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Boundary Regularity and Compactness for Overdetermined Problems

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Abstract. Let D be either the unit ball $B_1(0)$ or the half ball $B_1^+(0)$, let f be a strictly positive and continuous function, and let u and $\Omega \subset D$ solve the following overdetermined problem:

$$\Delta u(x) = \chi_\Omega(x) f(x) \text{ in } D, \quad 0 \in \partial\Omega, \quad u = |\nabla u| = 0 \text{ in } \Omega^c,$$

where χ_Ω denotes the characteristic function of Ω, Ω^c denotes the set $D \setminus \Omega$, and the equation is satisfied in the sense of distributions. When $D = B_1^+(0)$, then we impose in addition that

$$u(x) \equiv 0 \text{ on } \{ (x', x_n) \mid x_n = 0 \}.$$

We show that a fairly mild thickness assumption on Ω^c will ensure enough compactness on u to give us "blow-up" limits, and we show how this compactness leads to regularity of $\partial\Omega$. In the case where f is positive and Lipschitz, the methods developed in Caffarelli, Karp, and Shahgholian (2000) lead to regularity of $\partial\Omega$ under a weaker thickness assumption.

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1. – Introduction

In this paper we study the regularity properties of solutions to certain overdetermined problems which are similar to the obstacle problem, but which do not have a sign assumption. Specifically, we assume that we are given an open set $\Omega \subset D$, where D is the open unit ball $B_1(0) \subset \mathbb{R}^n$, or the half ball $B_1^+(0)$, and we assume that we are given a function u which satisfies:

$$\begin{cases} \Delta u(x) = \chi_\Omega(x) f(x) \quad \text{in} \quad D, \\ u = |\nabla u| = 0 \quad \text{in} \quad \Omega^c = D \setminus \Omega, \\ 0 \in \partial\Omega. \end{cases} \tag{1.1}$$

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Here f is a positive continuous function. In the case where D is a half ball, we impose the additional assumption on the boundary:

$$u(x) \equiv 0 \quad \text{on} \quad \{ (x',\ x_n) \mid x_n = 0 \} . \tag{1.2}$$

We want to examine the smoothness of $\partial\Omega$. (Note that by standard elliptic estimates, we know that $u \in W^{2,p}(D)$ for all $p < \infty$, and therefore by the Sobolev Embedding Theorem, we have $u \in C^{1,\alpha}(\overline{D})$ for all $\alpha < 1$. See [GT].)

These problems have been studied in many recent papers with either constant f or with $u \geq 0$. (See for example [B], [CKS], and [SU].) These types of problems have also arisen in various problems of mathematical physics. In the case without the fixed boundary this type of problem comes up in geophysics and inverse potential theory (see [I], [M], and [St]). Problems where a free boundary comes into contact with a fixed boundary appear in filtration, and motion by mean curvature with nonconvex obstacles.

Before we can state our results, we need some definitions. We will fix $0 < \lambda \leq \mu$ and we will assume that all constants are automatically allowed to depend on λ, μ, and n in addition to any other dependence which is indicated. (In other words, if we state that a constant depends on only α, for example, then it is actually allowed to depend on α, λ, μ, and n.) Any nonnegative increasing continuous function σ defined on the nonnegative real numbers with $\sigma(0) = 0$ is called a *modulus of continuity.* If σ is a modulus of continuity, then we define $\mathcal{P}_r(M, \sigma)$ to be the set of functions u which satisfy

1. Equation (1.1) with $D = B_r$, and a continuous function f which satisfies

$$\lambda \leq f \leq \mu \qquad \text{and} \qquad |f(x) - f(y)| \leq \sigma(||x - y||),$$

(the f and the Ω from Equation (1.1) are allowed to depend on u)

2. $|u| \leq M$ in D.

For such a function u we will say that the function "f" and the domain "Ω" correspond to u, and refer to Ω as the *nonzero set* for u and its complement will be called the *zero set*. We define $\mathcal{P}_r^+(M, \sigma)$ like $\mathcal{P}_r(M, \sigma)$ but with $D := B_r^+$, and with Equation (1.2) also assumed. For the situation on the half ball we distinguish different parts of the boundary of Ω as follows:

$$\begin{aligned} XB(u) &:= \partial\Omega \cap \{x_n = 0\} && \textit{the fixed boundary,} \\ FB(u) &:= \partial\Omega \cap \{x_n > 0\} \setminus \partial B_1 && \textit{the free boundary, } \text{and} \\ IB(u) &:= \overline{XB(u)} \cap \overline{FB(u)} && \textit{the interface.} \end{aligned} \tag{1.3}$$

We will assume that the origin is part of the free boundary or the interface in all of the sets of functions we define for the sake of simplicity of notation, but the equations themselves are translation invariant.

For a bounded set $S \subset \mathbb{R}^n$, we define its *minimum diameter* (denoted $m.d.(S)$) to be the infimum among the distances between pairs of parallel hyperplanes enclosing S. We use the minimum diameter to define two "thickness" conditions which we will use repeatedly.

