

Peak Solutions for an Elliptic System of FitzHugh-Nagumo Type

EDWARD NORMAN DANCER – SHUSEN YAN

Abstract. The aim of this paper is to study the existence of various types of peak solutions for an elliptic system of FitzHugh-Nagumo type. We prove that the system has a single peak solution, which concentrates near the boundary of the domain. Under some extra assumptions, we also construct multi-peak solutions with all the peaks near the boundary, and a single peak solution with its peak near an interior point of the domain.

Mathematics Subject Classification (2000): 35J50 (primary), 93C15 (secondary).

1. – Introduction

In this paper we consider the following problem:

$$(1.1) \quad \begin{cases} -\varepsilon^2 \Delta u = f(u) - v, & \text{in } \Omega, \\ -\Delta v + \gamma v = \delta u, & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in \mathbb{R}^N , ε is a parameter, γ and δ are nonnegative constants, $f(t) = t(t - a)(1 - t)$, $a \in (0, \frac{1}{2})$.

Solutions of (1.1) are the steady state solutions of the following reaction diffusion systems of the FitzHugh-Nagumo type [8], [13]:

$$u_t = \varepsilon^2 \Delta u + f(u) - v, \quad v_t = \Delta v - \gamma v + \delta u,$$

which is a simplification of the original Hodgkin-Huxley nerve conduction equations [10]. This system can also be used as a model for other problems arising from the applied areas. The readers can find more references in [22] for background on the systems of FitzHugh-Nagumo type. Here we mention some early

This work was supported by the ARC.

Pervenuto alla Redazione il 10 marzo 2003.

results obtained by Klaasen and Troy [12], Klaasen and Mitidieri [11], and De-Figueiredo and Mitidieri [7]. Some recent results on these systems can be found in [17], [18], [19], [20], [6], [22].

For each $u \in H_0^1(\Omega)$, let $G_\gamma u$ be the unique solution of the following problem:

$$\begin{cases} -\Delta v + \gamma v = u, & \text{in } \Omega, \\ v = 0, & \text{on } \partial\Omega. \end{cases}$$

Then we see (1.1) is equivalent to the following nonlocal elliptic problem:

$$(1.2) \quad \begin{cases} -\varepsilon^2 \Delta u + \delta G_\gamma u = f(u), & \text{in } \Omega, \\ u \in H_0^1(\Omega). \end{cases}$$

The energy associated with (1.2) is

$$(1.3) \quad I(u) = \frac{1}{2} \int_{\Omega} (\varepsilon^2 |Du|^2 + \delta u G_\gamma u) - \int_{\Omega} F(u), \quad u \in H_0^1(\Omega).$$

It is easy to see from $\int_{\Omega} u G_\gamma u = \int_{\Omega} (|DG_\gamma u|^2 + \gamma |G_\gamma u|^2) \geq 0$, that $u = 0$ is a local minimizer of $I(u)$ in $H_0^1(\Omega)$. On the other hand, we can check easily that there are $0 < \tau_1 < \tau_2$ such that $f(\tau_1) < 0$, $f(\tau_2) > 0$, $f'(t) < 0$ if $t \in (-\infty, \tau_1) \cup (\tau_2, +\infty)$, and $f'(t) > 0$ if $t \in (\tau_1, \tau_2)$. Besides, $f(t)$ has exactly three zero points $0, a, 1$, and

$$\int_0^1 f(s) ds > 0,$$

because $a \in (0, \frac{1}{2})$. Moreover, $f(t) \rightarrow +\infty$ as $t \rightarrow -\infty$, $f(t) \rightarrow -\infty$ as $t \rightarrow +\infty$.

Suppose that $\delta = 0$. Then (1.1) becomes an equation. It follows from [2] that the global minimizer of (1.3) converges to 1 uniformly on any compact subset of Ω . On the other hand, it is proved in [14] that, if Ω is convex, the mountain pass type solution is a single peak solution with the peak locating near the center of the domain. Moreover, if Ω is a ball [9], or more generally if Ω has certain kind of symmetry [3], we know that (1.1) has exactly two nontrivial solutions for $\varepsilon > 0$ small. So in this special case, we know at least for the domains with some kind of symmetry that the solution set is quite simple and we also know the profile of these solutions.

Suppose that $\delta > 0$. We prove in [6] that if $\delta > 0$ is larger than some constant, then the global minimizer oscillates on a set of positive measure. Besides, $I(u)$ also has a nontrivial local minimizer. Thus, at least for $\delta > 0$ not too small, the solution set for the system is quite different from the equation case and the profile of the solutions for the system is much more complicated. On the other hand, if Ω is a ball and $\delta > 0$ is small enough, we may ask whether the solution set for (1.1) and the profile of these solutions are similar to the case $\delta = 0$.

In this paper, for any fixed $\delta > 0$, we will construct peak solutions for (1.2) for $\varepsilon > 0$ small. The result we obtain in this paper presents a striking contrast to the result in [14] for the single equation case because the mountain pass type solution for the system is a single peak solution with the peak close to the boundary. Moreover, if Ω is a ball, we will show that (1.2) also has a k -peak solution for any positive integer k . So we see that for any $\delta > 0$, the profile of some solutions is quite different from the case $\delta = 0$, and the number of the solutions in the system case is larger than that in the case $\delta = 0$.

Before we state our results, we give some notation.

Let $U(y) = U(|y|)$ be the unique positive solution of the following problem:

$$(1.4) \quad \begin{cases} -\Delta U = f(U), & \text{in } \mathbb{R}^N, \\ U \in H^1(\mathbb{R}^N). \end{cases}$$

Note also that this solution is nondegenerate. See for example [16]. Denote $U_{\varepsilon,x}(y) = U(\frac{y-x}{\varepsilon})$.

For any $u \in H^1(\Omega)$, let $P_{\varepsilon,\Omega}u$ be the solution of

$$\begin{cases} -\varepsilon^2 \Delta P_{\varepsilon,\Omega}u + m^2 P_{\varepsilon,\Omega}u = f(u) + m^2 u, & \text{in } \Omega, \\ P_{\varepsilon,\Omega}u \in H_0^1(\Omega), \end{cases}$$

where $m = \sqrt{-f'(0)}$.

We have:

THEOREM 1.1. *Suppose that $\delta > 0$. There is an $\varepsilon_0 > 0$, such that for each $\varepsilon \in (0, \varepsilon_0]$, (1.2) has a solution u_ε of the form:*

$$u_\varepsilon = P_{\varepsilon,\Omega}U_{\varepsilon,x_\varepsilon} + \omega_\varepsilon,$$

where $x_\varepsilon \in \Omega$ satisfies $d(x_\varepsilon, \partial\Omega) \rightarrow 0$, $\frac{d(x_\varepsilon, \partial\Omega)}{\varepsilon} \rightarrow +\infty$, as $\varepsilon \rightarrow 0$, and ω_ε satisfies

$$\int_{\Omega} (\varepsilon^2 |D\omega_\varepsilon|^2 + \omega_\varepsilon^2) = o(\varepsilon^N).$$

It is natural to ask which point on the boundary x_ε will converge to. This problem is quite technical because the contribution to the energy from the boundary is hardly seen due to the fact $\frac{d(x_\varepsilon, \partial\Omega)}{\varepsilon} \rightarrow +\infty$ as $\varepsilon \rightarrow 0$.

The solution obtained in Theorem 1.1 does not have least energy among all the nontrivial solutions of (1.1) because (1.1) has a nontrivial global minimizer. From the construction of the solution in Theorem 1.1, we see that this solution has least energy among all the possible solutions of the form $P_{\varepsilon,\Omega}U_{\varepsilon,x_\varepsilon} + \omega_\varepsilon$. On the other hand, if we replace $f(u)$ by the typical superlinear nonlinearity $u^{p-1} - u$, $p \in (2, 2N/(N-2))$, then it is standard to prove that the mountain pass solution has least energy among all the nontrivial solutions, and using the estimates in this paper, we can also show that this mountain pass solution has the form $P_{\varepsilon,\Omega}U_{\varepsilon,x_\varepsilon} + \omega_\varepsilon$ with $d(x_\varepsilon, \partial\Omega) \rightarrow 0$ and $\frac{d(x_\varepsilon, \partial\Omega)}{\varepsilon} \rightarrow +\infty$ as $\varepsilon \rightarrow 0$.

If $\partial\Omega$ has several disjoint connected components, using the same method as in the proof of Theorem 1.1, we can prove (1.1) also has multipeak solutions for $\varepsilon > 0$ small. See Remark 3.4.

Next, we show the existence of multipeak solutions for (1.2) if Ω has certain kind of symmetry. Let $\mathbf{O}(N)$ be the set of all orthogonal transformation in \mathbb{R}^N . Write $\mathbb{R}^N = \mathbb{R}^{\tilde{N}} \oplus \mathbb{R}^{N-\tilde{N}}$, where \tilde{N} is an integer satisfying $1 \leq \tilde{N} \leq N$. For any $x \in \mathbb{R}^N$, denote $x = (x', x'')$, $x' \in \mathbb{R}^{\tilde{N}}$, $x'' \in \mathbb{R}^{N-\tilde{N}}$. We assume that Ω has the symmetry defined as follows:

- (Ω_1) There are integer $1 \leq \tilde{N} \leq N$, and a finite cyclic subgroup G of $\mathbf{O}(N)$ generated by g , that is, $G = \{g, g^2, \dots, g^k = id\}$ for some integer $k > 1$, where $g \in \mathbf{O}(N)$ satisfies $gx = x$ for any $x = (0, x'') \in \mathbb{R}^N$, and $g^i x \neq x$, $\forall x = (x', 0) \in \partial\Omega$, $i = 1, \dots, k - 1$, such that $G\Omega = \Omega$;
- (Ω_2) If $(x', x_{\tilde{N}+1}, \dots, x_i, \dots, x_N) \in \Omega$, then $(x', x_{\tilde{N}+1}, \dots, -x_i, \dots, x_N) \in \Omega$.

THEOREM 1.2. *Suppose that $\delta > 0$. Assume that there are integer $1 \leq \tilde{N} \leq N$ and a group G , such that (Ω_1) and (Ω_2) hold. Then there is an $\varepsilon_0 > 0$, such that for each $\varepsilon \in (0, \varepsilon_0]$, (1.2) has a solution u_ε of the form:*

$$u_{\varepsilon,G} = \sum_{i=1}^k P_{\varepsilon,\Omega} U_{\varepsilon,g^i x_\varepsilon} + \omega_{\varepsilon,G},$$

where $x_\varepsilon = (x'_\varepsilon, 0) \in \Omega$ satisfies $d(x_\varepsilon, \partial\Omega) \rightarrow 0$, $\frac{d(x_\varepsilon, \partial\Omega)}{\varepsilon} \rightarrow +\infty$, as $\varepsilon \rightarrow 0$, $\omega_{\varepsilon,G}$ satisfies $\omega_{\varepsilon,G}(g^i y) = \omega_{\varepsilon,G}(y)$, $i = 1, \dots, k - 1$, $\omega_{\varepsilon,G}(y', -y'') = \omega_{\varepsilon,G}(y)$ and

$$\int_{\Omega} (\varepsilon^2 |D\omega_{\varepsilon,G}|^2 + \omega_{\varepsilon,G}^2) = o(\varepsilon^N).$$

EXAMPLE 1.3. Suppose that Ω is a ball in \mathbb{R}^N . Let $\tilde{N} = 2$. For any integer $k \geq 2$, we may choose g being the rotation of angle $2\pi/k$ in \mathbb{R}^2 . It is easy to see that $g^k = id$. So it follows from Theorem 1.2 that for any fixed $\delta > 0$, (1.2) has a k -peak solution if $\varepsilon > 0$ is small enough.

EXAMPLE 1.4. Suppose that Ω satisfies the following condition:

$$(x_1, \dots, -x_i, \dots, x_N) \in \Omega \text{ if } (x_1, \dots, x_i, \dots, x_N) \in \Omega, \forall i = 1, \dots, N.$$

We may take $\tilde{N} = 1$ and $g \in \mathbf{O}(\mathbb{R}^N)$ with $g(x_1, x_2, \dots, x_N) = (-x_1, x_2, \dots, x_N)$. By Theorem 1.2, we see that (1.2) has a double peak solution.

From the argument, we see that Theorems 1.1 and 1.2 hold for more general nonlinearities $f(t)$ if $f'(0) < 0$, and the corresponding problem (1.4) has a nondegenerate positive solution. Besides, Theorem 1.2 remains true for more general finite group, which fixes $\mathbb{R}^{N-\tilde{N}}$ and acts freely on $\partial\Omega \cap \mathbb{R}^{\tilde{N}}$.

Finally, we study the existence of solution for (1.1), which has a peak at some interior point of the domain.

Let $I_{-1} = (-\infty, \tau_1)$, $I_0 = (\tau_1, \tau_2)$ and $I_1 = (\tau_2, +\infty)$. Let $u = h_+(v)$, $u = h_0(v)$ and $u = h_-(v)$ be the inverse function of $v = f(u)$ restricted to I_1 , I_0 and I_{-1} respectively.

We extend $h_+(v)$ continuously into $v \in (f(\tau_2), +\infty)$ in such a way that $h_+(v)$ is decreasing. Then since $h_+(v)$ is decreasing, it is easy to see that the following problem has a unique solution v_δ :

$$(1.5) \quad \begin{cases} -\Delta v + \gamma v = \delta h_+(v), & \text{in } \Omega, \\ v \in H_0^1(\Omega). \end{cases}$$

Moreover, by using the maximum principle, we can deduce easily that $v_{\delta_1} < v_{\delta_2}$ if $\delta_1 < \delta_2$.

By the comparison theorem, it is easy to see that $\max_{x \in \Omega} v_\delta(x) \rightarrow +\infty$ as $\delta \rightarrow +\infty$. So, there is a unique $\delta_0 > 0$, such that

$$\max_{x \in \Omega} v_{\delta_0}(x) = \alpha_0,$$

where $\alpha_0 > 0$ is the constant such that $\int_{h_-(\alpha_0)}^{h_+(\alpha_0)} (f(s) - \alpha_0) ds = 0$. Let us emphasize here that from the definition of α_0 , we have $\int_{h_-(\alpha)}^{h_+(\alpha)} (f(s) - \alpha) ds > 0$ if $\alpha < \alpha_0$, and $\int_{h_-(\alpha)}^{h_+(\alpha)} (f(s) - \alpha) ds < 0$ if $\alpha > \alpha_0$.

In [6], we proved the following existence of a nontrivial local minimizer of $I(u)$.

