparing the base loci, one can see that $h^0(\omega_X \otimes L \otimes Q) = h^0(\omega_X^{\otimes 2} \otimes Q) = 1$ (cf. [CH1], Lemma 2.1 and Proposition 2.8) and so

$$h^0(Y, f_*(\omega_X \otimes L \otimes Q) \otimes \alpha) = h^0(f_*(\omega_X \otimes L \otimes Q)) = 1 \quad \forall \alpha \in \mathrm{Pic}^0(Y).$$

Since $f_*(\omega_X \otimes L \otimes Q)$ is a torsion free sheaf of generic rank one, by [Hac] it is a principal polarization M.

Since one may arrange that $\lfloor \frac{B}{m} \rfloor \prec K_X$, there is an inclusion $\omega_X \otimes Q \hookrightarrow \omega_X \otimes L \otimes Q$. Pushing forward to Y, it induces an inclusion

$$\varphi : f_*(\omega_X \otimes Q) \hookrightarrow M.$$

Therefore, $f_*(\omega_X \otimes Q)$ is of the form $M \otimes \mathcal{I}_Z$ for some ideal sheaf $\mathcal{I}_Z$. However, $h^0(Y, f_*(\omega_X \otimes Q) \otimes P) = h^0(M \otimes P \otimes \mathcal{I}_Z) > 0$ for all $P \in \mathrm{Pic}^0(Y)$ and M is a principal polarization. It follows that $\mathcal{I}_Z = \mathcal{O}_Y$ and thus $f_*(\omega_X \otimes Q) = M$. Therefore, one has an inclusion

$$f^*M^{\otimes 2} \hookrightarrow (\omega_X \otimes Q) \otimes (\omega_X \otimes L \otimes Q) \hookrightarrow \omega_X^{\otimes 3} \otimes Q^{\otimes 2}.$$

It follows that

$$4 = P_3(X) = h^0(X, \omega_X^{\otimes 3} \otimes Q^{\otimes 2}) \geq h^0(Y, M^{\otimes 2}) \geq 2^{\dim(S)}.$$

This is the required contradiction. □

CLAIM 3.13. Any two components of $V^0(\omega_X)$ of dimension at least 2 must be parallel.

PROOF OF CLAIM 3.13. For $i = 1, 2$, let $p_i : X \longrightarrow T_i^\vee$ be the induced morphism. Assume that $\delta_1, \delta_2 \geq 2$ and T_1, T_2 are not parallel. By Lemma 2.10, one may write $K_X = G_i + D_i$ where D_i is vertical with respect to $p_i : X \longrightarrow T_i^\vee$ and for general $P \in S_i$, one has $|K_X + P| = G_i + |D_i + P|$ is a 0-dimensional linear system (see Claim 3.11).

Recall that we may assume that the image of the Iitaka fibration $f : X \longrightarrow Y$ is an abelian variety. Pick H an ample divisor on Y and for m sufficiently big and divisible integer, let

$$B \in |mK_X - f^*H|.$$

After replacing X by an appropriate birational model, we may assume that B has normal crossings support. Let

$$L := \omega_X\left(-\left\lfloor \frac{B}{m}\right\rfloor\right) \equiv \left\{\frac{B}{m}\right\} + f^*\left(\frac{H}{m}\right).$$

It follows that

$$h^i(f_*(\omega_X \otimes L \otimes P) \otimes \alpha) = 0 \quad \text{for all} \quad i > 0,\ \alpha \in \mathrm{Pic}^0(Y), \quad P \in \mathrm{Pic}^0(X).$$

The quantity $h^0(\omega_X \otimes L \otimes P \otimes f^*\alpha)$ is independent of $\alpha \in \mathrm{Pic}^0(Y)$. For some fixed $P \in S_1$ as above, and $\alpha \in \mathrm{Pic}^0(T_1^\vee)$, one has a morphism

$$|D_1 + P + \alpha| \times |D_1 + P - \alpha| \longrightarrow |2D_1 + 2P|$$

and hence $h^0(\mathcal{O}_X(2D_1) \otimes P^{\otimes 2}) \geq 3$. Similarly for some fixed $Q \in S_2$, and $\alpha' \in \mathrm{Pic}^0(T_2^\vee)$, one has a morphism

$$|D_2 + Q + \alpha'| \times |K_X + L - Q + 2P - \alpha'| \longrightarrow |K_X + L + D_2 + 2P|$$

and hence $h^0(\omega_X(D_2) \otimes L \otimes P^{\otimes 2}) \geq 3$. It follows that since $h^0(\omega_X^{\otimes 3} \otimes P^{\otimes 2}) = 4$, there is a 1 dimensional intersection between the images of the 2 morphisms above which are contained in the loci

$$|2D_1 + 2P| + 2G_1 + K_X, \qquad |K_X + L + D_2 + 2P| + \left\lfloor \frac{B}{m} \right\rfloor + G_2.$$

It is easy to see that for all but finitely many $P \in \mathrm{Pic}^0(X)$, one has $h^0(\omega_X \otimes P) \leq 1$. So there is a 1 parameter family $\tau_2 \subset \mathrm{Pic}^0(T_2^\vee)$ such that for $\alpha' \in \tau_2$, one has that the divisor $D_{Q+\alpha'} = |D_2 + Q + \alpha'|$ is contained in $D_{P+\alpha} + D_{P-\alpha} + 2G_1 + K_X$ where $\alpha \in \tau_1$ a 1 parameter family in $\mathrm{Pic}^0(T_1^\vee)$. Let $D^*_{Q+\alpha'}$ be the components of $D_{Q+\alpha'}$ which are not fixed for general $\alpha' \in \tau_2$, then $D^*_{Q+\alpha'}$ is not contained in the fixed divisor $2G_1 + K_X$ and hence is contained in some divisor of the form $D^*_{P+\alpha} + D^*_{P-\alpha}$ and hence is $T_1^\vee$ vertical.

If $\mathrm{Pic}^0(T_1^\vee) \cap \mathrm{Pic}^0(T_2^\vee) = \{\mathcal{O}_X\}$, then $D^*_{Q+\alpha'}$ is a-exceptional, and this is impossible by Lemma 2.9.

If there is a 1-dimensional component $\Gamma \subset \mathrm{Pic}^0(T_1^\vee) \cap \mathrm{Pic}^0(T_2^\vee)$. Let $E = \Gamma^\vee$ and $\pi : X \longrightarrow E$ be the induced morphism. The divisors $D^*_{Q+\alpha'}$ are E-vertical. We may assume that π has connected fibers. Since the $D^*_{Q+\alpha'}$ vary with $\alpha' \in \tau_2$, for general $\alpha' \in \tau_2$, they contain a smooth fiber of π. So for general $\alpha' \in \tau_2$ there is an inclusion $\pi^* M \longrightarrow \omega_X \otimes Q \otimes \pi^* \alpha'$ where M is a line bundle of degree at least 1. By Claim 3.10, one has $\kappa(X) = 2$ and hence T_1, T_2 are parallel.