DEFINITION 1.1 ((ϵ, r)-Thickness). We will say that a point x of the free boundary is *(ϵ, r)-thick* if

$$\inf_{r \geq s > 0} \frac{m.d.(\{\Omega^c \cup D^c\} \cap B_s(x))}{s} \geq \epsilon. \tag{1.4}$$

(Better terminology might be that *the zero set is (ϵ, r)-thick at x*, but we prefer brevity.)

DEFINITION 1.2 ((ϵ, r)-Pthickness). We will say that an x of the interface is *(ϵ, r)-Pthick* if

$$\inf_{r \geq s > 0} \frac{m.d.(Proj_{\{ x_n = 0 \}}(\Omega^c \cap B_s(x)))}{s} \geq \epsilon , \tag{1.5}$$

where $Proj_\pi(S)$ is the orthogonal projection of a set S into the plane π, and our minimum diameter in (1.5) is taken with respect to $n-2$ dimensional planes in $\{ x_n = 0 \}$.

For $\epsilon > 0$, we define $\mathcal{P}_r^0(M, \sigma, \epsilon)$ to be the $u \in \mathcal{P}_r(M, \sigma)$ such that 0 is $(\epsilon, r/4)$-*thick*, and we define $\mathcal{P}_r^{0,+}(M, \sigma, \epsilon)$ to be the $u \in \mathcal{P}_r^+(M, \sigma)$ such that 0 is $(\epsilon, r/4)$-*Pthick*.

THEOREM 1.3 (Quadratic growth). *If either $u \in \mathcal{P}_1^0(M, \sigma, \epsilon)$ or $u \in \mathcal{P}_1^{0,+}(M, \sigma, \epsilon)$, then there exists a constant γ which depends on only M, σ, and ϵ such that for any $x \in B_{1/4}$ (or $B_{1/4}^+$ as appropriate)*

$$|u(x)| \leq \gamma ||x||^2. \tag{1.6}$$

REMARK 1.4 (First generalization). Our proof of the theorem above will not use the continuity or the positivity of f. It only requires $f \in L^\infty$.

REMARK 1.5 (Second generalization). Our proof of the theorem above will also not make full use of the minimum diameter condition. We need the minimum diameter condition for other parts of this work, but there is a weaker sufficient condition for the quadratic bound which can be found in the appendix at the end of this paper.

We define $\widehat{\mathcal{P}}_r(M, \sigma, \epsilon)$ to be the set of functions $u \in \mathcal{P}_r(M, \sigma)$ whose free boundary points within $r/2$ of the origin are all $(\epsilon, r/4)$-*thick.*

THEOREM 1.6 (Nonnegativity near Ω^c). *If $u \in \widehat{\mathcal{P}}_1(M, \sigma, \epsilon)$, then there exists a constant α depending on M, σ, and ϵ such that $u \geq 0$ in an α neighborhood of $\partial\Omega \cap B_{1/4}$.*

REMARK 1.7 (Necessity of thickness for nonnegativity). Examples where u becomes negative can be found in the literature. Examples where Ω^c has codimension of at least two can be given explicitly by $2x^2 - y^2$ in $\mathbb{R}^3$ or (for the contact problem) by $x^2 + xy$ in the set $\{x \geq 0\}$. On the other hand, in [KN] (387–390) there are local examples (of codimension one) where $\partial\Omega$ is given by the curves

$$x_2 = \pm x_1^{\gamma/2}, \quad 0 \leq x_1 \leq 1 \tag{1.7}$$

where $\gamma = 4k + 3$. These examples can be adapted for the problem where there is contact with a fixed boundary, and a description of a way to produce such an adaptation can be found in [SU].

After these theorems have been established, we will be able to invoke the results of the first author in [B] to conclude the following corollary. Before we state it, however, we define some notions of flatness. Let $S \subset \mathbb{R}^n$ be a compact set, and let $\gamma > 0$. Then S is γ*−Reifenberg flat* if there exists a constant $R > 0$ such that for every $x \in S$ and every $r \in (0, R]$ we have a hyperplane $L(x, r)$ containing x such that

$$D(L(x, r) \cap B_r(x),\ S \cap B_r(x)) \leq 2r\gamma\ . \tag{1.8}$$

Here D denotes the Hausdorff distance: If $A,\ B \subset \mathbb{R}^n$, then

$$D(A, B) := \max\{\ \sup_{a\in A} d(a, B)\ ,\ \sup_{b\in B} d(b, A)\ \}\ . \tag{1.9}$$

We also define the following quantity, which we call the *modulus of flatness,* to get a more quantitative and uniform measure of flatness:

$$\theta(r) := \sup_{0<\rho\leq r} \left(\sup_{x\in S} \frac{D(L(x, \rho) \cap B_\rho(x),\ S \cap B_\rho(x))}{\rho} \right)\ . \tag{1.10}$$

Finally, we will say that S is a *Reifenberg vanishing* set, if

$$\lim_{r\to 0} \theta(r) = 0\ . \tag{1.11}$$

Reifenberg flatness was introduced by Reifenberg in 1960 (see [R]), and has appeared in the work of Kenig and Toro relating boundary regularity to the regularity of the Poisson kernel (see [KT] and [T]). In particular, they introduced the notion of a Reifenberg vanishing set (see [KT]).