THEOREM 1.5. *Let $\bar{\delta} > \delta_0$ be the number such that $\max_{x \in \Omega} v_{\bar{\delta}}(x) = f(\tau_2)$, where $v_{\bar{\delta}}$ is the solution of (1.5) with $\delta = \bar{\delta}$. Suppose that $\delta \in (\delta_0, \bar{\delta})$. Then there is an $\varepsilon_0 > 0$, such that for $\varepsilon \in (0, \varepsilon_0]$, (1.1) has a solution $(\bar{u}_\varepsilon, \bar{v}_\varepsilon)$, satisfying*

- (i) $\bar{v}_\varepsilon \rightarrow \bar{v}$ in $C^{1,\sigma}(\Omega)$, for any $\sigma \in (0, 1)$, where \bar{v} is the solution of (1.5);
- (ii) $\bar{u}_\varepsilon \rightarrow h_+(\bar{v})$ uniformly in any compact subset of Ω ;
- (iii) \bar{u}_ε is a local minimizer of $I_\varepsilon(u)$.

Our next result shows we can attach downward a peak to this nontrivial local minimizer to obtain a new solution for (1.2). Let $b = \max_{x \in \Omega} \bar{v}(x)$. Consider

$$(1.6) \quad \begin{cases} -\Delta U_b = q_b(U_b), & \text{in } \mathbb{R}^N, \\ U_b \in H^1(\mathbb{R}^N), \end{cases}$$

where $q_b(t) = t(t - (h_+(b) - h_0(b)))(h_+(b) - h_-(b) - t)$. Since $\delta \in (\delta_0, \bar{\delta})$, we know that $b \in (\alpha_0, f(\tau_2))$. So, noting that $f(t) - b = (t - h_-(b))(t - h_0(b))(h_+(b) - t)$ and $\int_{h_-(b)}^{h_+(b)} (f(s) - b) ds < 0$ if $b > \alpha_0$, we see easily that (using $\tau = h_+(b) - t$),

$$\int_0^{h_+(b)-h_-(b)} q_b(s) ds = - \int_{h_-(b)}^{h_+(b)} (f(\tau) - b) d\tau > 0.$$

As a result, (1.6) has a unique solution which is also nondegenerate. See for example [16].

Let $U_b(x) = U_b(|x|)$ be the solution (1.6) and let $U_{\varepsilon,x,b}(y) = U_b(\frac{y-x}{\varepsilon})$. Denote $\bar{m}^2 = -q'_b(0)$. For any $u \in H^1(\Omega)$, let $\bar{P}_{\varepsilon,\Omega}u$ be the solution of

$$\begin{cases} -\varepsilon^2 \Delta \bar{P}_{\varepsilon,\Omega}u + \bar{m}^2 \bar{P}_{\varepsilon,\Omega}u = q_b(u) + \bar{m}^2 u, & \text{in } \Omega, \\ \bar{P}_{\varepsilon,\Omega}u \in H_0^1(\Omega). \end{cases}$$

Then we have:

THEOREM 1.6. *Suppose that $\delta \in (\delta_0, \bar{\delta})$. There is an $\varepsilon_0 > 0$, such that for each $\varepsilon \in (0, \varepsilon_0]$, (1.2) has a solution \tilde{u}_ε of the form*

$$\tilde{u}_\varepsilon = \bar{u}_\varepsilon - \bar{P}_{\varepsilon,\Omega}U_{\varepsilon,x_\varepsilon,b} + \omega_\varepsilon,$$

where \bar{u}_ε is the nontrivial local minimum obtained in Theorem 1.5, $x_\varepsilon \in \Omega$ satisfies $x_\varepsilon \rightarrow x_0 \in \Omega$, with $\bar{v}(x_0) = \max_{x \in \Omega} \bar{v}(x)$, as $\varepsilon \rightarrow 0$, and

$$\int_{\Omega} (\varepsilon^2 |D\omega_\varepsilon|^2 + \omega_\varepsilon^2) = o(\varepsilon^N).$$

This paper is arranged as follows. In Section 2, we estimate $G_\gamma U_{\varepsilon,x}$. As we will see that the main contributions to the energy $I(P_{\varepsilon,\Omega}U_{\varepsilon,x})$ are from the term $\int_{\Omega} U_{\varepsilon,x} G_\gamma U_{\varepsilon,x}$ as well as the geometry of the domain. Thus the estimates in Section 2 play a very important role in the proof of Theorem 1.1. Theorems 1.1 and 1.2 and Theorem 1.6 are proved in Section 3 and Section 4 and Section 5 respectively by using the reduction method.

2. – Preliminaries

In this section we will estimate $G_\gamma U_{\varepsilon,x}$. We need to treat the case $N \geq 3$, the case $N = 2$ and the case $N = 1$ differently due to the different behaviours of the corresponding fundamental solutions.

Suppose that $N \geq 3$. Let W_ε be the solution of the following problem:

$$(2.1) \quad \begin{cases} -\Delta w + \varepsilon^2 \gamma w = U, & \text{in } \mathbb{R}^N, \\ w(|x|) \rightarrow 0 & \text{as } |x| \rightarrow +\infty. \end{cases}$$

LEMMA 2.1. *If $N \geq 3$, we have*

$$\begin{aligned} G_\gamma U_{\varepsilon,x}(y) &= \varepsilon^2 W_\varepsilon \left(\frac{y-x}{\varepsilon} \right) - \varepsilon^N B H(x, y) \\ &+ O \left(\frac{\varepsilon^{N+2}}{d(x, \partial\Omega)^N} + \frac{\varepsilon^N e^{-(m-\theta)d(x, \partial\Omega)/\varepsilon}}{d(x, \partial\Omega)^{N-2}} \right), \end{aligned}$$

where $\theta > 0$ is any small constant, $B = \int_{\mathbb{R}^N} U$, $H(y, x)$ is the regular part of the Green function of the operator $-\Delta + \gamma$ with Dirichlet boundary condition.

PROOF. Let $S(y, x)$ be the solution of

$$\begin{cases} -\Delta S + \gamma S = \delta_x, & \text{in } \mathbb{R}^N, \\ S(y, x) \rightarrow 0, & \text{as } |y| \rightarrow +\infty, \end{cases}$$

where δ_x is the Dirac measure. Then

$$G_\gamma U_{\varepsilon,x}(y) = \int_{\Omega} (S(z, y) - H(z, y)) U_{\varepsilon,x}(z) dz.$$

Noting that $|D^l H(z, y)| \leq \frac{C}{d(z, \partial\Omega)^{N-2+l}}$, $l = 0, 1, 2$, using the mean value theorem, we have

$$\begin{aligned} & \int_{\Omega} H(z, y) U_{\varepsilon,x}(z) dz \\ &= \int_{B_{(1-\theta)d(x, \partial\Omega)}(x)} H(z, y) U_{\varepsilon,x}(z) dz + \int_{\Omega \setminus B_{(1-\theta)d(x, \partial\Omega)}(x)} H(z, y) U_{\varepsilon,x}(z) dz \\ &= \int_{B_{(1-\theta)d(x, \partial\Omega)}(x)} H(x, y) U_{\varepsilon,x}(z) dz \\ & \quad + \frac{1}{2} \int_{B_{(1-\theta)d(x, \partial\Omega)}(x)} \langle D^2 H(\xi(z), y)(z - x), z - x \rangle U_{\varepsilon,x}(z) dz \\ & \quad + O\left(\frac{\varepsilon^N e^{-(m-\theta)d(x, \partial\Omega)/\varepsilon}}{d(x, \partial\Omega)^{N-2}}\right) \\ &= \varepsilon^N H(x, y) \int_{\mathbb{R}^N} U(z) dz + O\left(\frac{\varepsilon^{N+2}}{d(x, \partial\Omega)^N} + \frac{\varepsilon^N e^{-(m-\theta)d(x, \partial\Omega)/\varepsilon}}{d(x, \partial\Omega)^{N-2}}\right). \end{aligned}$$

On the other hand,

$$\begin{aligned} \int_{\Omega} S(z, y) U_{\varepsilon,x}(z) dz &= \varepsilon^N \int_{\Omega_{\varepsilon,x}} S(\varepsilon z + x, y) U(z) dz \\ &= \varepsilon^N \int_{\mathbb{R}^N} S(\varepsilon z + x, y) U(z) dz + \varepsilon^N O(e^{-(m-\theta)d(x, \partial\Omega)/\varepsilon}). \end{aligned}$$

But $\varepsilon^{N-2} S(\varepsilon z + x, y)$ is the solution of

$$\begin{cases} -\Delta w + \varepsilon^2 \gamma w = \delta_{(y-x)/\varepsilon}, & \text{in } \mathbb{R}^N, \\ w(y) \rightarrow 0, & \text{as } |y| \rightarrow +\infty. \end{cases}$$

As a result,

$$\varepsilon^{N-2} \int_{\mathbb{R}^N} S(\varepsilon z + x, y) U(z) dz = W_\varepsilon \left(\frac{y-x}{\varepsilon} \right)$$

and W_ε is the solution of (2.1). Thus the result follows. □

To obtain an expansion for $G_\gamma U_{\varepsilon,x}(y)$ in the case $N = 2$, we need to introduce some notation first.

For any $x \in \Omega$, let $\bar{S}(y, x)$ be the solution of

$$(2.2) \quad \begin{cases} -\Delta u + \gamma u = \frac{\gamma}{2\pi} \ln \frac{1}{|y-x|}, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

By the L^p estimate of the elliptic equation, we know that there is a $C > 0$, independent of $x \in \Omega$, such that $\|\bar{S}\|_{C^1(\Omega)} \leq C$. Let $\bar{H}(y, x)$ be the solution of

$$(2.3) \quad \begin{cases} -\Delta u + \gamma u = 0, & \text{in } \Omega, \\ u = \frac{1}{2\pi} \ln \frac{1}{|y-x|}, & \text{on } \partial\Omega. \end{cases}$$

Then, we see that $\frac{1}{2\pi} \ln \frac{1}{|y-x|} - \bar{H}(y, x)$, where $\tilde{H}(y, x) = \bar{S}(y, x) + \bar{H}(y, x)$, satisfies

$$(2.4) \quad \begin{cases} -\Delta u + \gamma u = \delta_x, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

LEMMA 2.2. *If $N = 2$, then*

$$G_\gamma U_{\varepsilon,x}(y) = \frac{1}{2\pi} B \varepsilon^2 \ln \frac{1}{\varepsilon} + \varepsilon^2 \tilde{W} \left(\frac{y-x}{\varepsilon} \right) - \varepsilon^2 B \tilde{H}(y, x) + \varepsilon^2 O \left(\frac{\varepsilon^2}{d(x, \partial\Omega)^2} + e^{-(m-\theta)d(x, \partial\Omega)/\varepsilon} \ln \frac{1}{d(x, \partial\Omega)} \right),$$

where $B = \int_{\mathbb{R}^2} U$, $\tilde{W}(y) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \ln \frac{1}{|z-y|} U(z) dz$.

PROOF. We have

$$(2.5) \quad G_\gamma U_{\varepsilon,x}(y) = \int_{\Omega} \left(\frac{1}{2\pi} \ln \frac{1}{|z-y|} - \tilde{H}(z, y) \right) U_{\varepsilon,x}(z) dz.$$

Similarly to the proof of Lemma 2.1, we have

$$\begin{aligned} & \int_{\Omega} \tilde{H}(z, y) U_{\varepsilon,x}(z) dz \\ &= \varepsilon^2 \tilde{H}(x, y) \int_{\mathbb{R}^2} U + \varepsilon^2 O \left(\frac{\varepsilon^2}{d(x, \partial\Omega)^2} + e^{-(m-\theta)d(x, \partial\Omega)/\varepsilon} \ln \frac{1}{d(x, \partial\Omega)} \right). \end{aligned}$$

But

$$\begin{aligned} & \int_{\Omega} \ln \frac{1}{|z-y|} U_{\varepsilon,x}(z) dz = \varepsilon^2 \int_{\Omega_{\varepsilon,x}} \left(\ln \frac{1}{\varepsilon} + \ln \frac{1}{|z - \frac{y-x}{\varepsilon}|} \right) U(z) \\ &= \varepsilon^2 \ln \frac{1}{\varepsilon} \int_{\mathbb{R}^2} U + \varepsilon^2 \int_{\mathbb{R}^2} \ln \frac{1}{|z - \frac{y-x}{\varepsilon}|} U(z) dz + \varepsilon^2 \ln \frac{1}{\varepsilon} O(e^{-(m-\theta)d(x, \partial\Omega)/\varepsilon}) \\ &= B \varepsilon^2 \ln \frac{1}{\varepsilon} + \varepsilon^2 \tilde{W} \left(\frac{y-x}{\varepsilon} \right) + \varepsilon^2 \ln \frac{1}{\varepsilon} O(e^{-(m-\theta)d(x, \partial\Omega)/\varepsilon}). \end{aligned}$$

Thus the result follows. □

Now we deal with the case $N = 1$.

LEMMA 2.3. *Suppose that $N = 1$. Then*

$$G_\gamma U_{\varepsilon,x}(y) = BG_\gamma(x, y)\varepsilon + O(\varepsilon^2),$$

where $G_\gamma(x, y)$ is the Green function of $-\Delta + \gamma$ subject to the Dirichlet boundary condition.