If there is a 2-dimensional component $\Gamma \subset \mathrm{Pic}^0(T_1^\vee) \cap \mathrm{Pic}^0(T_2^\vee)$, then $\delta_1 = \delta_2 \geq 3$. By (2), one sees that $P_{2,Q_1+Q_2} \geq 3$. By Lemma 3.4, this is impossible. □

By Claim 3.10, if there is a one dimensional component, then $\kappa(X) = 2$. Therefore, we may assume that $\delta_i \geq 2$ for all $i \neq 0$. By Claim 3.13, since $\delta_i \geq 2$ for all $i \neq 0$, then S_i, S_j are parallel for all $i, j \neq 0$. By Theorem 2.6, for an appropriate $i \neq 0$, $\kappa(X) = \dim(S_i)$ and so by Claim 3.12, one has $\kappa(X) = 2$. □

4. – Varieties with $P_3(X) = 4, q(X) = \dim(X)$ and $\kappa(X) = 2$

In this section, we classify varieties with $P_3(X) = 4, q(X) = \dim(X)$ and $\kappa(X) = 2$. The first step is to describe the cohomological support loci of these varieties. We must show that the only possible cases are the following (which corresponds to Examples 2 and 3 respectively):

(1) $\bar{G} \cong \mathbb{Z}_2$, $V_0(X, \omega_X) = \{\mathcal{O}_X\} \cup S_\eta$, $\delta_\eta = 2$.
(2) $\bar{G} \cong \mathbb{Z}_2^2$, $V_0(X, \omega_X) = \{\mathcal{O}_X\} \cup S_\eta \cup S_\zeta \cup S_{\eta+\zeta}$, $\delta_\eta = \delta_\zeta = 1, \delta_{\eta+\zeta} = 2$.

Using this information, we will determine the sheaves $\mathrm{a}_*(\omega_X)$ and this will enable us to prove the following:

THEOREM 4.1. *Let X be a smooth projective variety with $P_3(X) = 4$, $q(X) = \dim(X)$ and $\kappa(X) = 2$, then X is one of the varieties described in Examples* 2 *and* 3.

PROOF. Recall that $f : X \longrightarrow Y$ is a morphism birational to the Iitaka fibration, Y is an abelian surface and $f = q \circ \mathrm{a}$ where $q : \mathrm{A} \longrightarrow Y$.

CLAIM 4.2. One has that $f_* \omega_X = \mathcal{O}_Y$.

PROOF OF CLAIM 4.2. By Proposition 3.6, one has that $V^0(\omega_X) \cap f^* \mathrm{Pic}^0(Y) = \{\mathcal{O}_X\}$. By the proof of [CH3] Theorem 4, one sees that $f_* \omega_X \cong \mathcal{O}_Y \otimes H^0(\omega_X)$. Since $h^0(\omega_X|_{F_{X/Y}}) = 1$, it follows that $\mathrm{rank}(f_* \omega_X) = 1$ and hence $f_* \omega_X \cong \mathcal{O}_Y$. □

CLAIM 4.3. Let S_1, S_2 be distinct components of $V^0(\omega_X)$ such that $S_1 \cap S_2 \neq \emptyset$, then $S_1 \cap S_2 = P$ and

$$f_*(\omega_X \otimes P) = L_1 \boxtimes L_2 \otimes \mathcal{I}_p$$

where $Y = E_1 \times E_2$ and L_i are line bundles of degree 1 on the elliptic curves E_i and p is a point of Y.

PROOF OF CLAIM 4.3. Assume that $P \in S_1 \cap S_2$. Since $\kappa(X) = 2$, by Proposition 2.7, the T_i are 1-dimensional. Let $\pi_i : X \longrightarrow E_i := T_i^\vee$ be the induced morphisms. There are line bundles of degree 1, L_i on E_i and inclusions $\pi_i^* L_i \longrightarrow \omega_X \otimes P$ (cf. Corollary 2.11).

We claim that $\text{rank}(\pi_{1,*}(\omega_X \otimes P)) = 1$. If this were not the case, then by Lemma 2.13

$$P_1(F_{X/E_1}) = \text{rank}(\pi_{1,*}(\omega_X \otimes P)) \geq 2, \qquad P_2(F_{X/E_1}) = \text{rank}(\pi_{1,*}(\omega_X^{\otimes 2} \otimes P)) \geq 3$$

and so

$$\begin{aligned} P_3(X) = h^0(\omega_X^{\otimes 3} \otimes P^{\otimes 2}) &\geq h^0(\omega_X^{\otimes 2} \otimes P \otimes \pi_1^* L_1) \\ &= h^0(\pi_{1,*}(\omega_X^{\otimes 2} \otimes P) \otimes L_1) \\ &\geq \text{rank}(\pi_{1,*}(\omega_X^{\otimes 2} \otimes P)) + \deg(\pi_{1,*}(\omega_X^{\otimes 2} \otimes P)) \end{aligned}$$

and therefore

$$\text{rank}(\pi_{1,*}(\omega_X^{\otimes 2} \otimes P)) = 3, \qquad \deg(\pi_{1,*}(\omega_X^{\otimes 2} \otimes P)) = 1.$$

Since $\text{rank}(\pi_{1,*}(\omega_X)) = \text{rank}(\pi_{1,*}(\omega_X \otimes P))$, one has

$$\deg(\pi_{1,*}(\omega_X^{\otimes 2} \otimes P)) \geq \deg(\pi_{1,*}(\omega_X) \otimes L_1) \geq \text{rank}(\pi_{1,*}(\omega_X)) \geq 2,$$

which is impossible. Therefore, we may assume that

$$\text{rank}(\pi_{i,*}(\omega_X \otimes P)) = 1 \qquad for\ \ i = 1, 2.$$

For any $P_i \in S_i$, one has that $P_i \otimes P^\vee = \pi_i^* \alpha_i$ with $\alpha_i \in \text{Pic}^0(E_i)$. One sees that

$$h^0(\omega_X \otimes P_i) = h^0(\pi_{i,*}(\omega_X \otimes P) \otimes \alpha_i) = h^0(\pi_{i,*}(\omega_X \otimes P)) = h^0(\omega_X \otimes P).$$

If $h^0(\omega_X \otimes P) \geq 2$, then we may assume that $L_1 := \pi_{1,*}(\omega_X \otimes P)$ is an ample line bundle of degree at least 2. From the inclusion $\phi : L_1^{\otimes 2} \longrightarrow \pi_{1,*}(\omega_X^{\otimes 2} \otimes P^{\otimes 2})$, one sees that $h^0(\omega_X^{\otimes 2} \otimes P^{\otimes 2}) = 4$ and ϕ is an I.T. 0 isomorphism (cf. Lemma 2.4) and so

$$P_2(F_{X/E_1}) = h^0(\omega_X^{\otimes 2} \otimes P^{\otimes 2}|_F) = 1.$$

By Theorem 1.1, $\kappa(F_{X/E_1}) = 0$ and hence by easy addition, $\kappa(X) \leq 1$ which is impossible. Therefore we may assume that $h^0(\omega_X \otimes P) = 1$.