COROLLARY 1.8 (Boundary regularity). *If* $u \in \widehat{\mathcal{P}}_1(M, \sigma, \epsilon)$, *then* $\partial\Omega \cap B_{1/4}$ *is a Reifenberg vanishing set with* σ *as its modulus of flatness, and in particular we can conclude the following:*

1. *For any* $x \in \partial\Omega \cap B_{1/4}$

$$\lim_{s\to 0} \frac{|\Omega^c \cap B_s(x)|}{|B_s|} = \frac{1}{2}. \tag{1.12}$$

2. *If* σ *is a Hölder modulus, then* $\partial\Omega \cap B_{1/4}$ *is* $C^{1,\alpha}$.
3. *If* σ *is a Dini modulus (i.e.* $\int_0^1 (\sigma(r)/r)\, dr < \infty$*), then* $\partial\Omega \cap B_{1/4}$ *is* C^1.

Incidently, even in the case where we assume that $u \geq 0$, we still need a thickness assumption on Ω^c to get regularity. Schaeffer constructed counter-examples in [Sc] when no thickness is assumed, and Caffarelli's celebrated results in [C1] and [C2] showed that for Hölder continuous and positive f, Ω^c would be either $C^{1,\alpha}$ or "cusp-like" at any given point of its boundary. (In fact $\gamma = 4k + 1$ in Equation (1.7) admits nonnegative solutions.) For functions f which are not Dini continuous the counter-example due to the first author in [B] shows that given our assumptions above, our conclusions are sharp. (Whether the hypotheses we have above can be weakened is a subject of further investigation.)

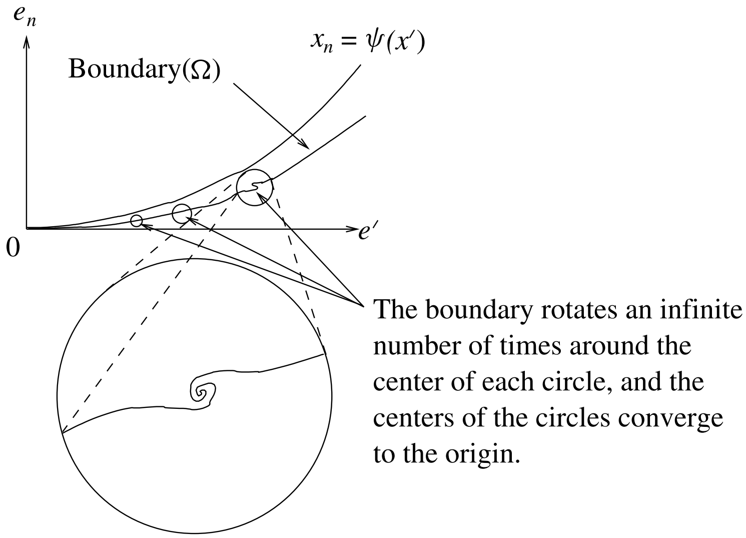
From Theorem 1.3 combined with the characterization of global solutions found in [SU] we obtain the following corollary by a simple blow-up argument. (The theorem from [SU] which we are referring to is stated as Theorem 2.2 in the next section, and several other theorems we cite are also also stated there for the reader's convenience.)

COROLLARY 1.9 (Free and fixed boundaries touch tangentially). *If $u \in \mathcal{P}_1^{0,+}(M, \sigma, \epsilon)$, then there exists a modulus of continuity ψ, and a positive constant γ which depend on only M, σ, and ϵ, such that*

$$\partial\Omega \cap B_\gamma \subset \{ (x',\ x_n) \ : x_n \leq ||x'||\psi(||x'||) \} \ . \tag{1.13}$$

REMARK 1.10. The proof of this corollary does not use more than the fact that the fixed boundary, $\{ x_n = 0 \}$, is a C^1 manifold. In [SU] the fixed boundary has to be C^3 in order to prove the necessary compactness, but here we can assume only C^1 as we get our compactness from the assumption on thickness of the zero set Ω^c.

In [SU] the fact that the free boundary touches the fixed boundary tangentially leads to a proof that $\partial\Omega$ is C^1 in a neighborhood of the point of contact. In the current situation, the counter-example in [B] can be adapted to show that even though the free boundary touches the fixed boundary tangentially, it will not in general be C^1 in a neighborhood of zero if the Laplacian is not Dini continuous. The counter-example from [B] is a function which satisfies $\Delta w = \chi_\Omega f$ for a nonnegative continuous function f, and a set Ω which has density $1/2$ on every point of its boundary, but which "spins" an infinite number of times around zero. It can be constructed by taking $u(x) := (x_1^+)^2$ and taking a σ which does not satisfy the Dini condition, and then rotating $u(x)$ in alternating dyadic annuli by an amount proportional to σ of the inner radius. It can then be shown by a short computation that the Laplacian of $u(x)$ will deviate from constant by an amount proportional to the rotation. It can also be shown that failing the Dini condition is then exactly equivalent to an infinite total rotation. In particular (since $w \geq 0$), Caffarelli's free boundary regularity alternative of [C1] and [C2] does not hold in its classical form. To adapt the example to suit our purposes, it needs to be iterated at a sequence of boundary points which converge to zero. The following picture shows how to construct



the counter-example for the current situation using the counter-example from [B], and the details of the computation are essentially contained in [B].