PROOF. Since $G_\gamma(z, y)$ is Lipschitz continuous, we have

$$\begin{aligned} G_\gamma U_{\varepsilon,x}(y) &= \int_\Omega G_\gamma(z, y)U_{\varepsilon,x}(z) dz = G_\gamma(x, y) \int_\Omega U_{\varepsilon,x}(z) dz + O(\varepsilon^2) \\ &= BG_\gamma(x, y)\varepsilon + O(\varepsilon^2). \end{aligned}$$

□

Let

$$\tau_{\varepsilon,x} = \int_\Omega h(U_{\varepsilon,x})(U_{\varepsilon,x} - P_{\varepsilon,\Omega}U_{\varepsilon,x}),$$

where $h(t) = f(t) + mt$. Then, from [4], [5], we know that for any small $\eta > 0$, there are $c_2 \geq c_1 > 0$, such that

$$c_1\varepsilon^N e^{-m(2+\eta)d(x,\partial\Omega)/\varepsilon} \leq \tau_{\varepsilon,x} \leq c_2\varepsilon^N e^{-m(2-\eta)d(x,\partial\Omega)/\varepsilon}.$$

We have:

PROPOSITION 2.4. *If $N \geq 3$, we have*

$$\begin{aligned} I(P_{\varepsilon,\Omega}U_{\varepsilon,x}) &= \varepsilon^N A + \delta A_\varepsilon \varepsilon^{N+2} + \tau_{\varepsilon,x} - \frac{\delta}{2} B^2 \varepsilon^{N+2} \varepsilon^{N-2} H(x, x) \\ (2.6) \quad &+ \varepsilon^N O\left(e^{-(2+\sigma)md(x,\partial\Omega)/\varepsilon} + \frac{\varepsilon^{N+2}}{d(x,\partial\Omega)^N} + \frac{\varepsilon^N e^{-(m-\theta)d(x,\partial\Omega)/\varepsilon}}{d(x,\partial\Omega)^{N-2}}\right), \end{aligned}$$

where $A = \frac{1}{2} \int_{\mathbb{R}^N} |DU|^2 - \int_{\mathbb{R}^N} F(U)$, $A_\varepsilon = \frac{1}{2} \int_{\mathbb{R}^N} U W_\varepsilon$, $\sigma > 0$ is a constant. If $N = 2$, we have

$$\begin{aligned} I(P_{\varepsilon,\Omega}U_{\varepsilon,x}) &= \varepsilon^2 A + \frac{\delta}{4\pi} B \varepsilon^4 \ln \frac{1}{\varepsilon} + \delta \tilde{A} \varepsilon^4 + \tau_{\varepsilon,x} - \frac{\delta}{2} B^2 \varepsilon^4 \tilde{H}(x, x) \\ (2.7) \quad &+ \varepsilon^2 O\left(e^{-(2+\sigma)md(x,\partial\Omega)/\varepsilon} + \frac{\varepsilon^4}{d(x,\partial\Omega)^2} + \varepsilon^2 e^{-(m-\theta)d(x,\partial\Omega)/\varepsilon} \ln \frac{1}{d(x,\partial\Omega)}\right), \end{aligned}$$

where $\tilde{A} = \frac{1}{2} \int_{\mathbb{R}^2} U \tilde{W}$. If $N = 1$, we have

$$\begin{aligned} (2.8) \quad I(P_{\varepsilon,\Omega}U_{\varepsilon,x}) &= \varepsilon A + \frac{1}{2} \delta B^2 G_\gamma(x, x) \varepsilon^2 + \tau_{\varepsilon,x} \\ &+ \varepsilon O(e^{-(2+\sigma)md(x,\partial\Omega)/\varepsilon} + \varepsilon^{1+\sigma}). \end{aligned}$$

PROOF. We know that [4], [5]

$$(2.9) \quad \frac{1}{2} \int_{\Omega} \varepsilon^2 |DP_{\varepsilon, \Omega} U_{\varepsilon, x}|^2 - \int_{\Omega} F(P_{\varepsilon, \Omega} U_{\varepsilon, x}) = \varepsilon^N A + \tau_{\varepsilon, x} + \varepsilon^N O(e^{-(2+\sigma)md(x, \partial\Omega)/\varepsilon}).$$

So it remains to estimate

$$I = \int_{\Omega} P_{\varepsilon, \Omega} U_{\varepsilon, x} G_{\gamma} P_{\varepsilon, \Omega} U_{\varepsilon, x}.$$

We have

$$(2.10) \quad \begin{aligned} I &= \int_{\Omega} U_{\varepsilon, x} G_{\gamma} U_{\varepsilon, x} + 2 \int_{\Omega} (P_{\varepsilon, \Omega} U_{\varepsilon, x} - U_{\varepsilon, x}) G_{\gamma} U_{\varepsilon, x} \\ &\quad + \int_{\Omega} (P_{\varepsilon, \Omega} U_{\varepsilon, x} - U_{\varepsilon, x}) G_{\gamma} (P_{\varepsilon, \Omega} U_{\varepsilon, x} - U_{\varepsilon, x}) \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

On the other hand, from Lemmas 2.1 and 2.2, we see that for $N \geq 2$,

$$(2.11) \quad \begin{aligned} |I_2| &\leq C\varepsilon^2 \ln \frac{1}{\varepsilon} \int_{\Omega_{\varepsilon, x}} |P_{\Omega_{\varepsilon, x}} U - U| \\ &\leq \varepsilon^2 \ln \frac{1}{\varepsilon} |P_{\Omega_{\varepsilon, x}} U - U|_{\infty}^{1-\theta} \int_{\Omega_{\varepsilon, x}} |P_{\Omega_{\varepsilon, x}} U - U|^{\theta} \\ &\leq \varepsilon^2 \ln \frac{1}{\varepsilon} |P_{\Omega_{\varepsilon, x}} U - U|_{\infty}^{1-\theta} \int_{\Omega_{\varepsilon, x}} |U|^{\theta} \leq C\varepsilon^{N+2} \ln \frac{1}{\varepsilon} e^{-(1-\theta)md(x, \partial\Omega)/\varepsilon} \\ &= \varepsilon^N O(\varepsilon^{2+\sigma} + e^{-(2+\sigma)md(x, \partial\Omega)/\varepsilon}), \end{aligned}$$

and

$$(2.12) \quad |I_3| \leq \int_{\Omega} |P_{\varepsilon, \Omega} U_{\varepsilon, x} - U_{\varepsilon, x}| G_{\gamma} U_{\varepsilon, x} = \varepsilon^N O(\varepsilon^{2+\sigma} + e^{-(2+\sigma)md(x, \partial\Omega)/\varepsilon}).$$

As for the estimate of I_1 , using Lemma 2.1, we obtain that if $N \geq 3$,

$$(2.13) \quad \begin{aligned} I_1 &= \varepsilon^2 \int_{\Omega} U_{\varepsilon, x} W_{\varepsilon} \left(\frac{y-x}{\varepsilon} \right) \\ &\quad - \varepsilon^N \int_{\Omega} U_{\varepsilon, x} \left(BH(y, x) + O \left(\frac{\varepsilon^2}{d(x, \partial\Omega)^N} + \frac{e^{-(m-\theta)d(x, \partial\Omega)/\varepsilon}}{d(x, \partial\Omega)^{N-2}} \right) \right) \\ &= \varepsilon^{N+2} \int_{\Omega_{\varepsilon, x}} U W_{\varepsilon} \\ &\quad - \varepsilon^{2N} \left(BH(x, x) \int_{\mathbb{R}^N} U + O \left(\frac{\varepsilon^2}{d(x, \partial\Omega)^N} + \frac{e^{-(m-\theta)d(x, \partial\Omega)/\varepsilon}}{d(x, \partial\Omega)^{N-2}} \right) \right) \\ &= \varepsilon^{N+2} \int_{\mathbb{R}^N} U W_{\varepsilon} - \varepsilon^{N+2} B^2 H(x, x) \varepsilon^{N-2} \\ &\quad + \varepsilon^{2N} O \left(\frac{\varepsilon^2}{d(x, \partial\Omega)^N} + \frac{e^{-(m-\theta)d(x, \partial\Omega)/\varepsilon}}{d(x, \partial\Omega)^{N-2}} \right). \end{aligned}$$

So the (2.6) follows from (2.9)–(2.13).

If $N = 2$, using Lemma 2.2, we have

$$\begin{aligned}
 I_1 &= \frac{1}{2\pi} B \varepsilon^2 \ln \frac{1}{\varepsilon} \int_{\Omega} U_{\varepsilon,x} + \varepsilon^2 \int_{\Omega} U_{\varepsilon,x} \tilde{W} \left(\frac{y-x}{\varepsilon} \right) \\
 &\quad - \varepsilon^2 \int_{\Omega} U_{\varepsilon,x} \left(B \tilde{H}(y,x) + O \left(\frac{\varepsilon^2}{d(x, \partial\Omega)^2} + e^{-(m-\theta)d(x, \partial\Omega)/\varepsilon} \ln \frac{1}{d(x, \partial\Omega)} \right) \right) \\
 (2.14) \quad &= \frac{1}{2\pi} B \varepsilon^4 \ln \frac{1}{\varepsilon} \int_{\Omega_{\varepsilon,x}} U + \varepsilon^4 \int_{\Omega_{\varepsilon,x}} U \tilde{W} \\
 &\quad - \varepsilon^4 \left(B \tilde{H}(x,x) \int_{\mathbb{R}^2} U + O \left(\frac{\varepsilon^2}{d(x, \partial\Omega)^2} + e^{-(m-\theta)d(x, \partial\Omega)/\varepsilon} \ln \frac{1}{d(x, \partial\Omega)} \right) \right) \\
 &= \frac{1}{2\pi} B \varepsilon^4 \ln \frac{1}{\varepsilon} \int_{\mathbb{R}^2} U + \varepsilon^4 \int_{\mathbb{R}^2} U \tilde{W} - \varepsilon^4 B^2 \tilde{H}(x,x) \\
 &\quad + \varepsilon^4 O \left(\frac{\varepsilon^2}{d(x, \partial\Omega)^2} + e^{-(m-\theta)d(x, \partial\Omega)/\varepsilon} \ln \frac{1}{d(x, \partial\Omega)} \right).
 \end{aligned}$$

So the (2.7) follows from (2.9)–(2.12) and (2.14).

If $N = 1$, using Lemma 2.3, we have

$$I_1 = \varepsilon B G_{\gamma}(x, x) \int_{\Omega} U_{\varepsilon,x} + O(\varepsilon^3) = \varepsilon^2 B G_{\gamma}(x, x) \int_{\mathbb{R}^1} U + O(\varepsilon^3),$$

and

$$|I_2|, |I_3| \leq C \varepsilon \int_{\Omega} |P_{\varepsilon,\Omega} U_{\varepsilon,x} - U_{\varepsilon,x}| = \varepsilon O(\varepsilon e^{-m(1-\theta)d(x, \partial\Omega)}).$$

Thus the result follows. □

3. – Solution with peak near the boundary

In this section, we will use the reduction argument to prove Theorem 1.1.

The functional $I(u)$ may not be well defined in $H_0^1(\Omega)$. But it is easy to see that there is a constant $K > 0$, such that for any solution $(u_{\varepsilon}, v_{\varepsilon})$ of (1.1), we have $|u_{\varepsilon}|_{L^{\infty}(\Omega)}, |v_{\varepsilon}|_{L^{\infty}(\Omega)} \leq K$. Thus, we can truncate $f(t)$ for $|t| \geq K$ so that the new function is bounded in $C^2(\mathbb{R}^1)$. For simplicity, we still use $f(t)$ to denote this new function.

Let

$$\langle u, v \rangle_{\varepsilon} = \int_{\Omega} (\varepsilon^2 Du Dv + m^2 uv),$$

$\|u\|_{\varepsilon} = \langle u, u \rangle_{\varepsilon}^{1/2}$. Denote

$$E_{\varepsilon,x} = \left\{ \omega : \omega \in H_0^1(\Omega), \left\langle \frac{\partial P_{\varepsilon,\Omega} U_{\varepsilon,x}}{\partial x_i}, \omega \right\rangle_{\varepsilon} = 0, i = 1, \dots, N \right\}.$$

Let $D_{\varepsilon,R} = \{x : x \in \Omega, d(x, \partial\Omega) \geq \varepsilon R\}$, where $R > 0$ is a large constant.

PROPOSITION 3.1. *There is an $\varepsilon_0 > 0$, such that for each $\varepsilon \in (0, \varepsilon_0]$, there is a C^1 -map $\omega_{\varepsilon,x} : D_{\varepsilon,\mathbb{R}} \rightarrow H_0^1(\Omega)$ satisfying, $\omega_{\varepsilon,x} \in E_{\varepsilon,x}$,*

$$(3.1) \quad \left\langle \frac{\partial I(P_{\varepsilon,\Omega}U_{\varepsilon,x} + \omega_{\varepsilon,x})}{\partial \omega}, \eta \right\rangle_{\varepsilon} = 0, \quad \forall \eta \in E_{\varepsilon,x}.$$

Moreover, if $N \geq 3$,

$$\|\omega_{\varepsilon,x}\|_{\varepsilon} = \varepsilon^{N/2} O \left(\varepsilon^{1+\sigma} + e^{-(1+\sigma)m \frac{d(x,\partial\Omega)}{\varepsilon}} \right),$$

where $\sigma > 0$ is some constant.

If $N = 2$,

$$\|\omega_{\varepsilon,x}\|_{\varepsilon} = \varepsilon O \left(\varepsilon \ln \ln \frac{1}{\varepsilon} + e^{-(1+\sigma)m \frac{d(x,\partial\Omega)}{\varepsilon}} \right).$$

If $N = 1$,

$$\|\omega_{\varepsilon,x}\|_{\varepsilon} = \varepsilon^{1/2} O \left(\varepsilon^{1/2} d(x, \partial\Omega) + e^{-(1+\sigma)m \frac{d(x,\partial\Omega)}{\varepsilon}} \right).$$

PROOF. Let $h(t) = f(t) + m^2t$. As in [1], [21], we expand $I(P_{\varepsilon,\Omega}U_{\varepsilon,x} + \omega)$, $\omega \in E_{\varepsilon,x}$, as follows:

$$(3.2) \quad I(P_{\varepsilon,\Omega}U_{\varepsilon,x} + \omega) = I(P_{\varepsilon,\Omega}U_{\varepsilon,x}) + \langle l_{\varepsilon}, \omega \rangle_{\varepsilon} + \frac{1}{2} \langle L_{\varepsilon}\omega, \omega \rangle_{\varepsilon} + R_{\varepsilon}(\omega),$$

where $l_{\varepsilon} \in E_{\varepsilon,x}$ satisfying

$$\langle l_{\varepsilon}, \omega \rangle_{\varepsilon} = \delta \int_{\Omega} \omega G_{\gamma} P_{\varepsilon,\Omega} U_{\varepsilon,x} + \int_{\Omega} (h(U_{\varepsilon,x}) - h(P_{\varepsilon,\Omega}U_{\varepsilon,x})) \omega,$$

L_{ε} is a linear operator from $E_{\varepsilon,x}$ to $E_{\varepsilon,x}$, satisfying

$$\langle L_{\varepsilon}\omega, \omega \rangle_{\varepsilon} = \int_{\Omega} (\varepsilon^2 |D\omega|^2 + m^2\omega^2) + \delta \int_{\Omega} \omega G_{\gamma} \omega - \int_{\Omega} h'(P_{\varepsilon,\Omega}U_{\varepsilon,x}) \omega^2,$$

and $R_{\varepsilon}(\omega)$ satisfies

$$\begin{aligned} R_{\varepsilon}(\omega) &= \varepsilon^N O(\varepsilon^{-Np/2} \|\omega\|_{\varepsilon}^p), \\ R'_{\varepsilon}(\omega) &= \varepsilon^{N/2} O(\varepsilon^{-N(p-1)/2} \|\omega\|_{\varepsilon}^{p-1}), \\ R''_{\varepsilon}(\omega) &= O(\varepsilon^{-N(p-2)/2} \|\omega\|_{\varepsilon}^{p-2}), \end{aligned}$$

for some $p \in (2, 2N/(N - 2))$. Thus, (3.1) is equivalent to

$$(3.3) \quad l_{\varepsilon} + L_{\varepsilon}\omega + R'(\omega) = 0.$$

By Lemma 3.2 below, we see that L_{ε} is invertible in $E_{\varepsilon,x}$. So it follows from the implicit function theorem that (3.3) has a solution $\omega_{\varepsilon} \in E_{\varepsilon,x}$, satisfying

$$\|\omega_{\varepsilon}\|_{\varepsilon} \leq C \|l_{\varepsilon}\|_{\varepsilon}.$$

Thus the estimate for $\|\omega_{\varepsilon}\|_{\varepsilon}$ follows from Lemma 3.3. □

LEMMA 3.2. *There are $\varepsilon_0 > 0$, $\tau > 0$ and $R_0 > 0$, such that for $\varepsilon \in (0, \varepsilon_0]$, $x \in D_{\varepsilon,R}$ with $R \geq R_0$, we have*

$$\|L_\varepsilon \omega\| \geq \tau \|\omega\|_\varepsilon, \quad \forall \omega \in E_{\varepsilon,x}.$$

PROOF. We argue by contradiction. Suppose that there are $\varepsilon_j \rightarrow 0$, $x_j \in \Omega$ with $\frac{d(x_j, \partial\Omega)}{\varepsilon_j} \rightarrow +\infty$, $\omega_j \in E_{\varepsilon_j, x_j}$, such that

$$(3.4) \quad \|L_{\varepsilon_j} \omega_j\|_{\varepsilon_j} \leq o(1) \|\omega_j\|_{\varepsilon_j}.$$

We may assume that $\|\omega_j\|_{\varepsilon_j} = \varepsilon_j^{N/2}$.