The coherent sheaf $f_*(\omega_X \otimes P)$ is torsion free of generic rank 1 on Y and hence is isomorphic to $L \otimes \mathcal{I}$ where L is a line bundle and $\mathcal{I}$ is an ideal sheaf cosupported at finitely many points. Let $q_i : Y \longrightarrow E_i$, so that $\pi_i = q_i \circ f$. Since

$$1 = \text{rank}(\pi_{i,*}(\omega_X \otimes P)) = \text{rank}(q_{i,*}(L \otimes \mathcal{I})) = \text{rank}(q_{i,*} L),$$

one sees that $L.F_{Y/E_i} = 1$ and it easily follows that $L = L_1 \boxtimes L_2$ where $L_i = q_{i,*}(L)$ is a line bundle of degree 1 on E_i. Clearly, $\mathcal{I}$ is the ideal sheaf of a point. □

We will now consider the case in which $\bar{G} = \mathbb{Z}_2$. Let B be the branch locus of $\mathrm{a} : X \longrightarrow \mathrm{A}$. The divisor B is vertical with respect to $q : \mathrm{A} \longrightarrow Y$ and hence we may write $B = q^*\bar{B}$. Let $g \circ h : X \longrightarrow Z \longrightarrow A$ be the Stein factorization of a. Then Z is a normal variety and g is finite of degree 2 and so $g_*\mathcal{O}_Z = \mathcal{O}_A \oplus M^\vee$ where M is a line bundle and the branch locus B is a divisor in $|2M|$. The map $F_{Z/Y} \longrightarrow F_{\mathrm{A}/Y}$ is étale of degree 2 and so $M = q^*L \otimes P$ where P is a 2-torsion element of $\mathrm{Pic}^0(X)$. Let $\nu : \mathrm{A}' \longrightarrow \mathrm{A}$ be a birational morphism so that ν^*B is a divisor with simple normal crossings support. Let $B' = \nu^*B - 2\lfloor \frac{\nu^*B}{2} \rfloor$ and $M' = \nu^*(M)(-\lfloor \frac{\nu^*B}{2} \rfloor)$. Let Z' be the normalization of $Z \times_{\mathrm{A}} \mathrm{A}'$, and $g' : Z' \longrightarrow \mathrm{A}'$ be the induced morphism. Then g' is finite of degree 2, Z' is normal with rational singularities and $g'_*(\mathcal{O}_{Z'}) = \mathcal{O}_{\mathrm{A}'} \oplus (M')^\vee$. Let $\tilde{X}$ be an appropriate birational model of X such that there are morphisms $\alpha : \tilde{X} \longrightarrow \mathrm{A}'$, $\upsilon : \tilde{X} \longrightarrow X$, $\tilde{\mathrm{a}} : \tilde{X} \longrightarrow \mathrm{A}$ and $\beta : \tilde{X} \longrightarrow Z'$. For all $n \geq 0$, one has that $\beta_*(\omega_{\tilde{X}}^{\otimes n}) \cong \omega_{Z'}^{\otimes n}$. It follows that

$$\alpha_*(\omega_{\tilde{X}}^{\otimes m}) = \omega_{\mathrm{A}'}^{\otimes m} \otimes (M'^{\otimes m-1} \oplus M'^{\otimes m}).$$

Therefore

$$\begin{aligned}
\mathrm{a}_*(\omega_X) &= \tilde{\mathrm{a}}_*(\omega_{\tilde{X}}) \\
&= \nu_*(\omega_{\mathrm{A}'} \oplus \omega_{\mathrm{A}'} \otimes M') \\
&= \mathcal{O}_{\mathrm{A}} \oplus \nu_* \left(\omega_{\mathrm{A}'} \otimes \nu^*(q^*L) \left(- \left\lfloor \frac{\nu^*B}{2} \right\rfloor \right) \right) \\
&= \mathcal{O}_{\mathrm{A}} \oplus q^*L \otimes P \otimes \mathcal{I}\left(\frac{B}{2}\right).
\end{aligned}$$

CLAIM 4.4. If $\bar{G} = \mathbb{Z}_2$, then for any $P \in V^0(\omega_X)$, one has

$$f_*(\omega_X \otimes P) \neq L_1 \boxtimes L_2 \otimes \mathcal{I}_p$$

where $Y = E_1 \times E_2$ and L_i are ample line bundles of degree 1 on E_i and p is a point of Y.

PROOF OF CLAIM 4.4. If $f_*(\omega_X \otimes P) = L_1 \boxtimes L_2 \otimes \mathcal{I}_p$, then $\frac{B}{2}$ is not log terminal. By [Hac3] Theorem 1, one sees that since $\frac{B}{2}$ is not log terminal, one has that $\lfloor \frac{B}{2} \rfloor \neq 0$ and this is impossible as then Z is not normal. □

Combining Claim 4.3 and Claim 4.4, one sees that if $\bar{G} = \mathbb{Z}_2$, then $V_0(X, \omega_X) = \{\mathcal{O}_X\} \cup S_\eta$ with $\delta_\eta = 2$. We then have the following:

CLAIM 4.5. If $\bar{G} = \mathbb{Z}_2$, then $h^0(X, \omega_X \otimes P) = 1$ for all $P \in S_\eta$.

PROOF OF CLAIM 4.5. It is clear that $h^0(\tilde{X}, \omega_{\tilde{X}} \otimes P) = h^0(A', \omega_{A'} \otimes M' \otimes P)$ for all $P \in S_\eta$, and $h^0(\tilde{X}, \omega_{\tilde{X}} \otimes P) = 1$ for general $P \in S_\eta$.

If $h^0(\tilde{X}, \omega_{\tilde{X}} \otimes Q_0) \geq 2$ for some $Q_0 \in S_\eta$, then $h^0(\tilde{X}, \omega_{\tilde{X}} \otimes Q_0) = 2$ as otherwise $h^0(\omega_{\tilde{X}}^{\otimes 2} \otimes Q_0^{\otimes 2}) \geq 3 + 3 - 1$ which is impossible.

Consider the linear series $|K_{A'}+M'+Q_0|$. Let $\mu:\tilde{A}\to A'$ be a log resolution of this linear series. We have

$$\mu^*|K_{A'}+M'+Q_0| = |D|+F,$$

where $|D|$ is base point free and F has simple normal crossings support. There is an induced map $\phi_{|D|}:\tilde{A}\to\mathbb{P}^1$ such that $|D|=\phi^*_{|D|}|\mathcal{O}_{\mathbb{P}^1}(1)|$. We have an inclusion

$$\varphi_1:\phi^*_{|D|}|\mathcal{O}_{\mathbb{P}^1}(2)|+G\hookrightarrow \mu^*|2K_{A'}+2M'+2Q_0|.$$

For all $\alpha\in \mathrm{Pic}^0(Y)$, there is a morphism

$$\varphi_2:\mu^*|K_{A'}+M'+Q_0+\alpha|+\mu^*|K_{A'}+M'+Q_0-\alpha|\longrightarrow \mu^*|2K_{A'}+2M'+2Q_0|.$$

Notice that $h^0(A',\omega_{A'}^{\otimes 2}\otimes M'^{\otimes 2}\otimes Q_0^{\otimes 2})\le h^0(X,\omega_X^{\otimes 2}\otimes Q_0^{\otimes 2})\le 4$.
Since $h^0(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(2))=3$, φ_1 has a 2-dimensional image. Since α varies in a 2-dimensional family, φ_2 also has 2-dimensional image. In particular, there is a positive dimensional family $\mathcal{N}\subset \mathrm{Pic}^0(Y)$ such that for general $\alpha\in\mathcal{N}$, one has

$$D_{\pm\alpha}+F_{\pm\alpha}\in\mu^*|K_{A'}+M'+Q_0\pm\alpha|$$

where $G=F_\alpha+F_{-\alpha}$ and $D_\alpha+D_{-\alpha}\in\phi^*_{|D|}|\mathcal{O}_{\mathbb{P}^1}(2)|$. Since G is a fixed divisor, it decomposes in at most finitely many ways as the sum of two effective divisors and so we may assume that $F_\alpha, F_{-\alpha}$ do not depend on $\alpha\in\mathcal{N}$.