In the Dini case we establish C^1 contact between the fixed and free boundaries with a very mild smoothness condition. We let $\delta(x)$ denote the distance from a free boundary point x to the fixed boundary, $XB(u)$. (Note that in general we expect $\delta(x) > x_n$.) We define one final class of functions before we state our corollary. Let $\widehat{\mathcal{P}}_r^+(M, \sigma, \epsilon)$ be the set of functions $u \in \mathcal{P}_r^+(M, \sigma)$ such that all $x \in FB(u) \cap B_\epsilon$, are $(\epsilon, \delta(x)\epsilon)$-*thick*, and all $x \in IB(u) \cap B_{r/2}$, are $(\epsilon, r/4)$-*Pthick*.

COROLLARY 1.11 (C^1 contact in the Dini case). *Suppose $u \in \widehat{\mathcal{P}}_r^+(M, \sigma, \epsilon)$, and σ is a Dini modulus of continuity. Then the boundary of Ω is C^1 in a neighborhood of the origin, where the C^1 norm of the local parametrization and the size of the neighborhood will depend on M, σ, and ϵ.*

REMARK 1.12. The presence of the $\delta(x)$ term in the thickness assumption allows this condition to degenerate as we approach the fixed boundary. Allowing $\mathbb{R}^n \setminus \Omega = \Omega^c \cup D^c$ instead of just $\Omega^c = B_1^+(0) \setminus \Omega$ in Equation (1.4) is necessary for the fixed boundary case because Corollary 1.9 guarantees that otherwise this equation would never be satisfied by free boundary points sufficiently close to the fixed boundary.

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2. – Preliminary results

We collect here several results from [B], [CKS], and [SU] which are essential to this paper. We begin with two fundamental classification theorems.

THEOREM 2.1 ([CKS]). *If there exists an* $M > 0$ *such that*

$$\begin{cases} \Delta u = \chi_\Omega \text{ in } \mathbb{R}^n, \quad m.d.(\{u = |\nabla u| = 0\} \cap B_r) > 0 \text{ for an } r > 0, \\ u = |\nabla u| = 0 \text{ in } \Omega^c, \quad |u(x)| \leq M(1 + |x|^2) \text{ in } \mathbb{R}^n\,, \end{cases} \tag{2.1}$$

then $u \geq 0$ *in* $\mathbb{R}^n$, *and* $D_{ee}u \geq 0$ *in* Ω, *for any direction* e, *i.e.* Ω^c *is convex. Moreover, if*

$$\limsup_{r\to\infty} \frac{m.d.(\{u = |\nabla u| = 0\} \cap B_r)}{r} > 0, \tag{2.2}$$

then $u(x) = (\max\{x_1, 0\})^2/2$ *in some coordinate system.*

THEOREM 2.2 ([SU]). *Let* $\mathbb{R}^n_+ := \mathbb{R}^n \cap \{x_1 > 0\}$, *and let* $\Pi := \{x_1 = 0\}$. *If there exists an* $M > 0$ *such that*

$$\begin{cases} \Delta u = \chi_\Omega \quad \text{in } \mathbb{R}^n_+\,, \qquad u = |\nabla u| = 0 \qquad \text{in } \mathbb{R}^n_+ \setminus \Omega\,, \\ u = 0 \qquad \text{on } \Pi\,, \qquad |u(x)| \leq M(1 + |x|^2) \quad \text{in } \mathbb{R}^n\,, \end{cases} \tag{2.3}$$

then

$$\partial\Omega \cap \{x_1 > 0\} = \phi \;\; \text{implies} \;\; u(x) = \frac{x_1^2}{2} + a x_1 x_2 + b x_1$$

for some real numbers a, b *in some rotated coordinate system of* Π, *and*

$$\partial\Omega \cap \{x_1 > 0\} \neq \phi \;\; \text{implies} \;\; u(x) = \frac{((x_1 - a)_+)^2}{2}$$

for an $a > 0$.

Observe that if $0 \in IB(u)$ then we can conclude that

$$u(x) = \frac{x_1^2}{2} + a x_1 x_2\,.$$

We turn now to local results we use. Essentially what we need is the following theorem which combines parts of Theorems 7.1 and 7.2 of [B]:

THEOREM 2.3 ([B]). *If* $u \in \widehat{\mathcal{P}}_1(M, \sigma, \epsilon)$ *and* $u \geq 0$, *then* $\partial\Omega \cap B_{1/4}$ *is a Reifenberg vanishing set with* σ *as its modulus of flatness, and in particular we can conclude the following:*

1. *For any* $x \in \partial\Omega \cap B_{1/4}$

$$\lim_{s \to 0} \frac{|\Omega^c \cap B_s(x)|}{|B_s|} = \frac{1}{2}. \tag{2.4}$$

2. *If* σ *is a Hölder modulus, then* $\partial\Omega \cap B_{1/4}$ *is* $C^{1,\alpha}$.
3. *If* σ *is a Dini modulus (i.e.* $\int_0^1 (\sigma(r)/r)\, dr < \infty$*), then* $\partial\Omega \cap B_{1/4}$ *is* C^1.