Let $\tilde{\omega}_j(y) = \omega_j(\varepsilon_j y + x_j)$. Then it follows from (3.4) that

$$(3.5) \quad \int_{\Omega_{\varepsilon_j, x_j}} (D\tilde{\omega}_j D\xi + m^2 \tilde{\omega}_j \xi) + \delta \int_{\Omega_{\varepsilon_j, x_j}} \xi \tilde{G}_\gamma \tilde{\omega}_j - \int_{\Omega_{\varepsilon_j, x_j}} h'(P_{\Omega_{\varepsilon_j, x_j}} U) \tilde{\omega}_j \xi = o(1) \|\xi\|$$

for any $\xi \in \tilde{E}_{\varepsilon_j} = \{\xi \in H_0^1(\Omega_{\varepsilon_j, x_j}) : \int_{\Omega_{\varepsilon_j, x_j}} (DP_{\Omega_{\varepsilon_j, x_j}} U D\xi + m^2 P_{\Omega_{\varepsilon_j, x_j}} U \xi) = 0\}$, where $\Omega_{\varepsilon_j, x_j} = \varepsilon_j^{-1}(\Omega - x_j)$, $P_{\Omega_{\varepsilon_j, x_j}} U \in H_0^1(\Omega_{\varepsilon_j, x_j})$ satisfies

$$-\Delta P_{\Omega_{\varepsilon_j, x_j}} U + m^2 P_{\Omega_{\varepsilon_j, x_j}} U = h(U),$$

and $\tilde{G}_\gamma \tilde{\omega}_j(y) = G_\gamma \omega_j(\varepsilon_j y + x_j) \in H_0^1(\Omega_{\varepsilon_j, x_j})$.

Since $\tilde{\omega}_j$ is bounded in $H^1(\mathbb{R}^N)$, we may assume that there is an $\omega \in H^1(\mathbb{R}^N)$, such that

$$\tilde{\omega}_j \rightharpoonup \omega, \quad \text{weakly in } H^1(\mathbb{R}^N).$$

Note that

$$(3.6) \quad -\Delta G_\gamma \omega_j + \gamma G_\gamma \omega_j = \omega_j.$$

From (3.6), we see

$$\int_{\Omega} |DG_\gamma \omega_j|^2 \leq C \int_{\Omega} \omega_j^2 \leq C \varepsilon_j^N,$$

which implies that $\int_{\Omega_{\varepsilon_j, x_j}} |D\tilde{G}_\gamma \tilde{\omega}_j|^2 \leq C \varepsilon_j^2 \rightarrow 0$ as $j \rightarrow +\infty$. Thus we have $\tilde{G}_\gamma \tilde{\omega}_j \rightarrow 0$ in $L^2_{loc}(\mathbb{R}^N)$ as $j \rightarrow +\infty$ if $N \geq 3$. On the other hand, if $N = 1, 2$, it follows from (3.6) that

$$|G_\gamma \omega_j|_{L^\infty(\Omega)} \leq C |G_\gamma \omega_j|_{H^2(\Omega)} \leq C' |\omega_j|_{L^2(\Omega)} \rightarrow 0$$

as $\varepsilon_j \rightarrow 0$. Thus we also have $\tilde{G}_\gamma \tilde{\omega}_j \rightarrow 0$ in $L^2_{loc}(\mathbb{R}^N)$ as $j \rightarrow +\infty$ if $N = 1, 2$. As a result, we deduce from (3.5) that ω satisfies

$$(3.7) \quad \int_{\mathbb{R}^N} (D\omega D\xi + m^2 \omega \xi) - \int_{\mathbb{R}^N} h'(U) \omega \xi = 0,$$

for all $\xi \in E = \{\xi \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} (DU D\xi + m^2 U \xi) = 0\}$. Since U is nondegenerate, we know that $\omega = 0$. Thus, noting that $h'(P_{\Omega_{\varepsilon_j, x_j}} U_{\varepsilon_j, x_j})$ is small in $\Omega \setminus B_{\varepsilon_j R}(x_j)$ if $R > 0$ is large, we see

$$\begin{aligned} & \int_{\Omega} (\varepsilon_j^2 |D\omega_j|^2 + m^2 \omega_j^2) + \delta \int_{\Omega} \omega_j \tilde{G}_\gamma \omega_j - \int_{\Omega} h'(P_{\Omega_{\varepsilon_j, x_j}} U) \omega_j^2 \\ & \geq \int_{\Omega} (\varepsilon_j^2 |D\omega_j|^2 + m^2 \omega_j^2) - \int_{\Omega} h'(P_{\Omega_{\varepsilon_j, x_j}} U) \omega_j^2 \\ & = \int_{\Omega} (\varepsilon_j^2 |D\omega_j|^2 + m^2 \omega_j^2) - \int_{B_{\varepsilon_j R}(x_j)} h'(P_{\Omega_{\varepsilon_j, x_j}} U) \omega_j^2 - \int_{\Omega \setminus B_{\varepsilon_j R}(x_j)} h'(P_{\Omega_{\varepsilon_j, x_j}} U) \omega_j^2 \\ & = \int_{\Omega} (\varepsilon_j^2 |D\omega_j|^2 + m^2 \omega_j^2) - o(\varepsilon_j^N) - o(1) \int_{\Omega} \omega_j^2 \geq c_0 \varepsilon_j^N. \end{aligned}$$

This is a contradiction to (3.4). □

LEMMA 3.3. Assume that $x \in D_{\varepsilon, R}$, where $R > 0$ is a large constant. We have

$$(3.8) \quad \left| \int_{\Omega} (h(U_{\varepsilon, x}) - h(P_{\varepsilon, x} U_{\varepsilon, x})) \omega \right| = \varepsilon^{N/2} O(e^{-(1+\sigma)d(x, \partial\Omega)/\varepsilon}) \|\omega\|_{\varepsilon}.$$

If $N \geq 3$,

$$(3.9) \quad \left| \int_{\Omega} \omega G_\gamma U_{\varepsilon, x} \right| = \varepsilon^{N/2} O(\varepsilon^{1+\sigma}) \|\omega\|_{\varepsilon},$$

where $\sigma > 0$ is some constant. If $N = 2$,

$$(3.10) \quad \left| \int_{\Omega} \omega G_\gamma U_{\varepsilon, x} \right| = \varepsilon^2 O\left(\ln \ln \frac{1}{\varepsilon}\right) \|\omega\|_{\varepsilon}.$$

If $N = 1$,

$$(3.11) \quad \left| \int_{\Omega} \omega G_\gamma U_{\varepsilon, x} \right| = \varepsilon O(d(x, \partial\Omega) + \varepsilon) \|\omega\|_{\varepsilon}.$$

PROOF. The estimate in (3.8) is known. See for example [4], [5], [14], [15]. To prove (3.9), we note that $(G_\gamma U_{\varepsilon, x})(\varepsilon y + x) = \varepsilon^2 v_\varepsilon(y)$ and v_ε satisfies

$$-\Delta v_\varepsilon + \varepsilon^2 \gamma v_\varepsilon = U, \quad v_\varepsilon \in H_0^1(\Omega_{\varepsilon, x}).$$

Let \bar{v} be the solution of

$$-\Delta v = U, \quad v \in H^1(\mathbb{R}^N).$$

Then $\bar{v}(y) = \int_{\mathbb{R}^N} \frac{1}{|y-x|^{N-2}} U(x) dx$. Thus $\bar{v}(y) \leq \frac{C}{1+|y|^{N-2}}$, since $U(x)$ decays exponentially. By the comparison theorem, we have

$$0 < v_\varepsilon(y) < \bar{v}(y).$$

Thus, we have the following estimate for v_ε :

$$v_\varepsilon(y) \leq \frac{C}{1+|y|^{N-2}}, \forall y \in \mathbb{R}^N.$$

As a result, for any $\theta > 0$, if $N = 3, 4$,

$$\begin{aligned} & \left| \int_{\Omega} \omega G_\gamma U_{\varepsilon,x} \right| \\ & \leq C \left(\int_{\Omega} |G_\gamma U_{\varepsilon,x}|^{(N+\theta)/(N-2)} \right)^{(N-2)/(N+\theta)} \left(\int_{\Omega} |\omega|^{(N+\theta)/(2+\theta)} \right)^{(2+\theta)/(N+\theta)} \\ & \leq \varepsilon^{2+N(N-2)/(N+\theta)} \|\omega\|_\varepsilon \leq \varepsilon^{1+\sigma+N/2} \|\omega\|_\varepsilon, \end{aligned}$$

because $\frac{N+\theta}{2+\theta} \leq 2$ if $N = 3, 4$.

If $N \geq 5$, then

$$\begin{aligned} \left| \int_{\Omega} \omega G_\gamma U_{\varepsilon,x} \right| & \leq C \left(\int_{\Omega} |G_\gamma U_{\varepsilon,x}|^2 \right)^{1/2} \left(\int_{\Omega} |\omega|^2 \right)^{1/2} \\ & \leq \varepsilon^{2+N/2} \|\omega\|_\varepsilon. \end{aligned}$$

Now, we prove (3.10).

We have

$$(3.12) \quad \left| \int_{\Omega} \omega G_\gamma U_{\varepsilon,x} \right| \leq \int_{B_{1/\ln \frac{1}{\varepsilon}}(x)} G_\gamma U_{\varepsilon,x} |\omega| + \int_{\Omega \setminus B_{1/\ln \frac{1}{\varepsilon}}(x)} G_\gamma U_{\varepsilon,x} |\omega|.$$

It follows from Lemma 2.2 that $|G_\gamma U_{\varepsilon,x}(y)| \leq C\varepsilon^2 \ln \frac{1}{\varepsilon}, \forall x \in D_{\varepsilon,R}$. Thus,

$$(3.13) \quad \int_{B_{1/\ln \frac{1}{\varepsilon}}(x)} G_\gamma U_{\varepsilon,x} |\omega| \leq C\varepsilon^2 \ln \frac{1}{\varepsilon} \int_{B_{1/\ln \frac{1}{\varepsilon}}(x)} |\omega| \leq C\varepsilon^2 \|\omega\|_\varepsilon.$$

On the other hand, from (2.5), we see that for $|y - x| \geq \frac{1}{\ln \frac{1}{\varepsilon}}$, we have

$$\begin{aligned}
 G_\gamma U_{\varepsilon,x}(y) &= \frac{1}{2\pi} \int_\Omega \ln \frac{1}{|z - y|} U_{\varepsilon,x}(z) dz + O(\varepsilon^2 |\tilde{H}(y, x)| + \varepsilon^{2+\sigma}) \\
 &= \frac{1}{2\pi} \int_{B_{|y-x|/2}(x)} \ln \frac{1}{|z - y|} U_{\varepsilon,x}(z) dz \\
 &\quad + \frac{1}{2\pi} \int_{\Omega \setminus B_{|y-x|/2}(x)} \ln \frac{1}{|z - y|} U_{\varepsilon,x}(z) dz + O\left(\varepsilon^2 \ln \ln \frac{1}{\varepsilon}\right) \\
 (3.14) \quad &= \frac{1}{2\pi} \int_{B_{|y-x|/2}(x)} \ln \frac{1}{|z - y|} U_{\varepsilon,x}(z) dz \\
 &\quad + O\left(e^{-(m-\theta) \ln \frac{1}{\varepsilon}/(2\varepsilon)} + \varepsilon^2 \ln \ln \frac{1}{\varepsilon}\right) \\
 &\leq \ln\left(2 \ln \frac{1}{\varepsilon}\right) \frac{1}{2\pi} \int_{B_{|y-x|/2}(x)} U_{\varepsilon,x}(z) dz + O\left(\varepsilon^2 \ln \ln \frac{1}{\varepsilon}\right) \\
 &= O\left(\varepsilon^2 \ln \ln \frac{1}{\varepsilon}\right),
 \end{aligned}$$

since $|\tilde{H}(y, x)| \leq C \ln \frac{1}{|y-x|}$.

Using (3.14), we obtain

$$(3.15) \quad \int_{\Omega \setminus B_{1/\ln \frac{1}{\varepsilon}}(x)} G_\gamma U_{\varepsilon,x} |\omega| = O\left(\varepsilon^2 \ln \ln \frac{1}{\varepsilon}\right) \|\omega\|_\varepsilon.$$

Thus, (3.10) follows from (3.12), (3.13) and (3.15).

Finally, we prove (3.11). Using Lemma 2.3, we have

$$\begin{aligned}
 (3.16) \quad \left| \int_\Omega \omega G_\gamma U_{\varepsilon,x} \right| &= O\left(\varepsilon \int_\Omega |\omega| G_\gamma(x, y) dy + \varepsilon^2 \|\omega\|_\varepsilon\right) \\
 &= O\left(\varepsilon \left(\int_\Omega G_\gamma^2(x, y) dy\right)^{1/2} + \varepsilon^2\right) \|\omega\|_\varepsilon.
 \end{aligned}$$

Without loss of generality, we assume $\Omega = [0, 1]$.

Suppose that $\gamma = 0$. Then

$$(3.17) \quad G_\gamma(x, y) = \begin{cases} (1-x)y, & y \in [0, x], \\ x(1-y), & y \in [x, 1]. \end{cases}$$

Thus,

$$(3.18) \quad \int_0^1 G_\gamma^2(x, y) dy = \frac{1}{3} x^2 (1-x)^2 = O(d^2(x, \partial\Omega)).$$

So (3.11) follows from (3.16) and (3.18).