Take any $\alpha\neq\alpha'\in\mathcal{N}$ with $F_\alpha=F_{\alpha'}$. One has that $D_\alpha=\phi^*_{|D|}H$ is numerically equivalent to $D_{\alpha'}=\phi^*_{|D|}H'$. It follows that H and H' are numerically equivalent on $\mathbb{P}^1$ hence linearly equivalent. Thus D_α and $D_{\alpha'}$ are linearly equivalent which is a contradiction. □

CLAIM 4.6. If $\bar{G}=\mathbb{Z}_2$, then $\mathrm{a}:X\longrightarrow \mathrm{A}$ has generic degree 2 and is branched over a divisor $B\in|2f^*\Theta|$ where $\mathcal{O}_Y(\Theta)$ is an ample line bundle of degree 1. Furthermore, $\mathrm{a}_*(\mathcal{O}_X)\cong\mathcal{O}_{\mathrm{A}}\oplus q^*\mathcal{O}_Y(\Theta)\otimes P$ where $P\notin\mathrm{Pic}^0(Y)$ and $P^{\otimes 2}=\mathcal{O}_{\mathrm{A}}$. See Example 2.

PROOF OF CLAIM 4.6. For all $\alpha\in\mathrm{Pic}^0(Y)$ and $P\in S_\eta$, one has that

$$h^0(\omega_X\otimes P\otimes\alpha)=h^0(\omega_{\mathrm{A}'}\otimes M'\otimes P\otimes\alpha)=1.$$

The sheaf $q_*\nu_*(\omega_{\mathrm{A}'}\otimes M'\otimes P)$ is torsion free of generic rank 1 and

$$h^0(q_*\nu_*(\omega_{\mathrm{A}'}\otimes M'\otimes P)\otimes\alpha)=1\quad\text{for all}\quad\alpha\in\mathrm{Pic}^0(Y).$$

Following the proof of Proposition 4.2 of [HP], one sees that higher cohomologies vanish. By [Hac], $q_*\nu_*(\omega_{\mathrm{A}'}\otimes M'\otimes P)$ is a principal polarization $\mathcal{O}_Y(\Theta)$. From the isomorphism $\nu_*(\omega_{\mathrm{A}'}\otimes M'\otimes P)\cong\bar{L}\otimes\mathcal{I}(\frac{\bar{B}}{2})$, one sees that $\bar{L}=\mathcal{O}_Y(\Theta)$ and $\mathcal{I}(\frac{\bar{B}}{2})=\mathcal{O}_Y$. Therefore, $\nu_*(\omega_{\mathrm{A}'}\otimes M'\otimes P)\cong q^*\mathcal{O}_Y(\Theta)$. It follows that

$$\mathrm{a}_*(\omega_X)\cong\mathcal{O}_{\mathrm{A}}\oplus q^*\mathcal{O}_Y(\Theta)\otimes P.$$ □

From now on we therefore assume that $\bar{G}\neq\mathbb{Z}_2$.

CLAIM 4.7. $V^0(\omega_X)$ has at most one 2-dimensional component.

PROOF OF CLAIM 4.7. Let S_η, S_ζ be 2-dimensional components of $V^0(\omega_X)$ with $\eta\neq\zeta$. Since $\kappa(X)=2$, one has $\delta_{\eta,\zeta}=2$. Thus by (2), $P_{2,\eta+\zeta}\ge 3$. By Lemma 3.4, this is impossible. □

CLAIM 4.8. Let S_1, S_2 be two parallel 1-dimensional components of $V^0(\omega_X)$, then $S_1 + \mathrm{Pic}^0(Y) = S_2 + \mathrm{Pic}^0(Y)$.

PROOF OF CLAIM 4.8. Let $P_i \in S_i$, $\pi : X \longrightarrow E := T_1^\vee = T_2^\vee$ the induced morphism and L_i ample line bundles on E_i with inclusions $\phi_i : \pi^* L_i \longrightarrow \omega_X \otimes P_i$. By Lemma 2.12, one sees that $h^0(\omega_X^{\otimes 2} \otimes P_1 \otimes P_2) \geq 2$. If it were equal, then the inclusion

$$L_1 \otimes L_2 \longrightarrow \pi_*(\omega_X^{\otimes 2} \otimes P_1 \otimes P_2)$$

would be an I.T. 0 isomorphisms and this would imply that $P_2(F_{X/E}) = 1$ and hence that $\kappa(X) \leq 1$. So $h^0(\omega_X^{\otimes 2} \otimes P_1 \otimes P_2) \geq 3$. By Lemma 3.4, this is impossible. □

CLAIM 4.9. If $\bar{G} \neq \mathbb{Z}_2$, let S_η be a 2-dimensional component of $V^0(\omega_X)$, then $h^0(\omega_X \otimes P) = 1$ for all $P \in S_\eta$. In particular $f_*(\omega_X \otimes P)$ is a principal polarization.

PROOF OF CLAIM 4.9 Let $f : X \longrightarrow (T_\eta)^\vee$ be the induced morphism. Then f is birational to the Iitaka fibration of X i.e. $(T_\eta)^\vee = Y$. By Claim 4.7, $V^0(\omega_X)$ has at most one 2-dimensional component, and so there must exist a 1-dimensional component S_ζ of $V^0(\omega_X)$. Let $\pi : X \longrightarrow E := T_\zeta^\vee$ be the induced morphism. There is an ample line bundle L on E and an inclusion $\pi^* L \longrightarrow \omega_X \otimes Q_\zeta$ for some general $Q_\zeta \in S_\zeta$.

Assume that $P \in S_\eta$ and $h^0(\omega_X \otimes P) \geq 2$. If $\mathrm{rank}(\pi_*(\omega_X \otimes P)) = 1$, then $\pi_*(\omega_X \otimes P)$ is an ample line bundle of degree at least 2 and hence $h^0(\pi_*(\omega_X \otimes P) \otimes \alpha) \geq 2$ for all $\alpha \in \mathrm{Pic}^0(E)$. It follows that

$$h^0(\omega_X^{\otimes 2} \otimes P \otimes Q_\zeta) \geq h^0(\omega_X \otimes P \otimes \pi^* L) = h^0(\pi_*(\omega_X \otimes P) \otimes L) \geq 3.$$

By Lemma 3.4, this is impossible.
Therefore, we may assume that $\mathrm{rank}(\pi_*(\omega_X \otimes P)) \geq 2$. Proceeding as above, since

$$h^0(\pi_*(\omega_X \otimes P) \otimes L) \geq \mathrm{rank}(\pi_*(\omega_X \otimes P)) + \deg(\pi_*(\omega_X \otimes P)),$$

it follows that $\pi_*(\omega_X \otimes P)$ is a sheaf of degree 0. Since $h^0(\pi_*(\omega_X \otimes P) \otimes \alpha) > 0$ for all $\alpha \in \mathrm{Pic}^0(E)$, By Riemann-Roch one sees that also $h^1(\pi_*(\omega_X \otimes P) \otimes \alpha) > 0$ for all $\alpha \in \mathrm{Pic}^0(E)$. By Theorem 2.5, this is impossible.