Notice that this theorem is exactly Corollary 1.8, but with the additional assumption that $u \geq 0$. (This assumption, of course, leads to a quadratic bound on the function u, and also makes it much easier to construct solutions which can be used as barriers.)

3. – Compactness

PROOF OF THEOREM 1.3. We first deal with the case when $u \in \mathcal{P}_1^0(M, \sigma, \epsilon)$. Fix ϵ, M, and n, and define the following notation:

$$S_j(u) := \sup_{B_{2^{-j}}} |u| \qquad \text{and} \qquad \tilde{S}_r(u) := \sup_{B_r} |u|\,.$$

We claim that there exists a constant γ such that

$$S_{j+1}(u) \leq \max\left\{ \frac{\gamma 2^{-2j}}{4}, \frac{S_j(u)}{4^1}, \ldots, \frac{S_0(u)}{4^{j+1}} \right\} \tag{3.1}$$

for all j and for all $u \in \mathcal{P}_1^0(M, \sigma, \epsilon)$. In this case, if we let $\tilde{M} = \max\{\gamma, M\}$, then we have $\tilde{S}_r(u) \leq \tilde{M} r^2$, which is all we need. So we will suppose that Equation (3.1) does not hold. In this case, there exists a sequence $\{u_j\} \subset \mathcal{P}_1^0(M, \sigma, \epsilon)$, and a sequence of integers $\{k_j\}$ such that

$$S_{k_j+1}(u_j) > \max\left\{ \frac{j 2^{-2k_j}}{4}, \frac{S_{k_j}(u_j)}{4^1}, \ldots, \frac{S_0(u_j)}{4^{k_j+1}} \right\}. \tag{3.2}$$

Note that since $|u_j| \leq M$, we can conclude that we must have $k_j \to \infty$. Now define

$$v_j(x) := \frac{u_j(2^{-k_j} x)}{S_{k_j+1}(u_j)}\,, \tag{3.3}$$

and observe that in $B_{2^{k_j}}$ we have

$$|\Delta v_j| \leq \frac{||f||_\infty}{j} \to 0 \ , \tag{3.4}$$

and in B_{2^m} (for $0 \leq m \leq k_j$) we have

$$|v_j(x)| \leq \frac{S_{k_j-m}(u_j)}{S_{k_j+1}(u_j)} \leq 4(2^m)^2 \ . \tag{3.5}$$

Thus, for $||x|| \geq 1$ we have $|v_j(x)| \leq C||x||^2$, but we also have the following nondegeneracy: $||v_j||_{L^\infty(B_{1/2})} = 1$. After a rotation of coordinates we have $v_j \to v_0$ with $\Delta v_0 = 0$, $v_0(0) = |\nabla v_0(0)| = 0$, and $v_0(\frac{e_1}{2}) = 1$. So v_0 is a harmonic polynomial of degree 2, and in some system of coordinates we have

$$v_0(x) = \sum_{i=1}^{n} \alpha_i x_i^2 \qquad \text{with} \qquad \sum_{i=1}^{n} \alpha_i = 0. \tag{3.6}$$

Since v_0 is nontrivial, one of the α_i's must be nonzero. Since the sum of the α_i is zero, we must therefore have at least 2 nonzero α_i. Using this fact and the expression above, we see that ∇v_0 must vanish on a set which is the intersection of at least two orthogonal hyperplanes. (Because it must vanish on two hyperplanes and not one, we get the improvements alluded to in Remark 1.5 and explained fully in the appendix.) On the other hand,

$$m.d.(\{v_0 = |\nabla v_0| \equiv 0\} \cap B_1) \geq \epsilon \ , \tag{3.7}$$

since $\limsup \Omega^c(v_j) \subset \{v_0 = |\nabla v_0| \equiv 0\}$ which gives us the desired contradiction. (The lim sup of sets $S_j \in \mathbb{R}^n$ is defined by $S := \limsup S_j$ if $\chi_S = \limsup \chi_{S_j}$. This definition makes sense, since the lim sup of characteristic functions is still a characteristic function.)

The case when $u \in \mathcal{P}_1^{0,+}(M, \sigma, \epsilon)$ is very similar. In order to conclude that we have a quadratic harmonic polynomial in this case after the blow up, it is useful to reflect it in an odd fashion across the plane $\{ x_n = 0 \}$, and use the fact that $|\Delta u| \leq ||f||_\infty$ in B_1. The contradiction comes from Equation (1.5) for $\Omega^c(v_0)$. □

As a consequence, we obtain "blow-up" limits, and a uniform rate of convergence to these limits. First we define a rescaling:

$$u_s(x) := s^2 u(x/s) \tag{3.8}$$

and note that by Theorem 1.3 all of the "$\mathcal{P}$" classes are closed under this rescaling for $s > 1$.