Suppose that $\gamma > 0$. Then

$$(3.19) \quad G_\gamma(x, y) = \begin{cases} \frac{e^{\sqrt{\gamma}(1-x)} - e^{-\sqrt{\gamma}(1-x)}}{2\sqrt{\gamma}(e^{\sqrt{\gamma}} - e^{-\sqrt{\gamma}})}(e^{\sqrt{\gamma}y} - e^{-\sqrt{\gamma}y}), & y \in [0, x], \\ \frac{e^{\sqrt{\gamma}(1-x)} - e^{-\sqrt{\gamma}(1-x)}}{2\sqrt{\gamma}(e^{\sqrt{\gamma}} - e^{-\sqrt{\gamma}})}(e^{\sqrt{\gamma}y} - e^{-\sqrt{\gamma}y}) \\ - \frac{1}{2\sqrt{\gamma}}(e^{\sqrt{\gamma}(y-x)} - e^{-\sqrt{\gamma}(y-x)}), & y \in [x, 1]. \end{cases}$$

So, if $x \in [\frac{1}{2}, 1]$, we have

$$(3.20) \quad \begin{aligned} \int_0^1 G_\gamma^2(x, y) dy &= \int_0^x G_\gamma^2(x, y) dy + \int_x^1 G_\gamma^2(x, y) dy \\ &= O(|1-x|^2) + O\left(\int_x^1 |e^{\sqrt{\gamma}(y-x)} - e^{-\sqrt{\gamma}(y-x)}|^2\right) \\ &= O(|1-x|^2) + O(|1-x|^3) = O(d^2(x, \partial\Omega)). \end{aligned}$$

If $x \in [0, \frac{1}{2}]$, then

$$(3.21) \quad \begin{aligned} \int_0^1 G_\gamma^2(x, y) dy &= \int_0^x G_\gamma^2(x, y) dy + \int_x^1 G_\gamma^2(x, y) dy \\ &= O(|x|^3) + O\left(\int_x^1 |x|^2 dy\right) = O(|x|^2) = O(d^2(x, \partial\Omega)), \end{aligned}$$

since $G(0, y) = 0$. Thus (3.11) follows. □

PROOF OF THEOREM 1.1. Let

$$D_\varepsilon = \left\{ x : x \in \Omega, d(x, \partial\Omega) \in \left[c_0\varepsilon \ln \frac{1}{\varepsilon}, \varepsilon^{1-\eta} \right] \right\},$$

where c_0 and η are two fixed small constants.

Consider

$$(3.22) \quad \inf_{x \in D_\varepsilon} I(P_{\varepsilon, \Omega} U_{\varepsilon, x} + \omega_{\varepsilon, x}).$$

Let $x_\varepsilon \in D_\varepsilon$ be a minimum point of $I(P_{\varepsilon, \Omega} U_{\varepsilon, x} + \omega_{\varepsilon, x})$ in D_ε . As usual, if x_ε is an interior point of D_ε , then $P_{\varepsilon, \Omega} U_{\varepsilon, x} + \omega_{\varepsilon, x}$ is a solution of (1.1).

Suppose that $N \geq 3$. Then from Propositions 3.1 and 2.4, we have

$$(3.23) \quad \begin{aligned} I(P_{\varepsilon, \Omega} U_{\varepsilon, x} + \omega_{\varepsilon, x}) &= \varepsilon^N A + \delta A_\varepsilon \varepsilon^{N+2} + \tau_{\varepsilon, x} - \frac{\delta}{2} B^2 \varepsilon^{N+2} \varepsilon^{N-2} H(x, x) \\ &+ \varepsilon^N O(e^{-(2+\sigma)md(x, \partial\Omega)/\varepsilon}) + o\left(\frac{\varepsilon^{N+2}}{\left(\ln \frac{1}{\varepsilon}\right)^{N-2}}\right). \end{aligned}$$

Let $x^* \in D_\varepsilon$ with $d(x^*, \partial\Omega) = C\varepsilon \ln \frac{1}{\varepsilon}$ and $C > 0$ large. Noting that for any small $\theta > 0$,

$$\tau_{\varepsilon, x^*} \leq \varepsilon^N c' e^{-2m(1-\theta)d(x^*, \partial\Omega)/\varepsilon} = c' \varepsilon^{N+2Cm(1-\theta)},$$

and $H(x^*, x^*) \sim \frac{1}{d(x^*, \partial\Omega)^{N-2}}$, we obtain from Proposition 2.4 that

$$\begin{aligned} I(P_{\varepsilon, \Omega} U_{\varepsilon, x^*} + \omega_{\varepsilon, x}) &\leq \varepsilon^N A + \delta A_\varepsilon \varepsilon^{N+2} + c' \varepsilon^{N+2Cm(1-\theta)} \\ &\quad - \frac{\delta}{2} B^2 \tilde{c} \varepsilon^{N+2} \frac{1}{\left(C \ln \frac{1}{\varepsilon}\right)^{N-2}} + o\left(\frac{\varepsilon^{N+2}}{\left(\ln \frac{1}{\varepsilon}\right)^{N-2}}\right) \\ (3.24) \quad &\leq \varepsilon^N A + \delta A_\varepsilon \varepsilon^{N+2} - \frac{\delta}{2} B^2 c'' \varepsilon^{N+2} \frac{1}{\left(C \ln \frac{1}{\varepsilon}\right)^{N-2}}, \end{aligned}$$

where $c'' > 0$ is a constant.

For any $x' \in \Omega$ with $d(x', \partial\Omega) = c_0 \varepsilon \ln \frac{1}{\varepsilon}$, we have

$$\tau_{\varepsilon, x'} \geq c' \varepsilon^N e^{-2m(1+\theta)d(x', \partial\Omega)/\varepsilon} = c' \varepsilon^{N+2m(1+\theta)c_0}.$$

Thus,

$$\begin{aligned} I(P_{\varepsilon, \Omega} U_{\varepsilon, x'} + \omega_{\varepsilon, x}) &\geq \varepsilon^N A + \delta A_\varepsilon \varepsilon^{N+2} + c' \varepsilon^{N+2c_0m(1+\theta)} \\ &\quad - \frac{\delta}{2} B^2 c^* \varepsilon^{N+2} \frac{1}{\left(c_0 \ln \frac{1}{\varepsilon}\right)^{N-2}} + o\left(\frac{\varepsilon^{N+2}}{\left(\ln \frac{1}{\varepsilon}\right)^{N-2}}\right) \\ (3.25) \quad &\geq \varepsilon^N A + \delta A_\varepsilon \varepsilon^{N+2} + c'' \varepsilon^{N+2c_0m(1+\theta)}. \end{aligned}$$

Combining (3.24) and (3.25), we see that if $c_0 > 0$ is small enough, $I(P_{\varepsilon, \Omega} U_{\varepsilon, x} + \omega_{\varepsilon, x})$ can not attain its minimum on $\{x : x \in \Omega, d(x, \partial\Omega) = c_0 \ln \frac{1}{\varepsilon}\}$.

On the other hand, for any $x \in \Omega$ with $d(x, \partial\Omega) = \varepsilon^{1-\eta}$, we have

$$\begin{aligned} I(P_{\varepsilon, \Omega} U_{\varepsilon, x} + \omega_{\varepsilon, x}) &\geq \varepsilon^N A + \delta A_\varepsilon \varepsilon^{N+2} + c' e^{-m(1+\theta)\varepsilon^{-\eta}} \\ &\quad - \frac{\delta}{2} B^2 c^* \varepsilon^{N+2+\eta(N-2)} + o\left(\frac{\varepsilon^{N+2}}{\left(\ln \frac{1}{\varepsilon}\right)^{N-2}}\right) \\ (3.26) \quad &\geq \varepsilon^N A + \delta A_\varepsilon \varepsilon^{N+2} + o\left(\frac{\varepsilon^{N+2}}{\left(\ln \frac{1}{\varepsilon}\right)^{N-2}}\right). \end{aligned}$$

Combining (3.24) and (3.26), we see that $I(P_{\varepsilon,\Omega}U_{\varepsilon,x} + \omega_{\varepsilon,x})$ can not attain its minimum on $\{x : x \in \Omega, d(x, \partial\Omega) = \varepsilon^{1-\eta}\}$. Thus we have proved the x_ε is an interior point of D_ε .

Suppose that $N = 2$. Then from Lemma 3.3 and Proposition 2.4, we have

$$(3.27) \quad \begin{aligned} I(P_{\varepsilon,\Omega}U_{\varepsilon,x} + \omega_{\varepsilon,x}) &= \varepsilon^2 A + \frac{\delta}{4\pi} \varepsilon^4 \ln \frac{1}{\varepsilon} + \delta \tilde{A} \varepsilon^4 + \tau_{\varepsilon,x} - \frac{\delta}{2} B^2 \varepsilon^4 \tilde{H}(x, x) \\ &+ \varepsilon^2 O \left(e^{-(2+\sigma)md(x,\partial\Omega)/\varepsilon} + \varepsilon^2 \ln^2 \left(\ln \frac{1}{\varepsilon} \right) \right), \end{aligned}$$

Noting that $\tilde{H}(x, x) \sim \ln \frac{1}{d(x,\partial\Omega)}$ and $d(x, \partial\Omega) \in (c_0 \varepsilon \ln \frac{1}{\varepsilon}, \varepsilon^{1-\eta})$, we see that $\varepsilon^4 \ln^2(\ln \frac{1}{\varepsilon})$ is a higher order term than $\varepsilon^4 \tilde{H}(x, x)$. As in the case $N \geq 3$, we can check that if $x \in D_\varepsilon$ with $d(x, \partial\Omega) = C\varepsilon \ln \frac{1}{\varepsilon}$ with $C > 0$ large, then

$$\begin{aligned} \tau_{\varepsilon,x} - \frac{\delta}{2} B^2 \varepsilon^4 \tilde{H}(x, x) &\sim -\varepsilon^4 \ln \frac{1}{C\varepsilon \ln \frac{1}{\varepsilon}} \\ &< -\varepsilon^4 \ln \frac{1}{\varepsilon^{1-\eta}} \sim \min_{z \in \partial D_\varepsilon} \left(\tau_{\varepsilon,z} - \frac{\delta}{2} B^2 \varepsilon^4 \tilde{H}(z, z) \right). \end{aligned}$$

So we see that $\tau_{\varepsilon,x} - \frac{\delta}{2} B^2 \varepsilon^4 \tilde{H}(x, x)$ attains its minimum at an interior point of D_ε . Thus x_ε is an interior point of D_ε .

Finally, if $N = 1$ then from Lemma 3.3 and Proposition 2.4, we have

$$(3.28) \quad \begin{aligned} I(P_{\varepsilon,\Omega}U_{\varepsilon,x} + \omega_{\varepsilon,x}) &= \varepsilon A + \frac{1}{2} \delta B^2 G_\gamma(x, x) \varepsilon^2 + \tau_{\varepsilon,x} \\ &+ \varepsilon O(e^{-(2+\sigma)md(x,\partial\Omega)/\varepsilon} + \varepsilon d^2(x, \partial\Omega) + \varepsilon^3). \end{aligned}$$

From (3.17) and (3.19), we know that $G_\gamma(x, x) \sim d(x, \partial\Omega)$. Thus $\varepsilon^2 d^2(x, \partial\Omega)$ is a higher order term than $\frac{1}{2} \delta B^2 G_\gamma(x, x) \varepsilon^2$. On the other hand, it is easy to see that if $d(x, \partial\Omega) = C\varepsilon \ln \frac{1}{\varepsilon}$, we have

$$\frac{1}{2} \delta B^2 G_\gamma(x, x) \varepsilon^2 + \tau_{\varepsilon,x} \sim \varepsilon^3 \ln \frac{1}{\varepsilon} < \varepsilon^{3-\eta} \sim \min_{z \in \partial D_\varepsilon} \left(\frac{1}{2} \delta B^2 G_\gamma(z, z) \varepsilon^2 + \tau_{\varepsilon,z} \right).$$

Thus $\tau_{\varepsilon,x} + \frac{1}{2} \delta B^2 G_\gamma(x, x) \varepsilon^2$ attains its minimum at an interior point of D_ε . So x_ε is an interior point of D_ε . \square

REMARK 3.4. Suppose that Γ is a connected component of $\partial\Omega$. The above argument shows that (1.1) has a solution of the form $P_{\varepsilon,\Omega}U_{\varepsilon,x_\varepsilon} + \omega_\varepsilon$ with $d(x_\varepsilon, \Gamma) \rightarrow 0$, $\frac{d(x_\varepsilon, \Gamma)}{\varepsilon} \rightarrow +\infty$ and $\|\omega_\varepsilon\|_\varepsilon = o(\varepsilon^{N/2})$ as $\varepsilon \rightarrow 0$. Moreover, if $\partial\Omega$ has k disjoint connected components $\Gamma_1, \dots, \Gamma_k$, then for $\varepsilon > 0$ small, (1.1) has a solution of the form

$$\sum_{i=1}^k P_{\varepsilon,\Omega}U_{\varepsilon,x_{\varepsilon,i}} + \omega_\varepsilon,$$

with $d(x_{\varepsilon,i}, \Gamma_i) \rightarrow 0$, $\frac{d(x_{\varepsilon,i}, \Gamma_i)}{\varepsilon} \rightarrow +\infty$ and $\|\omega_\varepsilon\|_\varepsilon = o(\varepsilon^{N/2})$ as $\varepsilon \rightarrow 0$.

4. – Solutions with several peaks near the boundary

This section is devoted to the proof of Theorem 1.2. Because the proof of Theorem 1.2 is quite similar to the proof of Theorem 1.1, we are a bit sketchy in this section.

For a positive integer k and $X = (x_1, \dots, x_k)$, $x_j \in \mathbb{R}^N$, denote

$$E_{\varepsilon, X, k} = \left\{ \omega : \omega \in H_0^1(\Omega), \left\langle \frac{\partial P_{\varepsilon, \Omega} U_{\varepsilon, x_j}}{\partial x_{ji}}, \omega \right\rangle_{\varepsilon} = 0, i = 1, \dots, N \right\},$$

for $j = 1, \dots, k$. Let $D_{\varepsilon, R, k} = \{X : x_j \in \Omega, d(x_j, \partial\Omega) \geq \varepsilon R, |x_j - x_h| \geq \varepsilon R\}$, where $R > 0$ is a large constant.