Finally, the sheaf $f_*(\omega_X \otimes P)$ is torsion free of generic rank 1 on Y and hence, by [Hac], it is a principal polarization. □

CLAIM 4.10. Assume that $\bar{G} \neq \mathbb{Z}_2$. Then, for any $P \in V^0(\omega_X) - \mathrm{Pic}^0(Y)$ one has that $f_*(\omega_X \otimes P)$ is either:

i) a principal polarization on Y,
ii) the pull-back of a line bundle of degree 1 on an elliptic curve or
iii) of the form $L \boxtimes L' \otimes \mathcal{I}_p$ where L, L' are ample line bundles of degree 1 on E, E', $Y = E \times E'$ and p is a point of Y.

In particular, there are no 2 distinct parallel components of $V^0(\omega_X)$.

PROOF OF CLAIM 4.10. By Claim 4.9, we only need to consider the case in which all the components of $(P+\mathrm{Pic}^0(Y))\cap V^0(\omega_X)$ are 1-dimensional. By Claim 4.3, we may also assume that these components are parallel.

For any 1 dimensional component S_i of $(P+\mathrm{Pic}^0(Y))\cap V^0(\omega_X)$, $P_i\in S_i$ and corresponding projection $\pi_i : X\longrightarrow E_i := T_i^\vee$, one has $\mathrm{rank}(\pi_{i,*}(\omega_X\otimes P_i))=1$ and hence $\pi_{i,*}(\omega_X\otimes P_i)=L_i$ is an ample line bundle of degree at least 1 on E_i. If this were not the case, then By Lemma 2.13,

$$\mathrm{rank}(\pi_{i,*}(\omega_X\otimes P_i))=h^0(\omega_F)\geq 2$$

and so

$$\mathrm{rank}(\pi_{i,*}(\omega_X^{\otimes 2}\otimes P_i))=h^0(\omega_F^{\otimes 2})\geq 3.$$

From the inclusion (cf. Corollary 2.11)

$$\pi_i^*L_i\longrightarrow \omega_X\otimes P_i\longrightarrow \omega_X^{\otimes 2}\otimes P_i,$$

one sees that $h^0(\omega_X^{\otimes 2}\otimes P_i)\geq 2$ (cf. Lemma 3.1).
By Lemma 2.4, $\deg(\pi_{i,*}(\omega_X^{\otimes 2}\otimes P_i))\geq 2$. By Riemann-Roch, one has

$$h^0(L\otimes\pi_{i,*}(\omega_X^{\otimes 2}\otimes P_i))\geq \deg(\pi_{i,*}(\omega_X^{\otimes 2}\otimes P_i))+\mathrm{rank}(\pi_{i,*}(\omega_X^{\otimes 2}\otimes P_i))\geq 5.$$

This is a contradiction and so $\mathrm{rank}(\pi_{i,*}(\omega_X\otimes P_i))=1$.

Since we assumed that all components of $V^0(\omega_X)\cap(P+\mathrm{Pic}^0(Y))$ are parallel, then one has $\pi_i=\pi$, $E=E_i$ are independent of i. Let $q:Y\longrightarrow E$. Since there are injections

$$\mathrm{Pic}^0(E)+P_1=S_1\hookrightarrow P_1+\mathrm{Pic}^0(Y)\hookrightarrow \mathrm{Pic}^0(X),$$

we may assume that q has connected fibers. The sheaf $f_*(\omega_X\otimes P_1)$ is torsion free of rank 1, and hence we may write $f_*(\omega_X\otimes P_1)\cong M\otimes\mathcal{I}$ where M is a line bundle and $\mathcal{I}$ is supported in codimension at least 2 (i.e. on points). Since $\mathrm{rank}(\pi_*(\omega_X\otimes P_1))=1$, one has that $h^0(M|_{F_{Y/E}})=1$.

For general $\alpha\in\mathrm{Pic}^0(Y)$, one has that $V^0(\omega_X)\cap P_1+\alpha+\mathrm{Pic}^0(E)=\emptyset$ and so the semi-positive torsion free sheaf $\pi_*(\omega_X\otimes P_1\otimes\alpha)$ must be the 0-sheaf. In particular $h^0(M\otimes\alpha|_{F_{Y/E}})=0$. It follows that $\deg(M|_{F_{Y/E}})=0$ and hence $M|_{F_{Y/E}}=\mathcal{O}_{F_{Y/E}}$. One easily sees that $h^0(M\otimes\alpha)=0$ for all $\alpha\in\mathrm{Pic}^0(Y)-\mathrm{Pic}^0(E)$ and hence

$$V^0(\omega_X)\cap(P_1+\mathrm{Pic}^0(Y))=P_1+\mathrm{Pic}^0(E)=T_1.$$

By Proposition 2.3, one has that q^*L_1 and $f_*(\omega_X\otimes P_1)$ are isomorphic if and only if the inclusion $q^*L_1\longrightarrow f_*(\omega_X\otimes P_1)$ induces isomorphisms

$$H^i(Y,q^*L_1\otimes\alpha)\longrightarrow H^i(Y,f_*(\omega_X\otimes P_1)\otimes\alpha)$$

for $i = 0, 1, 2$ and all $\alpha \in \mathrm{Pic}^0(Y)$. If $\alpha \in \mathrm{Pic}^0(Y) - \mathrm{Pic}^0(E)$, then both groups vanish and so the isomorphism follows. If $\alpha \in \mathrm{Pic}^0(E)$, we proceed as follows: Let $p : \mathrm{A} \longrightarrow E$ and $W \subset H^1(A, \mathcal{O}_A)$ a linear subspace complementary to the tangent space to T_1. By Proposition 2.12 of [Hac2], one has isomorphisms

$$\begin{aligned} H^i(\mathrm{a}_*(\omega_X \otimes P_1) \otimes p^*\alpha) &\cong H^0(\mathrm{a}_*(\omega_X \otimes P_1) \otimes p^*\alpha) \otimes \wedge^i W \\ &\cong H^0(q^*(L_1 \otimes \alpha)) \otimes \wedge^i W \\ &\cong H^i(q^* L_1 \otimes \alpha). \end{aligned}$$

Pushing forward to Y, one obtains the required isomorphisms. □

CLAIM 4.11. If $\bar{G} \neq \mathbb{Z}_2$, then $\bar{G} = (\mathbb{Z}_2)^2$ and

$$V_0(X, \omega_X) = \{\mathcal{O}_X\} \cup S_\alpha \cup S_\zeta \cup S_\xi$$

with $\delta_\alpha = 2$, $\delta_\zeta = \delta_\xi = 1$.