PROPOSITION 3.1 (Convergence to global solutions). *Suppose that* $u \in \mathcal{P}_r^0(M, \sigma, \epsilon)$ *or* $\mathcal{P}_r^{0,+}(M, \sigma, \epsilon)$ *with* f *and* Ω *denoting the corresponding Laplacian and nonzero set. Then there exists a function* u_0 *defined on either* $\mathbb{R}^n$ *or the half-space* $\mathbb{R}^n_+$ *according to our assumption about* u (*with a nonzero set* Ω_0) *whose growth is no worse than quadratic, and which satisfies:*

1. $\Delta u_0 = f(0)\chi_{\Omega_0}$,
2. $u_0 = |\nabla u_0| = 0$ *in* Ω_0^c, *and*
3. $0 \in \partial\Omega_0$,

and there exists a sequence $\{s_k\} \to \infty$ *such that* $u_{s_k} \to u_0$ *uniformly in* $C^{1,\alpha}$ *on any compact set.*

The proof is standard so we omit it. (For a reference, see [F] pages 163 and 164.) By noting that either

$$\liminf_{R\to\infty} \frac{m.d.(\Omega_0^c \cap B_R)}{R} \geq \epsilon > 0$$

or

$$\liminf_{R\to\infty} \frac{m.d.(Proj_{\{\,x_n=0\,\}}(\Omega_0^c \cap B_R))}{R} \geq \epsilon > 0$$

and invoking either Theorem 2.1 or Theorem 2.2 according to which case we are considering, the following proposition is immediate.

COROLLARY 3.2. *With* u_0 *as in the previous proposition, after a rotation of coordinates we have*

$$u_0(x) = \frac{f(0)}{2}(x_{\mathrm{n}}^+)^2.$$

Now to prove Corollary 1.9 by contradiction, simply assume that the contact is not tangential. In that case, by doing a blow-up we can contradict the classification of the last corollary.

PROPOSIZIONE 3.3 (Convergence of nonzero sets). *There exists a modulus of continuity* ψ *which depends on* M, σ, *and* ϵ *such that for any* $u \in \mathcal{P}_1^0(M, \sigma, \epsilon)$ (*where* $u_s(x) := s^2 u(x/s)$ *and* Ω_s *is the corresponding nonzero set*) *we have the estimate:*

$$\left|\frac{|\Omega \cap B_{1/s}|}{|B_{1/s}|} - \frac{1}{2}\right| = \left|\frac{|\Omega_s \cap B_1|}{|B_1|} - \frac{1}{2}\right| \leq \psi\left(\frac{1}{s}\right) . \tag{3.9}$$

PROOF. Suppose such a ψ does not exist. Then there exists a sequence of $u_k \in \mathcal{P}_1^0(M, \sigma, \epsilon)$, a sequence of real numbers $s_k \to \infty$, and a $\delta > 0$ such that if $v_k(x) := s_k^2 u_k(x/s_k)$, and $\tilde{\Omega}_k$ is the corresponding nonzero set, then

$$\left|\frac{|\tilde{\Omega}_k \cap B_1|}{|B_1|} - \frac{1}{2}\right| \geq \delta . \tag{3.10}$$

Now take a subsequence of the v_k such that $f_k(0)$ converges. Again for simplicity, we assume that it converges to 1. Now by the same argument as in the proof of Proposition 3.1 we can produce a global solution u_0 such that $v_k \to u_0$. We will have a constant $\tilde{M}$ such that $u_0 \in \mathcal{P}_r(\tilde{M}r^2, \sigma \equiv 0)$ for all $r > 0$ (having $\sigma \equiv 0$ is just a simple way to force the Laplacian to be constant in the set Ω), and since $\limsup \tilde{\Omega}_j^c \subset \Omega_0^c$, u_0 will inherit the following property from the v_k : The zero set Ω_0^c will satisfy

$$\liminf_{R\to\infty} \frac{m.d.(\Omega_0^c \cap B_R)}{R} \geq \epsilon > 0\,. \tag{3.11}$$

Now by using Equation (3.11) and applying Theorem 2.1 again, we can conclude that in the right coordinate system $u_0(x) = \frac{1}{2}(x_n^+)^2$. Finally, we can now get a contradiction with Equation (3.10) above by using standard nondegeneracy statements based on the weak maximum principle. (See Equation (4.1) of [CKS] for example.) □

REMARK 3.4. In fact, the argument above shows more: If we let $\Omega^+ := \{ u > 0 \}$ and $\Omega^- := \{ u < 0 \}$, then with the same hypotheses as above we will have:

$$\left| \frac{|\Omega^+ \cap B_r|}{|B_r|} - \frac{1}{2} \right|, \quad \left| \frac{|\Omega^c \cap B_r|}{|B_r|} - \frac{1}{2} \right|, \quad \text{and} \quad \left| \frac{|\Omega^- \cap B_r|}{|B_r|} \right|$$

all bounded by $\psi(r)$. To produce the proofs of these statements merely change Equation (3.10) above to suit the new situation.