PROPOSITION 4.1. *There is an $\varepsilon_0 > 0$, such that for each $\varepsilon \in (0, \varepsilon_0]$, there is a C^1 -map $\omega_{\varepsilon, X} : D_{\varepsilon, R, k} \rightarrow H_0^1(\Omega)$ satisfying, $\omega_{\varepsilon, X} \in E_{\varepsilon, X, k}$,*

$$(4.1) \quad \left\langle \frac{\partial I \left(\sum_{j=1}^k P_{\varepsilon, \Omega} U_{\varepsilon, x_j} + \omega_{\varepsilon} \right)}{\partial \omega}, \eta \right\rangle_{\varepsilon} = 0, \quad \forall \eta \in E_{\varepsilon, X, k}.$$

Moreover, if $N \geq 3$,

$$\|\omega_{\varepsilon, X}\|_{\varepsilon} = \varepsilon^{N/2} O \left(\varepsilon^{1+\sigma} + \sum_{j=1}^k e^{-(1+\sigma)m \frac{d(x, \partial\Omega)}{\varepsilon}} + \sum_{h \neq j} e^{-(1+\sigma)m|x_h - x_j|/(2\varepsilon)} \right),$$

where $\sigma > 0$ is some constant.

If $N = 2$,

$$\|\omega_{\varepsilon, X}\|_{\varepsilon} = \varepsilon O \left(\varepsilon \ln \ln \frac{1}{\varepsilon} + \sum_{j=1}^k e^{-(1+\sigma)m \frac{d(x, \partial\Omega)}{\varepsilon}} + \sum_{h \neq j} e^{-(1+\sigma)m|x_h - x_j|/(2\varepsilon)} \right).$$

If $N = 1$,

$$\|\omega_{\varepsilon, X}\|_{\varepsilon} = \varepsilon^{1/2} O \left(\varepsilon^{1/2} d(x, \partial\Omega) + \sum_{j=1}^k e^{-(1+\sigma)m \frac{d(x, \partial\Omega)}{\varepsilon}} + \sum_{h \neq j} e^{-(1+\sigma)m|x_h - x_j|/(2\varepsilon)} \right).$$

On the other hand, if Ω satisfies (Ω_1) and (Ω_2) for some finite group $G = \{id, g, \dots, \dots, g^{k-1}\}$ and $X = (x, gx, \dots, g^{k-1}x)$, where $x = (x', 0)$, then $\omega_{\varepsilon, X}$ satisfies

$$\omega_{\varepsilon, X}(gy) = \omega_{\varepsilon, X}(y), \quad \forall y \in \Omega,$$

and

$$\omega_{\varepsilon, X}(y', y_{\tilde{N}+1}, \dots, -y_i, \dots, y_N) = \omega_{\varepsilon, X}(y', y_{\tilde{N}+1}, \dots, y_i, \dots, y_N), \quad \forall y \in \Omega.$$

PROOF. The proof of the existence and the estimates of $\|\omega_{\varepsilon,X}\|_\varepsilon$ are very similar to those in Proposition 3.1. Thus we omit them.

For the proof of the last claim, we let $\tilde{\omega}_{\varepsilon,X}(y) = \omega_{\varepsilon,X}(gy)$. For any $\eta \in E_{\varepsilon,X,k}$, let $\tilde{\eta}(y) = \eta(g^{-1}y)$. Then it is easy to check that $\tilde{\eta} \in E_{\varepsilon,X,k}$. As a result,

$$\left\langle \frac{\partial I \left(\sum_{j=1}^k P_{\varepsilon,\Omega} U_{\varepsilon,x_j} + \tilde{\omega}_{\varepsilon,X} \right)}{\partial \omega}, \eta \right\rangle_\varepsilon = \left\langle \frac{\partial I \left(\sum_{j=1}^k P_{\varepsilon,\Omega} U_{\varepsilon,x_j} + \omega_{\varepsilon,X} \right)}{\partial \omega}, \tilde{\eta} \right\rangle_\varepsilon = 0.$$

By the uniqueness of $\omega_{\varepsilon,X}$ satisfying (4.1), we conclude that $\tilde{\omega}_{\varepsilon,X} = \omega_{\varepsilon,X}$. Similarly, we have $\omega_{\varepsilon,X}(y', y_{\tilde{N}+1}, \dots, -y_i, \dots, y_N) = \omega_{\varepsilon,X}(y', y_{\tilde{N}+1}, \dots, y_i, \dots, \dots, y_N)$. \square

In the rest of this section, we will work on the space of functions, each of which is even in y_i , $i > \tilde{N}$, and is invariant under the action of G .

PROOF OF THEOREM 1.2. Let

$$\tilde{D}_\varepsilon = \left\{ x : x = (x', 0) \in \Omega, d(x, \partial\Omega) \in \left[c_0\varepsilon \ln \frac{1}{\varepsilon}, \varepsilon^{1-\eta} \right] \right\},$$

where c_0 and η are two fixed small constants.

Consider

$$(4.2) \quad \inf_{x \in \tilde{D}_\varepsilon} I \left(\sum_{j=1}^k P_{\varepsilon,\Omega} U_{\varepsilon,g^j x} + \omega_{\varepsilon,X} \right),$$

where $X = (x, gx, \dots, g^{k-1}x)$.

Let x_ε be a minimum point of $I(\sum_{j=1}^k P_{\varepsilon,\Omega} U_{\varepsilon,g^j x} + \omega_{\varepsilon,X})$ in D_ε . We will prove that x'_ε is an interior point of $\tilde{D}_\varepsilon \cap \mathbb{R}^{\tilde{N}}$.

It follows from Proposition 4.1 that

$$(4.3) \quad I \left(\sum_{j=1}^k P_{\varepsilon,\Omega} U_{\varepsilon,g^j x} + \omega_{\varepsilon,X} \right) = I \left(\sum_{j=1}^k P_{\varepsilon,\Omega} U_{\varepsilon,g^j x} \right) + O(\|\omega_{\varepsilon,X}\|_\varepsilon^2).$$

On the other hand, using the exponentially decay of U at infinity, we can deduce

$$(4.4) \quad \begin{aligned} I \left(\sum_{j=1}^k P_{\varepsilon,\Omega} U_{\varepsilon,g^j x} \right) &= \sum_{j=1}^k I(P_{\varepsilon,\Omega} U_{\varepsilon,g^j x}) + 2 \sum_{h < j} \int_\Omega P_{\varepsilon,\Omega} U_{\varepsilon,g^h x} G_\gamma(P_{\varepsilon,\Omega} U_{\varepsilon,g^j x}) \\ &\quad + O \left(\varepsilon^N \sum_{h \neq j} e^{-m|g^h x - g^j x|/\varepsilon} \right). \end{aligned}$$

Suppose that $N \geq 3$. Then from Lemma 2.1, similar to the proof of Proposition 2.4, we deduce

$$\begin{aligned}
 & \int_{\Omega} P_{\varepsilon, \Omega} U_{\varepsilon, g^h x} G_{\gamma} (P_{\varepsilon, \Omega} U_{\varepsilon, g^j x}) \\
 &= \int_{\Omega} U_{\varepsilon, g^h x} G_{\gamma} U_{\varepsilon, g^j x} + \varepsilon^N O \left(\varepsilon^{2+\sigma} + \sum_{j=1}^k e^{-md(g^j x, \partial\Omega)/\varepsilon} \right) \\
 (4.5) \quad &= \varepsilon^{N+2} B W_{\varepsilon} \left(\frac{g^h x - g^j x}{\varepsilon} \right) - \varepsilon^{2N} B^2 H(g^h x, g^j x) \\
 &+ \varepsilon^N O \left(\frac{\varepsilon^{N+2}}{d(x, \partial\Omega)^N} + \frac{\varepsilon^N e^{-(m-\theta)d(x, \partial\Omega)/\varepsilon}}{d(x, \partial\Omega)^{N-2}} + e^{-m(2+\sigma)d(x, \partial\Omega)/\varepsilon} \right).
 \end{aligned}$$

Since $0 < W_{\varepsilon}(y) \leq C|y|^{-(N-2)}$, noting that $|g^h x - g^j x| > \tau > 0$ for any $x \in \tilde{D}_{\varepsilon}$ and $h \neq j$, we obtain from (4.5)

$$\begin{aligned}
 & \int_{\Omega} P_{\varepsilon, \Omega} U_{\varepsilon, g^h x} G_{\gamma} (P_{\varepsilon, \Omega} U_{\varepsilon, g^j x}) \\
 (4.6) \quad &= \varepsilon^N O \left(\frac{\varepsilon^{N+2}}{d(x, \partial\Omega)^N} + \frac{\varepsilon^N e^{-(m-\theta)d(x, \partial\Omega)/\varepsilon}}{d(x, \partial\Omega)^{N-2}} + e^{-m(2+\sigma)d(x, \partial\Omega)/\varepsilon} \right), \quad h \neq j.
 \end{aligned}$$

Combining (4.4) and (4.6), using Proposition 2.4, we obtain

$$\begin{aligned}
 & I \left(\sum_{j=1}^k P_{\varepsilon, \Omega} U_{\varepsilon, g^j x} \right) = k A \varepsilon^N + k \delta A_{\varepsilon} \varepsilon^{N+2} + k \tau_{\varepsilon, x} \\
 (4.7) \quad & - \frac{\delta}{2} k B^2 \varepsilon^{N+2} \varepsilon^{N-2} H(x, x) \\
 & + \varepsilon^N O \left(\frac{\varepsilon^{N+2}}{d(x, \partial\Omega)^N} + \frac{\varepsilon^N e^{-(m-\theta)d(x, \partial\Omega)/\varepsilon}}{d(x, \partial\Omega)^{N-2}} + e^{-m(2+\sigma)d(x, \partial\Omega)/\varepsilon} \right).
 \end{aligned}$$

Suppose that $N = 2$. Then

$$\begin{aligned}
 & \int_{\Omega} P_{\varepsilon, \Omega} U_{\varepsilon, g^h x} G_{\gamma} (P_{\varepsilon, \Omega} U_{\varepsilon, g^j x}) \\
 (4.8) \quad &= \frac{1}{2\pi} B^2 \varepsilon^4 \ln \frac{1}{\varepsilon} + \varepsilon^4 B \tilde{W} \left(\frac{g^h x - g^j x}{\varepsilon} \right) - \varepsilon^4 B^2 H(g^h x, g^j x) \\
 &+ \varepsilon^2 O \left(\frac{\varepsilon^4}{d(x, \partial\Omega)^2} + \varepsilon^2 e^{-(m-\theta)d(x, \partial\Omega)/\varepsilon} \ln \frac{1}{d(x, \partial\Omega)} + e^{-m(2+\sigma)d(x, \partial\Omega)/\varepsilon} \right).
 \end{aligned}$$

On the other hand, we see that

$$(4.9) \quad \tilde{W}(y) - \frac{B}{2\pi} \ln \frac{1}{|y|} = \frac{1}{2\pi} \int_{\mathbb{R}^2} \ln \frac{|y|}{|z-y|} U(z) dz$$

is bounded for $|y| \geq 1$.

In fact, from $\ln|x| \leq C(|x|^{1/2} + \frac{1}{|x|^{1/2}})$, we have

$$(4.10) \quad \left| \int_{\mathbb{R}^2} \ln \frac{|y|}{|z-y|} U(z) dz \right| \leq C \int_{\mathbb{R}^2} \left(\frac{|y|^{1/2}}{|z-y|^{1/2}} + \frac{|z-y|^{1/2}}{|y|^{1/2}} \right) U(z) dz.$$

Since $\frac{|z-y|}{|y|} \leq \frac{|z|+|y|}{|y|} \leq 1 + |z|$ for $|y| \geq 1$, we obtain

$$(4.11) \quad \int_{\mathbb{R}^2} \frac{|z-y|^{1/2}}{|y|^{1/2}} U(z) dz \leq C \int_{\mathbb{R}^2} (1 + |z|) U(z) dz \leq C'.$$

If $|z| \leq \frac{1}{2}|y|$, then $\frac{|y|}{|z-y|} \leq \frac{|y|}{|y|-|z|} \leq 2$. Thus

$$(4.12) \quad \int_{|z| \leq |y|/2} \frac{|y|^{1/2}}{|z-y|^{1/2}} U(z) dz \leq C \int_{|z| \leq |y|/2} U(z) dz \leq C.$$

But

$$(4.13) \quad \begin{aligned} & \int_{|z| \geq |y|/2} \frac{|y|^{1/2}}{|z-y|^{1/2}} U(z) dz \\ &= \int_{\{|z| \geq |y|/2\} \cap B_1(y)} \frac{|y|^{1/2}}{|z-y|^{1/2}} U(z) dz + \int_{\{|z| \geq |y|/2\} \setminus B_1(y)} \frac{|y|^{1/2}}{|z-y|^{1/2}} U(z) dz \\ &\leq \int_{\{|z| \geq |y|/2\} \cap B_1(y)} \frac{(|z|+1)^{1/2}}{|z-y|^{1/2}} U(z) dz + \int_{\{|z| \geq |y|/2\} \setminus B_1(y)} |y|^{1/2} U(z) dz \\ &\leq C \int_{\{|z| \geq |y|/2\} \cap B_1(y)} \frac{1}{|z-y|^{1/2}} dz + C \int_{\{|z| \geq |y|/2\} \setminus B_1(y)} |z|^{1/2} U(z) dz \leq C. \end{aligned}$$

Combining (4.10)–(4.13), we obtain (4.9).

Using (4.9), we obtain

$$\tilde{W} \left(\frac{g^h x - g^j x}{\varepsilon} \right) = -\frac{B}{2\pi} \ln \frac{|g^h y - g^j y|}{\varepsilon} + O(1),$$

which, together with (4.8), implies

$$(4.14) \quad \int_{\Omega} P_{\varepsilon, \Omega} U_{\varepsilon, g^h x} G_{\gamma}(P_{\varepsilon, \Omega} U_{\varepsilon, g^j x}) = \varepsilon^2 O(\varepsilon^2 + e^{-m(2+\sigma)d(x, \partial\Omega)/\varepsilon}), \quad h \neq j.$$

Thus,

$$(4.15) \quad I \left(\sum_{j=1}^k P_{\varepsilon, \Omega} U_{\varepsilon, g^j x} \right) = kA\varepsilon^2 + \frac{\delta}{4\pi} kB^2 \varepsilon^4 \ln \frac{1}{\varepsilon} + k\tau_{\varepsilon, x} - \frac{\delta}{2} kB^2 \varepsilon^4 H(x, x) \\ + \varepsilon^2 O(\varepsilon^2 + e^{-(2+\sigma)md(x, \partial\Omega)/\varepsilon}).$$

Suppose that $N = 1$. Then

$$(4.16) \quad \int_{\Omega} P_{\varepsilon, \Omega} U_{\varepsilon, g^h x} G_{\gamma}(P_{\varepsilon, \Omega} U_{\varepsilon, g^j x}) = B^2 G_{\gamma}(g^h x, g^j x) \varepsilon^2 + O(\varepsilon^4).$$

Since $G_{\gamma}(g^h x, g^j x) \sim d(g^h x, \partial\Omega)d(g^j x, \partial\Omega)$, we see from (4.16) that

$$\int_{\Omega} P_{\varepsilon, \Omega} U_{\varepsilon, g^h x} G_{\gamma}(P_{\varepsilon, \Omega} U_{\varepsilon, g^j x}) = \varepsilon^2 O(d^2(x, \partial\Omega)).$$

As a result, we have

$$(4.17) \quad I \left(\sum_{j=1}^2 P_{\varepsilon, \Omega} U_{\varepsilon, g^j x} \right) = 2A\varepsilon^2 + \delta B^2 G_{\gamma}(x, x) + 2\tau_{\varepsilon, x} \\ + \varepsilon O(\varepsilon^{1+\sigma} + e^{-(2+\sigma)md(x, \partial\Omega)/\varepsilon}).$$

Using (4.7), (4.15) and (4.17), we can prove that x'_ε is an interior point of $\tilde{D}_\varepsilon \cap \mathbb{R}^{\tilde{N}}$ in exactly the same way as we did in the proof of Theorem 1.1. So we have

$$(4.18) \quad \sum_{j=1}^k \left\langle I' \left(\sum_{h=1}^k P_{\varepsilon, \Omega} U_{\varepsilon, g^h x_\varepsilon} + \omega_{\varepsilon, X} \right), \frac{\partial P_{\varepsilon, \Omega} U_{\varepsilon, g^j x_\varepsilon}}{\partial x_i} + \frac{\partial \omega_{\varepsilon, X}}{\partial x_{ji}} \right\rangle_\varepsilon = 0,$$

for $i = 1, \dots, \tilde{N}$.