PROOF OF CLAIM 4.11. We have seen that $V^0(\omega_X)$ has at most one 2-dimensional component and there are no parallel 1-dimensional components. Since $\bar{G} \neq \mathbb{Z}_2$, then there are at least two 1-dimensional components of $V^0(\omega_X)$. We will show that given two one dimensional components contained in $Q_1 + \mathrm{Pic}^0(Y) \neq Q_2 + \mathrm{Pic}^0(Y)$, then

$$\Big(Q_1 + Q_2 + \mathrm{Pic}^0(Y)\Big) \cap V^0(\omega_X)$$

does not contain a 1-dimensional component. Grant this for the time being. Then, by Proposition 2.7, it follows that $Q_1 + Q_2 + \mathrm{Pic}^0(Y)$ is a 2-dimensional component of $V^0(\omega_X)$. If $|\bar{G}| > 4$, this implies that there are at least two 2-dimensional components, which is impossible, and so $|\bar{G}| = 4$ and the claim follows.

Suppose now that there are three 1-dimensional components of $V^0(\omega_X)$, say S_1, S_2, S_3, contained in $Q_1 + \mathrm{Pic}^0(Y)$, $Q_2 + \mathrm{Pic}^0(Y)$, $Q_3 + \mathrm{Pic}^0(Y)$ respectively with $Q_1 + Q_2 + Q_3 \in \mathrm{Pic}^0(Y)$. By Claim 4.10, these components are not parallel to each other. We may assume that $\pi_i : X \to E_i := S_i^\vee$ factors through $f : X \to Y$ and that Y is an abelian surface. Let $q_i : Y \longrightarrow E_i$ be the induced morphisms.

Let Q_1, Q_2, Q_3 be general torsion elements in S_1, S_2, S_3 and

$$\mathcal{G} := f_*(\omega_X^{\otimes 2} \otimes Q_2 \otimes Q_3), \qquad \mathcal{F} := f_*(\omega_X^{\otimes 3} \otimes Q_1 \otimes Q_2 \otimes Q_3).$$

From the inclusions $\pi_i^* L_i \longrightarrow \omega_X \otimes Q_i$, one sees that we have inclusions

$$\varphi : q_2^* L_2 \otimes q_3^* L_3 \to \mathcal{G}, \qquad \psi : q_1^* L_1 \otimes q_2^* L_2 \otimes q_3^* L_3 \to \mathcal{F}$$

where L_i are ample line bundles on E_i respectively. Since $\mathcal{F}$ is torsion free of generic rank one, we may write

$$\mathcal{F} = q_1^* L_1 \otimes q_2^* L_2 \otimes q_3^* L_3 \otimes N \otimes \mathcal{I}$$

where N is a semi-positive line bundle on Y and $\mathcal{I}$ is an ideal sheaf cosupported at points. If N is not numerically trivial (or if $F_{Y/E_1} \cdot q_i^* L_i > 1$ for $i = 2$ or $i = 3$), then N is not vertical with respect to one of the projections q_i, say q_1. Then

$$\operatorname{rank}(q_{1,*}(\mathcal{F})) = F_{Y/E_1} \cdot (q_1^* L_1 + q_2^* L_2 + q_3^* L_3 + N) \geq 3.$$

On the other hand, from the inclusion φ, one sees that $\operatorname{rank}(q_{1,*}(\mathcal{G})) \geq 2$. Consider the inclusion of I.T. 0 sheaves $L_1 \longrightarrow q_{1,*}(\mathcal{G} \otimes \alpha)$ with $\alpha = Q_1 \otimes Q_2^\vee \otimes Q_3^\vee \in \operatorname{Pic}^0(Y)$. Since it is not an isomorphism, one sees that

$$h^0(\mathcal{G}) = h^0(\mathcal{G} \otimes \alpha) > h^0(L_1) \geq 1.$$

From the inclusion

$$\rho : L_1 \otimes q_{1,*}(\mathcal{G}) \longrightarrow q_{1,*}(\mathcal{F}) = \pi_{1,*}(\omega_X^{\otimes 3} \otimes Q_1 \otimes Q_2 \otimes Q_3)$$

one sees that by Riemann-Roch

$$h^0(\mathcal{G}) + \operatorname{rank}(q_{1,*}(\mathcal{G})) \leq h^0(\omega_X^{\otimes 3} \otimes Q_1 \otimes Q_2 \otimes Q_3) = P_3(X)$$

and therefore

$$h^0(\mathcal{G}) = 2, \qquad \operatorname{rank}(q_{1,*}(\mathcal{G})) = 2.$$

In particular, ρ is an I.T. 0 isomorphism. So, $\operatorname{rank}(q_{1,*}(\mathcal{F})) = \operatorname{rank}(q_{1,*}(\mathcal{G})) = 2$ which is a contradiction. Therefore, we have that

$$N \in \operatorname{Pic}^0(Y) \quad \textit{and} \quad q_2^* L_2 . F_{Y/E_1} = q_3^* L_3 . F_{Y/E_1} = 1.$$

Since $\deg(L_i) = 1$, one has $q_i^* L_i \equiv F_{Y/E_i}$. Since $(q_1^* L_1 \otimes q_2^* L_2 \otimes q_3^* L_3)^2 \geq 8$, we have that $q_2^* L_2 \cdot q_3^* L_3 \geq 2$. Since

$$h^0(q_2^* L_2 \otimes q_3^* L_3) \leq h^0(\mathcal{G}) = 2,$$

one sees that $q_2^* L_2 . q_3^* L_3 = 2$ and hence $\mathcal{I} = \mathcal{O}_Y$.

Now let $\mathcal{G}' := f_*(\omega_X^{\otimes 2} \otimes Q_1 \otimes Q_3)$. Proceeding as above, one sees that

$$\operatorname{rank}(q_{2,*}\mathcal{G}') \geq F_{Y/E_2} \cdot (q_1^* L_1 + q_3^* L_3) = 3, \qquad h^0(q_{2,*}\mathcal{G}') > h^0(L_2) = 1.$$

By Riemann Roch, one has that

$$P_3(X) = h^0(\omega_X^{\otimes 3} \otimes Q_1 \otimes Q_2 \otimes Q_3) \geq h^0(L_2 \otimes q_{2,*}\mathcal{G}') \geq 5$$

which is the required contradiction. □

CLAIM 4.12. If $\bar{G} \cong (\mathbb{Z}_2)^2$, then $Y = E_1 \times E_2$ and there are line bundles L_i of degree 1 on E_i, projections $p_i : \mathrm{A} \longrightarrow E_i$ and 2−torsion elements $Q_1, Q_2 \in \operatorname{Pic}^0(X)$ that generate $\bar{G}$, such that

$$\mathrm{a}_*(\mathcal{O}_X) \cong \mathcal{O}_\mathrm{A} \oplus M_1^\vee \oplus M_2^\vee \oplus M_1^\vee \otimes M_2^\vee$$

with

$$M_1 = p_1^* L_1 \otimes Q_1^\vee, \qquad M_2 = p_2^* L_2 \otimes Q_2^\vee \quad \text{and} \qquad M_3 = M_1 \otimes M_2.$$

In particular X is birational to the fiber product of two degree 2 coverings $X_i \longrightarrow$ A with $P_3(X_i) = 2$.