4. – Nonnegativity

PROOF OF THEOREM 1.6. The crucial observation in the proof of this theorem, is that if $u \in \widehat{\mathcal{P}}_1(M, \sigma, \epsilon)$ and $\tilde{y} \in \partial\Omega \cap B_{1/4}$ then

$$\tilde{u}(x) := 16\, u\left(\frac{x}{4} + \tilde{y}\right) \in \mathcal{P}_1^0(4M, \sigma, \epsilon) \tag{4.1}$$

and so we can apply Proposition 3.3 in a manner independent of u and independent of $\tilde{y}$. We wish to argue by contradiction, and note that by the translation just mentioned, it will suffice to assume that there exists a sequence of functions $\{u_k\} \subset \widehat{\mathcal{P}}_1(M, \sigma, \epsilon)$ and a sequence of points $\{x_k\} \subset \Omega_k$ (where Ω_k is the nonzero set for u_k) with

1. $u_k(x_k) < 0$, and
2. $x_k \to 0 \in \partial\Omega_k$.

Now we claim, that if we take k sufficiently large, then there exists a sequence $\{y_k\} \subset \partial\Omega_k$ with the following properties:

1. $\mathrm{dist}(x_k,\ \partial\Omega_k) = \mathrm{dist}(x_k,\ y_k)$ so y_k is one of the points in $\partial\Omega_k$ which is "closest" to x_k,
2. $\mathrm{dist}(y_k,\ 0) < \tilde{\epsilon}$ which will be a very small constant independent of k, and
3. if $r_k := \mathrm{dist}(x_k,\ y_k)$, then

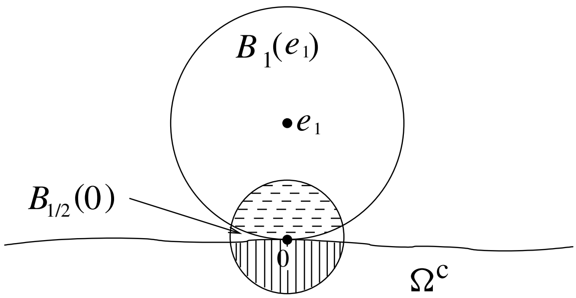
$$\left| \frac{|B_{r_k/2}(y_k) \cap \Omega_k|}{|B_{r_k/2}|} - \frac{1}{2} \right| \leq \tilde{\epsilon}\ . \tag{4.2}$$

The first two properties are fairly straightforward if k is sufficiently large. The fact that $\tilde{\epsilon}$ is still independent of u_k and k, and $\tilde{\epsilon}$ is also still as small as we like even after the assertion of the third property follows from Equation (4.1) and from the fact that ψ is independent of u in Proposition 3.3.

Now we make some rescalings and rename some things. We call $\tilde{u}_k$ the function in $\mathcal{P}_2^0(4M, \sigma, \epsilon)$ obtained by rescaling x_k to e_1 and y_k to 0, so we can obtain $\tilde{u}_k(x)$ as a rotation of the function $w_k(x)$ which we define by

$$w_k(x) := \frac{1}{(r_k)^2} u_k(y_k - r_k x).$$

We now have $\tilde{u}_k(e_1) = r_k^{-2} u_k(x_k) < 0$. Since $y_k \in \partial\Omega_k$, we now have $0 \in \partial\tilde{\Omega}_k$, where $\tilde{\Omega}_k$ is the nonzero set of $\tilde{u}_k$. At this point we have the following picture:



The dotted horizontal lines are in a region which must lie in $\tilde{\Omega}_k$ and the vertical striped region is a subset of $\tilde{\Omega}_k^c$ and has measure within $\tilde{\epsilon}$ of $\frac{1}{2}|B_{1/2}|$. After taking a limit (and using Theorem 2.1 again) we converge to $u_0(x) = \gamma(x_1^+)^2$ (for some constant γ), which contradicts the fact that $u_0(e_1) \leq 0$. $\square$

To prove Corollary 1.8 we observe that near the boundary we now have a solution of the obstacle problem. In particular, we can invoke the results from [B] immediately. (For a recent treatment of the obstacle problem see [C3] and [B].)

Before we prove Corollary 1.11, we prove a preliminary lemma.

LEMMA 4.1 (Convergence of normals at the fixed boundary). *Let $u \in \widehat{\mathcal{P}}_r^+(M, \sigma, \epsilon)$ and let σ be a Dini modulus. Then there exists a $\rho_0 = \rho_0(M, \sigma, \epsilon) > 0$ and a modulus of continuity ω_1 which also depends only on M, σ, and ϵ, such that if $n(x)$ is the interior unit normal at x (the normal must exist by Corollary 1.8), then*

$$||n(x) - e_n|| \leq \omega_1(\delta(x)) \tag{4.3}$$

(recall that $\delta(x)$ is the distance from x to the fixed boundary, $\partial\Omega \cap \{x_n = 0\}$), and $\partial\Omega(u) \cap B_{\rho_0}$ is a graph in the e_n direction.