On the other hand, in view of the symmetry of the domain, the functions $\sum_{h=1}^k P_{\varepsilon, \Omega} U_{\varepsilon, g^h x_\varepsilon}$ and $\omega_{\varepsilon, X}$, we can check easily that for $j = 1, \dots, k-1$,

$$(4.19) \quad \left\langle I' \left(\sum_{h=1}^k P_{\varepsilon, \Omega} U_{\varepsilon, g^h x_\varepsilon} + \omega_{\varepsilon, X} \right), \frac{\partial P_{\varepsilon, \Omega} U_{\varepsilon, g^j x_\varepsilon}}{\partial x_i} \right\rangle_\varepsilon \\ = \left\langle I' \left(\sum_{h=1}^k P_{\varepsilon, \Omega} U_{\varepsilon, g^h x_\varepsilon} + \omega_{\varepsilon, X} \right), \frac{\partial P_{\varepsilon, \Omega} U_{\varepsilon, x_\varepsilon}}{\partial x_i} \right\rangle_\varepsilon,$$

and

$$(4.20) \quad \left\langle I' \left(\sum_{h=1}^k P_{\varepsilon, \Omega} U_{\varepsilon, g^h x_\varepsilon} + \omega_{\varepsilon, X} \right), \frac{\partial \omega_{\varepsilon, X}}{\partial x_{ji}} \right\rangle_\varepsilon \\ = \left\langle I' \left(\sum_{h=1}^k P_{\varepsilon, \Omega} U_{\varepsilon, g^h x_\varepsilon} + \omega_{\varepsilon, X} \right), \frac{\partial \omega_{\varepsilon, X}}{\partial x_{1i}} \right\rangle_\varepsilon,$$

Combining (4.18)–(4.20), we obtain

$$(4.21) \quad \left\langle I' \left(\sum_{h=1}^k P_{\varepsilon, \Omega} U_{\varepsilon, g^h x_\varepsilon} + \omega_{\varepsilon, X} \right), \frac{\partial P_{\varepsilon, \Omega} U_{\varepsilon, g^j x_\varepsilon}}{\partial x_i} + \frac{\partial \omega_{\varepsilon, X}}{\partial x_{ji}} \right\rangle_\varepsilon = 0,$$

for $i = 1, \dots, \tilde{N}$, $j = 1, \dots, k$.

On the other hand, using the symmetry of the function $\omega_{\varepsilon, X}$, we can check easily that

$$(4.22) \quad \left\langle I' \left(\sum_{h=1}^k P_{\varepsilon, \Omega} U_{\varepsilon, g^h x_\varepsilon} + \omega_{\varepsilon, X} \right), \frac{\partial P_{\varepsilon, \Omega} U_{\varepsilon, g^j x_\varepsilon}}{\partial x_i} + \frac{\partial \omega_{\varepsilon, X}}{\partial x_{ji}} \right\rangle_\varepsilon = 0,$$

for $i = \tilde{N} + 1, \dots, N$, $j = 1, \dots, k$.

From (4.21) and (4.22), we see $X_\varepsilon = (x_\varepsilon, g x_\varepsilon, \dots, g^{k-1} x_\varepsilon)$ is a critical point of the function

$$I \left(\sum_{h=1}^k P_{\varepsilon, \Omega} U_{\varepsilon, x_h} + \omega_{\varepsilon, X} \right).$$

As usual, we see that $\sum_{h=1}^k P_{\varepsilon, \Omega} U_{\varepsilon, g^h x_\varepsilon} + \omega_{\varepsilon, X}$ is a solution for (1.2). □

5. – Solutions with interior peak

In this section, we will prove Theorem 1.6. Let $u = \bar{u}_\varepsilon - w$. Then

$$(5.1) \quad \begin{cases} -\varepsilon^2 \Delta w + \delta G_\gamma w = q(y, w) =: f(\bar{u}_\varepsilon) - f(\bar{u}_\varepsilon - w), & y \in \Omega \\ w \in H_0^1(\Omega). \end{cases}$$

Since for each $y \in \Omega$ with $d(y, \partial\Omega) \geq \bar{\tau} > 0$, we have $\bar{u}_\varepsilon(y) \in (\tau_2, 1)$. Thus we have $h_+(f(\bar{u}_\varepsilon(y))) = \bar{u}_\varepsilon(y)$. So it is easy to see that $q(y, t) = 0$ has exactly three solutions

$$\begin{aligned} & \bar{u}_\varepsilon(y) - h_+(f(\bar{u}_\varepsilon(y))) = 0, \\ t_1(y) &= \bar{u}_\varepsilon(y) - h_0(f(\bar{u}_\varepsilon(y))) = h_+(f(\bar{u}_\varepsilon(y))) - h_0(f(\bar{u}_\varepsilon(y))) \end{aligned}$$

and

$$t_2(y) = \bar{u}_\varepsilon(y) - h_-(f(\bar{u}_\varepsilon(y))) = h_+(f(\bar{u}_\varepsilon(y))) - h_-(f(\bar{u}_\varepsilon(y))).$$

So we have

$$(5.2) \quad q(y, t) = t(t - t_1(y))(t_2(y) - t), \quad \forall d(y, \partial\Omega) \geq \bar{\tau}.$$

Let $b = \max_{x \in \Omega} v$, where v is the solution of (1.5). Let $U_b(x) = U_b(|x|)$ be a solution of (1.6).

Let $S = \{x : x \in \Omega, v(x) = \max_{x \in \Omega} v(x)\}$. Then S is a compact subset of Ω . Let $S_\alpha = \{x : d(x, S) \leq \alpha\}$.

$$E_{\varepsilon,b} = \left\{ \omega \in H_0^1(\Omega) : \left\langle \omega, \frac{\partial \bar{P}_{\varepsilon,\Omega} U_{\varepsilon,x,b}}{\partial x_i} \right\rangle_\varepsilon = 0, i = 1, \dots, N \right\},$$

where $U_{\varepsilon,x,b} = U_b(\frac{y-x}{\varepsilon})$, $\langle u, v \rangle_\varepsilon = \int_\Omega (\varepsilon^2 Du Dv + \bar{m}^2 uv)$, $\bar{m}^2 = -q'_b(0)$.

We will construct a solution for (5.1), which has the form

$$w_\varepsilon = \bar{P}_{\varepsilon,\Omega} U_{\varepsilon,x_\varepsilon,b} + \omega_\varepsilon,$$

where $\omega_\varepsilon \in E_{\varepsilon,b}$ satisfies $\|\omega_\varepsilon\|_\varepsilon = o(\varepsilon^{N/2})$.

Let

$$K(w) = \frac{\varepsilon^2}{2} \int_\Omega |Dw|^2 + \frac{\delta}{2} \int_\Omega w G_\gamma w - \int_\Omega Q(y, w), \quad w \in H_0^1(\Omega),$$

where $Q(y, t) = \int_0^t q(y, s) ds$.

PROPOSITION 5.1. *There is an $\varepsilon_0 > 0$ small, such that for each $\varepsilon \in (0, \varepsilon_0]$, there is a C^1 map $\omega_{\varepsilon,x} : S_\alpha \rightarrow H_0^1(\Omega)$, such that $\omega_{\varepsilon,x} \in E_{\varepsilon,b}$,*

$$\left\langle \frac{\partial K(\bar{P}_{\varepsilon,\Omega} U_{\varepsilon,x,b} + \omega_{\varepsilon,x})}{\partial \omega}, \xi \right\rangle = 0, \quad \forall \xi \in E_{\varepsilon,b}.$$

Moreover, we have the following estimate:

$$\|\omega_{\varepsilon,x}\|_\varepsilon = \varepsilon^{N/2} O(|\bar{v}(x) - b| + o_\varepsilon(1)),$$

where $o_\varepsilon(1) \rightarrow 0$ as $\varepsilon \rightarrow 0$.

PROOF. Expand $K(\bar{P}_{\varepsilon,\Omega} U_{\varepsilon,x,b} + \omega)$ near $\omega = 0$ as follows:

$$K(\bar{P}_{\varepsilon,\Omega} U_{\varepsilon,x,b} + \omega) = K(\bar{P}_{\varepsilon,\Omega} U_{\varepsilon,x,b}) + \langle k_\varepsilon(x), \omega \rangle + \frac{1}{2} \langle \bar{Q}_\varepsilon(x) \omega, \omega \rangle + R_\varepsilon(\omega),$$

where

$$\begin{aligned} \langle k_\varepsilon(x), \omega \rangle &= \int_\Omega \varepsilon^2 D \bar{P}_{\varepsilon,\Omega} U_{\varepsilon,x,b} D \omega + \delta \int_\Omega \omega G_\gamma \bar{P}_{\varepsilon,\Omega} U_{\varepsilon,x,b} - \int_\Omega q(y, \bar{P}_{\varepsilon,\Omega} U_{\varepsilon,x,b}) \omega, \\ \langle \bar{Q}_\varepsilon(x) \omega, \omega \rangle &= \varepsilon^2 \int_\Omega |D \omega|^2 + \delta \int_\Omega \omega G_\gamma \omega - \int_\Omega f_u(\bar{u}_\varepsilon - \bar{P}_{\varepsilon,\Omega} U_{\varepsilon,x,b}) \omega^2, \end{aligned}$$

and $R_\varepsilon(\omega)$ satisfies

$$\begin{aligned} R_\varepsilon(\omega) &= \varepsilon^N O(\varepsilon^{-3N/2} \|\varphi\|_\varepsilon^3), \\ R'_\varepsilon(\omega) &= \varepsilon^{N/2} O(\varepsilon^{-N} \|\varphi\|_\varepsilon^2), \\ R''_\varepsilon(\omega) &= O(\varepsilon^{-N/2} \|\varphi\|_\varepsilon). \end{aligned}$$

In [6], we proved that for all $\omega \in H_0^1(\Omega)$,

$$\varepsilon^2 \int_{\Omega} |D\omega|^2 + \delta \int_{\Omega} \omega G_{\gamma} \omega - \int_{\Omega} f_u(\bar{u}_{\varepsilon}) \omega^2 \geq c_0 \|\omega\|_{\varepsilon}^2$$

for some $c_0 > 0$. But U_b is nondegenerate. So arguing in a similar way as in Proposition 3.1, we can prove that $\bar{Q}_{\varepsilon}(x)$ is invertible and $\|\bar{Q}_{\varepsilon}(x)^{-1}\| \leq C$. In fact, the only place we need to change in the argument of Proposition 3.1 is the last relation. In the present situation, we have

$$\begin{aligned} & \varepsilon_j^2 \int_{\Omega} |D\omega_j|^2 + \delta \int_{\Omega} \omega_j G_{\gamma} \omega_j - \int_{\Omega} f_u(\bar{u}_{\varepsilon} - \bar{P}_{\varepsilon, \Omega} U_{\varepsilon, x, b}) \omega_j^2 \\ &= \varepsilon_j^2 \int_{\Omega} |D\omega_j|^2 + \delta \int_{\Omega} \omega_j G_{\gamma} \omega_j - \int_{\Omega} f_u(\bar{u}_{\varepsilon}) \omega_j^2 - o(\varepsilon_j^N) - o(1) \int_{\Omega} \omega_j^2 \\ &\geq c_0 \|\omega_j\|_{\varepsilon_j}^2 - o(\varepsilon_j^N) - o(1) \int_{\Omega} \omega_j^2. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \langle k_{\varepsilon}(x), \omega \rangle &= \delta \int_{\Omega} \omega G_{\gamma} \bar{P}_{\varepsilon, \Omega} U_{\varepsilon, x, b} + \int_{\Omega} (q_b(U_{\varepsilon, x, b}) - q(y, \bar{P}_{\varepsilon, \Omega} U_{\varepsilon, x, b})) \omega \\ &\quad + \int_{\Omega} (U_{\varepsilon, x, b} - \bar{P}_{\varepsilon, \Omega} U_{\varepsilon, x, b}) \omega. \end{aligned}$$

Similar to Lemma 3.3, we have

$$\left| \int_{\Omega} \omega G_{\gamma} \bar{P}_{\varepsilon, \Omega} U_{\varepsilon, x, b} \right| = o(\varepsilon^{N/2}) \|\omega\|_{\varepsilon}^2.$$

Moreover,

$$\begin{aligned} & \left| \int_{\Omega} (q_b(U_{\varepsilon, x, b}) \omega - q(y, P_{\varepsilon, \Omega} U_{\varepsilon, x, b}) \omega) \right| \\ &= \left| \int_{\Omega} (q_b(U_{\varepsilon, x, b}) \omega - q(y, U_{\varepsilon, x, b}) \omega) \right| + o(\varepsilon^{N/2}) \|\omega\|_{\varepsilon} \\ &\leq C \int_{\Omega} U_{\varepsilon, x, b}(y) |f(\bar{u}_{\varepsilon}(y)) - b| |\omega| + o(\varepsilon^{N/2}) \|\omega\|_{\varepsilon} \\ &\leq C \varepsilon^{N/2} |f(\bar{u}_{\varepsilon}(x)) - b| \|\omega\|_{\varepsilon} = \varepsilon^{N/2} O(|f(\bar{u}_{\varepsilon}(x)) - b| + o(1)) \|\omega\|_{\varepsilon} \\ &= \varepsilon^{N/2} O(|\bar{v}(x) - b| + o(1)) \|\omega\|_{\varepsilon}, \end{aligned}$$

and

$$\int_{\Omega} (U_{\varepsilon, x, b} - \bar{P}_{\varepsilon, \Omega} U_{\varepsilon, x, b}) \omega = o(\varepsilon^{N/2}) \|\omega\|_{\varepsilon}.$$

Thus, we obtain

$$\|k_{\varepsilon}(x)\| = \varepsilon^{N/2} O(|\bar{v}(x) - b| + o(1)),$$

and the result follows. □

PROPOSITION 5.2. *We have*

$$K(\bar{P}_{\varepsilon, \Omega} U_{\varepsilon, x, b}) = \varepsilon^N \bar{A} + \varepsilon^N \bar{B}(b - \bar{v}(x)) + \varepsilon^N O(|\bar{v}(x) - b|^2 + o(1)),$$

where $\bar{A} = \frac{1}{2} \int_{\mathbb{R}^N} |DU_b|^2 - \int_{\mathbb{R}^N} G_b(U_b)$ and $\bar{B} > 0$ is a constant.