Proof of Claim 4.12. By Claim 4.11, the degree of a : $X \longrightarrow$ A is $|\bar{G}| = 4$ and there are two non parallel 1-dimensional components of $V^0(\omega_X)$ say S_1, S_2 such that $S_1 + \mathrm{Pic}^0(Y) \neq S_2 + \mathrm{Pic}^0(Y)$. Let $E_i := T_i^\vee$ and $q_i : Y \longrightarrow E_i$, $\pi_i : X \longrightarrow E_i$ be the induced morphisms. Then there are inclusions $\pi_i^* L_i \longrightarrow \omega_X \otimes Q_i$ where $Q_i \in S_i$. Moreover, by Claim 4.11, $Q_1 + Q_2 + \mathrm{Pic}^0(Y) \subset V^0(\omega_X)$. By Claim 4.9, one has that

$$L := f_*(\omega_X \otimes Q_1 \otimes Q_2)$$

is an ample line bundle of degree 1. Moreover,

$$V^0(\omega_X) = \{\mathcal{O}_X\} \cup S_1 \cup S_2 \cup (Q_1 + Q_2 + \mathrm{Pic}^0(Y)).$$

From the inclusion

$$q_1^* L_1 \otimes q_2^* L_2 \otimes L \longrightarrow f_*(\omega_X^{\otimes 3} \otimes Q_1^{\otimes 2} \otimes Q_2^{\otimes 2})$$

and the equality $4 = P_3(X) = h^0(\omega_X^{\otimes 3} \otimes Q_1^{\otimes 2} \otimes Q_2^{\otimes 2})$, one sees that

$$L^2 = 2, \qquad L.q_i^* L_i = q_1^* L_1 . q_2^* L_2 = 1.$$

By the Hodge Index Theorem, one sees that since

$$L^2 (q_1^* L_1 + q_2^* L_2)^2 = \big(L.(q_1^* L_1 + q_2^* L_2)\big)^2$$

then the principal polarization L is numerically equivalent to $q_1^* L_1 + q_2^* L_2$. Therefore,

$$(Y, q_1^* L_1 \otimes q_2^* L_2) \cong (E_1, L_1) \times (E_2, L_2),$$

and one sees that

$$L = q_1^*(L_1 \otimes P_1) \otimes q_2^*(L_2 \otimes P_2), \qquad P_i \in \mathrm{Pic}^0(E_i).$$

We have inclusions

$$L \longrightarrow f_*(\omega_X \otimes Q_1 \otimes Q_2) \longrightarrow f_*(\omega_X^{\otimes 2} \otimes Q_1 \otimes Q_2),$$

$$q_1^* L_1 \otimes q_2^* L_2 \longrightarrow f_*(\omega_X^{\otimes 2} \otimes Q_1 \otimes Q_2).$$

Let $\mathcal{G} := \omega_X^{\otimes 2}\otimes Q_1\otimes Q_2$. If $h^0(\mathcal{G}) = 1$, then $L = q_1^*L_1\otimes q_2^*L_2$ as required. If $h^0(\mathcal{G}) \geq 2$, then one sees that

$$h^0(\pi_{1,*}(\mathcal{G})\otimes L_1\otimes P_1) \geq \mathrm{rank}(\mathcal{G}) + \deg(\mathcal{G}) \geq 1 + 2.$$

Since

$$\mathrm{rank}(\pi_{2,*}(\mathcal{G}\otimes\pi_1^*(L_1\otimes P_1))) \geq \mathrm{rank}(q_{2,*}(q_1^*(L_1^{\otimes 2}\otimes P_1)\otimes q_2^*(L_2))) = 2,$$

one sees that

$$P_3(X) \geq h^0(\omega_X^{\otimes 2}\otimes Q_1\otimes Q_2\otimes L) = h^0(\pi_{2,*}(\mathcal{G}\otimes\pi_1^*(L_1\otimes P_1))\otimes L_2\otimes P_2) \geq 2 + 3$$

and this is impossible. Let $M_i := p_i^*L_i\otimes Q_i^\vee$. By Claim 4.10, one has

$$\mathrm{a}_*(\omega_X) \cong \mathcal{O}_{\mathrm{A}} \oplus M_1 \oplus M_2 \oplus M_1\otimes M_2$$

and hence by Groethendieck duality,

$$\mathrm{a}_*(\mathcal{O}_X) \cong \mathcal{O}_{\mathrm{A}} \oplus M_1^\vee \oplus M_2^\vee \oplus M_1^\vee\otimes M_2^\vee.$$

Let $X \longrightarrow Z \longrightarrow \mathrm{A}$ be the Stein factorization. Following [HM] Section 7, one sees that the only possible nonzero structure constants defining the $4-1$ cover $Z \longrightarrow \mathrm{A}$ are $c_{1,4} \in H^0(M_1\otimes M_2\otimes M_3^\vee)$, $c_{1,6} \in H^0(M_1\otimes M_2^\vee\otimes M_3)$ and $c_{4,6} \in H^0(M_1^\vee\otimes M_2\otimes M_3)$. So, $Z \longrightarrow \mathrm{A}$ is a bi-double cover. It is determined by two degree 2 covers $\mathrm{a}_i : X_i \longrightarrow \mathrm{A}$ defined by $\mathrm{a}_{i,*}(\mathcal{O}_{X_i}) = \mathcal{O}_{\mathrm{A}} \oplus p_i^*L_i\otimes Q_i^\vee$ and sections $-c_{1,4}c_{1,6} \in H^0(M_1^{\otimes 2})$ and $c_{1,4}c_{4,6} \in H^0(M_2^{\otimes 2})$. It is easy to see that X_1, X_2, Z are smooth. □

This completes the proof. □

5. – Varieties with $P_3(X) = 4, q(X) = \dim(X)$ and $\kappa(X) = 1$

THEOREM 5.1. *Let X be a smooth projective variety with $P_3(X) = 4$, $q(X) = \dim(X)$ and $\kappa(X) = 1$ then X is birational to $(C \times \tilde{K})/G$ where G is an abelian group acting faithfully by translations on an abelian variety $\tilde{K}$ and faithfully on a curve C. The Iitaka fibration of X is birational to $f : (C \times \tilde{K})/G \longrightarrow C/G = E$ where E is an elliptic curve and* $\dim H^0(C, \omega_C^{\otimes 3})^G = 4$.

PROOF. Let $f : X \longrightarrow Y$ be the Iitaka fibration. Since $\kappa(X) = 1$, and $\mathrm{a} : X \longrightarrow \mathrm{A}$ is generically finite, one has that Y is a curve of genus $g \geq 1$. If $g = 1$, then Y is an elliptic curve and by Proposition 2.1, $Y \longrightarrow \mathrm{A}(Y)$ is of degree 1 (i.e. an isomorphism). By Proposition 2.1 one sees that if $g \geq 2$, then $q(X) \geq \dim(X) + 1$ which is impossible.