PROOF. It suffices to prove Equation (4.3). To this end, we fix $\zeta > 0$ and show that once $\delta(x)$ is sufficiently small (where x is a point in a free boundary in our class, $\widehat{\mathcal{P}}_r^+(M, \sigma, \epsilon)$) we have

$$||n(x) - e_n|| \leq \zeta\ . \tag{4.4}$$

We take a sequence $\{u_k\} \in \widehat{\mathcal{P}}_r^+(M, \sigma, \epsilon)$, and a sequence $\{x^k\} \in \partial\Omega(u_k)$ which converges to zero, and we denote the corresponding sequence of normals by $\{n_k\}$. Let y^k be one of the points of $XB := \{x_n = 0\} \cap \partial\Omega(u_k)$ which is closest to x^k. Define

$$U_k(x) := \frac{u_k(\delta(x^k)x + y^k)}{\delta(x^k)^2}\ , \tag{4.5}$$

and define

$$X^k := \frac{x^k - y^k}{\delta(x^k)}\ . \tag{4.6}$$

Note that the normal to the free boundary at X^k is still n_k, note that $|X^k| = 1$, and note that if we extend U_k to be 0 in $\{x_n \leq 0\}$, then it is a solution of the local problem in a ball of radius one around X^k.

Our rescaling has eliminated the $\delta(x)$ term which appears in our assumptions, and so we have a uniform thickness of our zero set in a neighborhood of X^k. Because of the uniform thickness we have, we can apply Theorem 1.6 to conclude that $U_k \geq 0$ in a uniform neighborhood of X^k, and at that point, Theorem 7.2 of [B] shows that if x is a free boundary point in a small (but uniform) neighborhood around X^k and n is its normal vector, then

$$||n - n_k|| \leq C \sum_{m=0}^{\infty} \sigma(\delta(x_k)2^{-m}) \leq C \int_0^1 \frac{\sigma(\delta(x_k)t)}{t}\,dt\ . \tag{4.7}$$

By Corollary 1.9 we know that $X_n^k \leq \psi(\delta(x_k))$, where ψ is the modulus of continuity given in that corollary. On the other hand, we know that the free boundary cannot touch $x_n = 0$ within $B_1(X^k)$. This fact along with Equation (4.7) leads to an estimate of the form $||n_k - e_n|| \leq \zeta$, if $\delta(x_k)$ is sufficiently small. □

PROOF OF THEOREM 1.11. Fix $\epsilon > 0$. Choose d sufficiently small to ensure that (with ω_1 as given in the previous lemma) $\omega_1(d) \leq \epsilon/2$. If x and y are two points on the free boundary, with $\delta(x)$ and $\delta(y)$ less than d, and $n(x)$ and $n(y)$ are the corresponding normal vectors, then by the previous lemma we get

$$||n(x) - n(y)|| \leq ||n(x) - e_n|| + ||e_n - n(y)|| \leq \epsilon/2 + \epsilon/2 \ . \tag{4.8}$$

Now if $\delta(x)$ and $\delta(y)$ are both greater than $d/2$, then by assumption we have a uniform thickness estimate, and therefore we can apply Theorem 7.2 of [B] to conclude that we have a uniform modulus of continuity of the normal vectors which we will call ω_2. Let η be chosen small enough so that $\omega_2(\eta) \leq \epsilon$. If x and y are free boundary points, and

$$||x - y|| \leq \min\{d/2, \eta\} \tag{4.9}$$

then either $\delta(x)$ and $\delta(y)$ will both be less than d and Equation (4.8) will apply, or $\delta(x)$ and $\delta(y)$ will both be more than $d/2$ and we can use ω_2 to get the desired result. □

Appendix

Now we discuss a variant of the minimum diameter condition which leads to a weaker version of (ϵ, r)-thickness which is still strong enough to ensure a quadratic bound. One way of expressing minimum diameter of a set S, is to find the smallest γ such that after an orthonormal change of coordinates one has

$$S \subset \{x_1^2 \leq (\gamma/2)^2\}.$$

Obviously this condition forces the set to be "thin" in one direction. The content of Theorem 1.3 is that if the zero set is (ϵ, r)-*thick* at zero, then there is a quadratic bound. Recall that a point x of the free boundary is (ϵ, r)-*thick* if

$$\inf_{r \geq s > 0} \frac{m.d.(\{\Omega^c \cup D^c\} \cap B_s(x))}{s} \geq \epsilon \ .$$

For the sake of the quadratic bound only, this condition (the inequality in the definition of (ϵ, r)-thickness) can be improved by replacing $m.d.$ with the γ which is the minimum among the γ's where we have

$$S \subset \{x_1^2 + x_2^2 \leq (\gamma/2)^2\}$$

after an orthonormal change of coordinates. In other words, the sharper condition forces the zero set to be thin in two directions. In terms of the fact that the proof of Theorem 1.3 is still valid with our weaker assumptions, it is helpful to recall that the zero sets of nontrivial homogeneous quadratic harmonic polynomials were linear subspaces which always had codimension of at least two. (By zero set here we of course mean the set where the function and the gradient vanish.)

This condition is an improvement of the capacity density condition of [KS] which also gives a quadratic bound. A set with capacity greater than ϵ cannot be arbitrarily thin in two different directions, or in other words, sets of codimension two have capacity zero. If a set has small capacity, then one has control of its size, but our minimum diameter condition involves a control of size which is as strong as the control given by capacity, but also involves a control of shape. (Thus, more sets will end up being "thick" in this sense, and therefore our criterion for the quadratic bound is weaker.)

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