PROOF. We have

$$\begin{aligned} K(\bar{P}_{\varepsilon, \Omega} U_{\varepsilon, x, b}) &= \frac{1}{2} \int_{\Omega} \varepsilon^2 |D\bar{P}_{\varepsilon, \Omega} U_{\varepsilon, x, b}|^2 - \int_{\Omega} Q(y, \bar{P}_{\varepsilon, \Omega} U_{\varepsilon, x, b}) + o(\varepsilon^N) \\ (5.3) \quad &= \frac{1}{2} \int_{\mathbb{R}^N} \varepsilon^2 |DU_{\varepsilon, x, b}|^2 - \int_{\mathbb{R}^N} Q(y, U_{\varepsilon, x, b}) + o(\varepsilon^N) \\ &= \varepsilon^N \bar{A} + \int_{\mathbb{R}^N} (Q_b(U_{\varepsilon, x, b}) - Q(y, U_{\varepsilon, x, b})) + o(\varepsilon^N). \end{aligned}$$

On the other hand, it is easy to see

$$Q_b(t) - Q(y, t) = ((t_1(y) + t_2(y)) - (\tilde{t}_1 + \tilde{t}_2)) \frac{t^3}{3} - (t_1(y)t_2(y) - \tilde{t}_1\tilde{t}_2) \frac{t^2}{2},$$

where $t_2(y) = h_+(f(\bar{u}_\varepsilon(y))) - h_-(f(\bar{u}_\varepsilon(y)))$, $t_1(y) = h_+(f(\bar{u}_\varepsilon(y))) - h_0(f(\bar{u}_\varepsilon(y)))$, $\tilde{t}_2 = h_+(b) - h_-(b)$ and $\tilde{t}_1 = h_+(b) - h_0(b)$. Thus,

$$\begin{aligned} (5.4) \quad &\int_{\mathbb{R}^N} (Q_b(U_{\varepsilon, x, b}) - Q(y, U_{\varepsilon, x, b})) = \varepsilon^N \frac{1}{3} ((\tilde{t}_1 + \tilde{t}_2) - (t_1(x) + t_2(x))) \\ &\times \int_{\mathbb{R}^N} U_b^3 - \varepsilon^N \frac{1}{2} (\tilde{t}_1\tilde{t}_2 - t_1(x)t_2(x)) \int_{\mathbb{R}^N} U_b^2 + o(\varepsilon^N). \end{aligned}$$

Denote $\xi_1(t) = h_+(t) - h_0(t)$, $\xi_2(t) = h_+(t) - h_-(t)$. Then (5.4) can be written as

$$\begin{aligned} (5.5) \quad &\int_{\mathbb{R}^N} (Q_b(U_{\varepsilon, x, b}) - Q(y, U_{\varepsilon, x, b})) = -\varepsilon^N \\ &\times \left(\frac{1}{3} (\xi_1'(b) + \xi_2'(b)) \int_{\mathbb{R}^N} U_b^3 + \frac{1}{2} (\xi_1'(b)\xi_2(b) + \xi_1(b)\xi_2'(b)) \int_{\mathbb{R}^N} U_b^2 \right) \\ &\times (f(\bar{u}_\varepsilon(x)) - b) + \varepsilon^N O(|f(\bar{u}_\varepsilon(x)) - b|^2) + o(\varepsilon^N) = -\varepsilon^N \\ &\times \left(\frac{1}{3} (\xi_1'(b) + \xi_2'(b)) \int_{\mathbb{R}^N} U_b^3 + \frac{1}{2} (\xi_1'(b)\xi_2(b) + \xi_1(b)\xi_2'(b)) \int_{\mathbb{R}^N} U_b^2 \right) \\ &\times (\bar{v}(x) - b) + \varepsilon^N O(|\bar{v} - b|^2 + o(1)). \end{aligned}$$

Let

$$\bar{B} = \frac{1}{3} (\xi_1'(b) + \xi_2'(b)) \int_{\mathbb{R}^N} U_b^3 - \frac{1}{2} (\xi_1'(b)\xi_2(b) + \xi_1(b)\xi_2'(b)) \int_{\mathbb{R}^N} U_b^2.$$

We claim that $\bar{B} > 0$. Assuming this, we see that the result follows from (5.3) and (5.5).

Now we prove that $\bar{B} > 0$.

First, we have

$$\begin{aligned} & - \int_{\mathbb{R}^N} U_b^3 + (\xi_1(b) + \xi_2(b)) \int_{\mathbb{R}^N} U_b^2 - \xi_1(b)\xi_2(b) \int_{\mathbb{R}^N} U_b \\ & = \int_{\mathbb{R}^N} q_b(U_b) = - \int_{\mathbb{R}^N} \Delta U_b = 0. \end{aligned}$$

As a result, \bar{B} can be rewritten as

$$\begin{aligned} \bar{B} & = \left(\frac{1}{3}(\xi'_1(b) + \xi'_2(b))(\xi_1(b) + \xi_2(b)) - \frac{1}{2}(\xi'_1(b)\xi_2(b) + \xi_1(b)\xi'_2(b)) \right) \int_{\mathbb{R}^N} U_b^2 \\ & \quad - \frac{1}{3}(\xi'_1(b) + \xi'_2(b))\xi_1(b)\xi_2(b) \int_{\mathbb{R}^N} U_b \\ (5.6) \quad & = \frac{1}{6}(2\xi'_1(b)\xi_1(b) + 2\xi_2(b)\xi'_2(b) - \xi'_1(b)\xi_2(b) - \xi_1(b)\xi'_2(b)) \int_{\mathbb{R}^N} \frac{U_b^2}{\xi_2^2(b)} \\ & \quad - \frac{1}{3}(\xi'_1(b) + \xi'_2(b))\xi_1(b)\xi_2^2(b) \int_{\mathbb{R}^N} \frac{U_b}{\xi_2(b)}. \end{aligned}$$

We claim

$$\begin{aligned} (5.7) \quad & \frac{1}{6}(2\xi'_1(b)\xi_1(b) + 2\xi_2(b)\xi'_2(b) - \xi'_1(b)\xi_2(b) - \xi_1(b)\xi'_2(b)) \\ & > \frac{1}{3}(\xi'_1(b) + \xi'_2(b))\xi_1(b). \end{aligned}$$

In fact, (5.7) is equivalent to

$$(5.8) \quad -3\xi_1(b)\xi'_2(b) - \xi'_1(b)\xi_2(b) + 2\xi_2(b)\xi'_2(b) > 0,$$

which can also be rewritten as

$$(5.9) \quad -3(h'_+(b) - h'_-(b))\xi_1(b) - 2h'_-(b)\xi_2(b) + (h'_+(b) + h'_0(b))\xi_2(b) > 0.$$

Since by definition

$$f(t) - b = (t - h_-(b))(t - h_0(b))(h_+(b) - t),$$

we have

$$\begin{aligned} f'(h_+(b)) & = -(h_+(b) - h_-(b))(h_+(b) - h_0(b)), \\ f'(h_0(b)) & = (h_0(b) - h_-(b))(h_+(b) - h_0(b)), \end{aligned}$$

and

$$f'(h_-(b)) = -(h_0(b) - h_-(b))(h_+(b) - h_-(b)).$$

Thus,

$$h'_+(b) + h'_0(b) = \frac{f'(h_0(b)) + f'(h_+(b))}{f'(h_0(b))f'(h_+(b))} > 0,$$

$$h'_-(b) = \frac{1}{f'(h_-(b))} < 0,$$

and

$$\xi'_2(b) = h'_+(b) - h'_-(b) = \frac{f'(h_-(b)) - f'(h_+(b))}{f'(h_-(b))f'(h_+(b))} < 0, \quad \text{if } b > \alpha_0.$$

In the last relation, we have used the fact that $h_0(b) - h_-(b) > h_+(b) - h_0(b)$ if $b > \alpha_0$, which can be deduced directly from $\int_{h_-(b)}^{h_+(b)} (f(t) - b) dt < 0$. So we see that each term in the left hand hand of (5.9) is positive. Thus the claim follows.

Finally, combining (5.6) and (5.7), we obtain

$$\bar{B} > -\frac{\xi_1(b)\xi_2^2(b)}{3}(\xi'_1(b) + \xi'_2(b)) \int_{\mathbb{R}^N} \left(\frac{U_b}{\xi_2(b)} - \frac{U_b^2}{\xi_2^2(b)} \right) > 0$$

since $U_b \leq \xi_2(b)$. □

PROOF OF THEOREM 1.6. Consider

$$(5.10) \quad \inf\{K(P_{\varepsilon,\Omega}U_{\varepsilon,x,b} + \omega_\varepsilon) : x \in S_\alpha\}.$$

It follows from Propositions 5.1 and 5.2 that

$$(5.11) \quad K(P_{\varepsilon,\Omega}U_{\varepsilon,x,b} + \omega_\varepsilon) = \varepsilon^N \bar{A} + \varepsilon^N \bar{B}(b - \bar{v}(x)) + \varepsilon^N O(|\bar{v}(x) - b|^2 + o(1)).$$

Let $x_\varepsilon \in S_\alpha$ be a minimum point of (5.10). Then from (5.11), we see that $x_\varepsilon \rightarrow x_0 \in S$. So x_ε is an interior point of S_α . □

REMARK 5.3. From (5.11), we see that if $\bar{v}(x)$ attains its local maximum on an isolated set \tilde{S} , then we can attach to \bar{u}_ε a peak near \tilde{S} to obtain a solution for (1.1).

REFERENCES

- [1] A. BAHRI, "Critical points at infinity in some variational problems", Research Notes in Mathematics 182, Longman-Pitman, 1989.
- [2] P. CLÉMENT – G. SWEERS, *Existence and multiplicity results for a semilinear eigenvalue problem*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **14** (1987), 97-121.
- [3] E. N. DANCER, *A note on asymptotic uniqueness for some nonlinearities which change sign*, Bull. Austral. Math. Soc. **61** (2000), 305-312.

- [4] E. N. DANCER – S. YAN, *Interior and boundary peak solutions for a mixed boundary value problem*, Indiana Univ. Math. J. **48** (1999), 1177-1212.
- [5] E. N. DANCER – S. YAN, *Singularly perturbed elliptic problem in exterior domains*, J. Differential Integral Equations **13** (2000), 747-777.
- [6] E. N. DANCER – S. YAN, *A minimization problem associated with elliptic system of FitzHugh-Nagumo type*, Ann. Inst. H. Poincaré, Analyse Non Linéaire, to appear.
- [7] D. G. DEFIGUEIREDO – E. MITIDIERI, *A maximum principle for an elliptic system and applications to semilinear problems*, SIAM J. Math. Anal. **17** (1986), 836-849.
- [8] R. FITZHUGH, *Impulses and physiological states in theoretical models of nerve membrane*, Biophys. J. **1** (1961), 445-466.
- [9] R. GARDNER – L. A. PELETIER, *The set of positive solutions of semilinear equations in large ball*, Proc. Royal Soc. Edinburgh **104A** (1986), 53-72.
- [10] A. HODGKIN – A. HUXLEY, *A quantitative description of membrane current and its application to conduction and excitation in nerve*, J. Physiol. **117** (1952), 500-544.
- [11] G. A. KLAASEN – E. MITIDIERI, *Standing wave solutions for a system derived from the FitzHugh-Nagumo equation for nerve conduction*, SIAM J. Math. Anal. **17** (1986), 74-83.
- [12] G. A. KLAASEN – W. C. TROY, *Stationary wave solutions of a system of reaction-diffusion equations derived from the FitzHugh-Nagumo equations*, SIAM J. Appl. Math. **44** (1984), 96-110.
- [13] J. S. NAGUMO – S. ARIMOTO – Y. YOSHIZAWA, *An active pulse transmission line simulating nerve axon*, Proc. Inst. Radio. Engineers **50** (1962), 2061-2070.
- [14] W. M. NI – I. TAKAGI – J. WEI, *On the location and profile of spike-layer solutions to a singularly perturbed semilinear Dirichlet problem, intermediate solution*, Duke Math. J. **94** (1998), 597-618.
- [15] W. M. NI – J. WEI, *On the location and profile of spike-layer solutions to singularly perturbed semilinear Dirichlet problem*, Comm. Pure Appl. Math. **48** (1995), 731-768.
- [16] L. A. PELETIER – J. SERRIN, *Uniqueness of positive solutions of semilinear equations in \mathbb{R}^n* , Arch. Rat. Mech. Anal. **81** (1983), 181-197.
- [17] C. REINECKE – G. SWEERS, *A boundary layer solution to a semilinear elliptic system of FitzHugh-Nagumo type*, C. R. Acad. Sci. Paris Sér. I. Math. **329** (1999), 27-32.
- [18] C. REINECKE – G. SWEERS, *A positive solution on \mathbb{R}^N to a system of elliptic equations of FitzHugh-Nagumo type*, J. Differential Equations **153** (1999), 292-312.
- [19] C. REINECKE – G. SWEERS, *Existence and uniqueness of solutions on bounded domains to a FitzHugh-Nagumo type elliptic system*, Pacific J. Math. **197** (2001), 183-211.
- [20] C. REINECKE – G. SWEERS, *Solutions with internal jump for an autonomous elliptic system of FitzHugh-Nagumo type*, Math. Nachr. **251** (2003), 64-87.
- [21] O. REY, *The role of the Green's function in a non-linear elliptic equation involving the critical Sobolev exponent*, J. Funct. Anal. **89** (1990), 1-52.
- [22] G. SWEERS – W. TROY, *On the bifurcation curve for an elliptic system of FitzHugh-Nagumo type*, Physica D: Nonlinear Phenomena **177** (2003), 1-22.

School of Mathematics and Statistics
University of Sydney
NSW 2006, Australia
normd@maths.usyd.edu.au
shusen@maths.usyd.edu.au