From now on we will denote the elliptic curve $\mathrm{A}(Y)$ simply by E and $f : X \longrightarrow E$ will be the corresponding algebraic fiber space. Let $X \longrightarrow \bar{X} \longrightarrow A$ be the Stein factorization of the Albanese map. Since $\bar{X} \longrightarrow E$ is isotrivial, there is a generically finite cover $C \longrightarrow E$ such that $\bar{X} \times_E C$ is birational to $C \times \tilde{K}$. We may assume that $C \longrightarrow E$ is a Galois cover with group G. G acts by translations on $\tilde{K}$ and we may assume that the action of G is faithful on C and $\tilde{K}$. Since G acts freely on $C \times \tilde{K}$, one has that

$$H^0(X, \omega_X^{\otimes 3}) = H^0(C \times \tilde{K}, \omega_{C \times \tilde{K}}^{\otimes 3})^G = [H^0(\tilde{K}, \omega_{\tilde{K}}^{\otimes 3}) \otimes H^0(C, \omega_C^{\otimes 3})]^G.$$

Since G acts on $\tilde{K}$ by translations, G acts on $H^0(\tilde{K}, \omega_{\tilde{K}}^{\otimes 3})$ trivially. It follows that

$$4 = P_3(X) = \dim H^0(C, \omega_C^{\otimes 3})^G.$$

Similarly, one sees that $q(X) = q(C/G) + q(\tilde{K}/G)$ and so $q(C/G) = 1$. □

We now consider the induced morphism $\pi : C \to C/G =: E$. By the argument of [Be], Example VI.12, one has

$$4 = \dim H^0(C, \omega_C^{\otimes 3})^G = h^0\left(E, \mathcal{O}\left(\sum_{P \in E} \left\lfloor 3\left(1 - \frac{1}{e_P}\right)\right\rfloor\right)\right).$$

Where P is a branch points of π, and e_P is the ramification index of a ramification point lying over P. Note that $|G| = e_P s_P$, where s_P is the number of ramification points lying over P.

It is easy to see that since

$$\left\lfloor 3\left(1 - \frac{1}{e_P}\right)\right\rfloor = 1 \text{ (resp. } = 2) \quad \text{if} \quad e_P = 2 \text{ (resp. } e_P \geq 3),$$

we have the following cases:

Case 1. 4 branch points $P_1, ..., P_4$ with $e_{P_i} = 2$.

Case 2. 3 branch points P_1, P_2, P_3 with $e_{P_1} \geq 3, e_{P_2} = e_{P_3} = 2$.

Case 3. 2 branch points P_1, P_2 with $e_{P_i} \geq 3$.

We will follow the notation of [Pa]. Let $\pi : C \to E$ be an abelian cover with abelian Galois group G. There is a splitting

$$\pi_* \mathcal{O}_C = \oplus_{\chi \in G^*} L_\chi^\vee.$$

In particular, if $d_\chi := \deg(L_\chi)$, then

$$g = 1 + \sum_{\chi \in G^*,\ \chi \neq 1} d_\chi.$$

For every branch point P_i with $i = 1, ..., s$, the inertia group H_i, which is defined as the stabilizer subgroup at any point lying over P_i, is a cyclic subgroup of order $e_i := e_{P_i}$. We also associate a generator ψ_i of each H_i^* which corresponds to the character of P_i. For every $\chi \in G^*$, $\chi_{|H_i} = \psi_i^{n(\chi)}$ with $0 \le n(\chi) \le |H_i| - 1$. And define

$$\epsilon_{\chi,\chi'}^{H_i,\psi_i} := \left\lfloor \frac{n(\chi)+n(\chi')}{|H_i|} \right\rfloor .$$

Following [Pa], one sees that there is an abelian cover $C \to E$ with group G with building data L_χ if and only if the line bundles L_χ satisfy the following set of linear equivalences:

$$L_\chi + L_{\chi'} = L_{\chi\chi'} + \sum_{i=1,...,s} \epsilon_{\chi,\chi'}^{H_i,\psi_i} P_i. \tag{3}$$

If $\chi_{|H_i} = \psi_i^{n_i(\chi)}$, then

$$d_\chi + d_{\chi'} = d_{\chi\chi'} + \sum_{i=1,...,s} \left\lfloor \frac{n_i(\chi)+n_i(\chi')}{e_i} \right\rfloor . \tag{4}$$

Let H be the subgroup of G generated by the inertia subgroups H_i and let $Q = G/H$. One sees that there is an exact sequence of groups

$$1 \longrightarrow Q^* \longrightarrow G^* \longrightarrow H^* \longrightarrow 1.$$

The generators ψ_i of H_i^* define isomorphisms $H_i^* \cong \mathbb{Z}_{e_i}$ where $e_i := |H_i|$. Therefore, we have an induced injective homomorphism

$$\varphi : H^* \hookrightarrow \prod_{i=1,...,s} \mathbb{Z}_{e_i}$$

such that the induced maps $\varphi_i : H^* \longrightarrow \mathbb{Z}_{e_i}$ are surjective. By abuse of notation, we will also denote by φ the induced homomorphism $\varphi : G^* \longrightarrow \prod_{i=1,...,s} \mathbb{Z}_{e_i}$. We will write

$$\varphi(\chi) = (n_1(\chi), ..., n_s(\chi)) \quad \forall \chi \in G^*.$$

Let $\mu(\chi)$ be the order of χ. By [Pa] Proposition 2.1,

$$d_\chi = \sum_{i=1,...,s} \frac{n_i(\chi)}{e_i}.$$

We will now analyze all possible inertia groups H.

CASE 1: $s = 4$, and $e := e_i = 2$. Then $H^* \subset \mathbb{Z}_2^4$. Note that $H^* \neq \mathbb{Z}_2^4$ since $(1, 0, 0, 0) \notin H^*$. Thus $H^* \cong (\mathbb{Z}_2)^s$ with $1 \le s \le 3$.

By Example 1, all of these possibilities occur.

CASE 2: $s = 3$ and $e_1 \geq 3$, $e_2 = e_3 = 2$. There must be a character χ with $\varphi(\chi) = (1, n_2, n_3)$, and so

$$d_\chi = \frac{1}{e_1} + \frac{n_2}{2} + \frac{n_3}{2}$$

which is not an integer. Therefore this case is impossible.

CASE 3: $s = 2$ and $e_1, e_2 \geq 3$. Assume that $e_1 > e_2$. Since $G^* \to \mathbb{Z}_{e_1}$ is surjective, there is $\chi \in H^*$ with $\varphi(\chi) = (1, n_2)$. Then

$$d_\chi = \frac{1}{e_1} + \frac{n_2}{e_2} < 1$$

which is impossible. So we may assume that $e = e_1 = e_2 \geq 3$ and $H^* \subset \mathbb{Z}_e^2$. Let $\varphi(\chi) = (n_1, n_2)$. One has $d_\chi = \frac{n_1+n_2}{e}$. Thus $n_2 = e - n_1$ for any $\chi \neq 1$. Therefore, $H^* = \{(i, e-i) | 0 \leq i \leq e-1\} \cong \mathbb{Z}_e$. By Example 1, all of these possibilities occur.

From the above discussion, it follows that:

PROPOSITION 5.2. *Let $\phi : C \longrightarrow E$ be a G-cover with E an elliptic curve and* $\dim H^0(\omega_C^{\otimes 3})^G = 4$. *Then either ϕ is ramified over 4-points and the inertia group H is isomorphic to $(\mathbb{Z}_2)^s$ with $s \in \{1, 2, 3\}$ or ϕ is ramified over 2-points and the inertia group H is isomorphic to $\mathbb{Z}_m$ with $m \geq 3$.*

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Department of Mathematics
National Taiwan University
No.1, Sec. 4, Roosevelt Road
Taipei, 106, Taiwan
jkchen@math.ntu.edu.tw

Department of Mathematics
University of Utah
155 South 1400 East, JWB 233
Salt Lake City
Utah 84112-0090, USA
hacon@math.utah.edu