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On Singular Perturbation Problems with Robin Boundary Condition

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Abstract. We consider the following singularly perturbed elliptic problem

$$\epsilon^2 \Delta u - u + f(u) = 0,\ u > 0 \text{ in } \Omega,$$
$$\epsilon \frac{\partial u}{\partial \nu} + \lambda u = 0 \text{ on } \partial\Omega,$$

where f satisfies some growth conditions, $0 \leq \lambda \leq +\infty$, and $\Omega \subset \mathbb{R}^N$ $(N > 1)$ is a smooth and bounded domain. The cases $\lambda = 0$ (Neumann problem) and $\lambda = +\infty$ (Dirichlet problem) have been studied by many authors in recent years. We show that, there exists a generic constant $\lambda_* > 1$ such that, as $\epsilon \to 0$, the least energy solution has a spike near the boundary if $\lambda \leq \lambda_*$, and has an interior spike near the innermost part of the domain if $\lambda > \lambda_*$. Central to our study is the corresponding problem on the half space.

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1. – Introduction

In the recent years, many works have been devoted to the study of the following singularly perturbed problems:

$$\epsilon^2 \Delta u - u + f(u) = 0, u > 0 \text{ in } \Omega, \tag{1.1}$$

with either Neumann boundary condition

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

or Dirichlet boundary condition

$$u = 0 \text{ on } \partial\Omega. \tag{1.3}$$

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Here $\Omega \subset \mathbb{R}^N$ is a smooth and bounded domain, ν denotes the outward normal, and f satisfies some structure conditions. A typical f is $f(u) = u^p, 1 < p < \frac{N+2}{N-2}$ if $N \geq 3$, and $1 < p < \infty$ if $N = 1, 2$.

In [26], [27], Ni and Takagi showed that, under some conditions on $f(u)$, as $\epsilon \to 0$, the least energy solution for (1.1) with Neumann boundary condition (1.2) has a unique maximum point, say P_ϵ, on $\partial\Omega$. Moreover, $H(P_\epsilon) \to \max_{P \in \partial\Omega} H(P)$, where $H(P)$ is the mean curvature function on $\partial\Omega$. On the other hand, Ni and Wei in [30] showed that, as $\epsilon \to 0$, the least energy solution for (1.1) with Dirichlet boundary condition (1.3) has a unique maximum point, say Q_ϵ, in Ω. Furthermore, $d(Q_\epsilon, \partial\Omega) \to \max_{Q \in \Omega} d(Q, \partial\Omega)$, where $d(Q, \partial\Omega)$ is the distance function from Q to $\partial\Omega$.

Since then, many papers further investigated the higher energy solutions for (1.1) with either (1.2) or (1.3). These solutions are called spike layer solutions. A general principle is that the interior spike layer solutions are generated by distance functions. We refer the reader to the articles [1], [6], [8], [9], [10], [13], [14], [17], [20], [29], [31], [33], [34] and the references therein. On the other hand, the boundary peaked solutions are related to the boundary mean curvature function. This aspect is discussed in the papers [2], [5], [15], [19], [32], [35], [36], and the references therein. A good review of the subject is to be found in [25].

It is a natural question to ask what happens if we replace (1.2) or (1.3) by the following Robin boundary conditions (or boundary conditions of the third kind)

$$a\frac{\partial u}{\partial \nu} + (1-a)u = 0 \text{ on } \partial\Omega, \tag{1.4}$$

where $0 < a < 1$. Such Robin boundary conditions are particularly interesting in biological models where they often arise. We refer the reader to [7] for this aspect.

The main purpose of this paper is to answer the above question.

First of all, we rewrite (1.4) in the following form

$$\epsilon\frac{\partial u}{\partial \nu} + \lambda u = 0 \text{ on } \partial\Omega, \tag{1.5}$$

where $\lambda = \frac{\epsilon(1-a)}{a} > 0$. (The term $\epsilon\frac{\partial u}{\partial \nu}$ is an appropriate scaling with respect to $\epsilon^2 \Delta u$, as we shall see later.) We shall investigate the role of λ on the properties of *least-energy* solutions, which we shall define now.

Similar to [26] and [30], we can define the following energy functional associated with (1.5):

$$J_\epsilon[u] := \frac{\epsilon^2}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2}\int_\Omega u^2 - \int_\Omega F(u) + \frac{\epsilon\lambda}{2}\int_{\partial\Omega} u^2, \tag{1.6}$$

where $F(u) = \int_0^u f(s)ds, u \in H^1(\Omega)$.

Assume that f has superlinear growth. Then, for each fixed λ, by taking a function $e(x) \equiv k$ for some constant k in Ω, and choosing k large enough, we have $J_\epsilon(e) < 0$, for all $\epsilon \in (0, 1)$. Then for fixed λ, and for each $\epsilon \in (0, 1)$, we can define the so-called mountain-pass value

$$c_{\epsilon,\lambda} = \inf_{h\in\Gamma} \max_{0\le t\le 1} J_\epsilon[h(t)] \tag{1.7}$$

where $\Gamma = \{h : [0, 1] \to H^1(\Omega)|h(t)$ is continuous, $h(0) = 0, h(1) = e\}$.

Under some further conditions on f, which we will specify later, similar to [26] or [30], $c_{\epsilon,\lambda}$ can be characterized by

$$c_{\epsilon,\lambda} = \inf_{u\not\equiv 0, u\in H^1(\Omega)} \sup_{t>0} J_\epsilon[tu], \tag{1.8}$$

which can be shown to be the least among all nonzero critical values of J_ϵ. (This formulation is sometimes referred to as the Nehari manifold technique.) Moreover, $c_{\epsilon,\lambda}$ is attained by some function $u_{\epsilon,\lambda}$ which is then called a least-energy solution. Here and throughout this paper, we say that a function $u_{\epsilon,\lambda}$ achieves the maximum in (1.8) if it satisfies

$$c_{\epsilon,\lambda} = \sup_{t>0} J_\epsilon[tu_{\epsilon,\lambda}].$$

This also applies to similar variational formulations below.

For fixed ϵ small, as λ moves from 0 (which is (1.2)) to $+\infty$ (which is (1.3)), by the results of [26], [27] and [30], the asymptotic behavior of $u_{\epsilon,\lambda}$ changes dramatically: a boundary spike is displaced to become an interior spike. The question we shall answer is: where is the borderline of λ for spikes to move inwards?

Note that when $N = 1$, by ODE analysis, it is easy to see that the borderline is exactly at $\lambda = 1$. In fact, we may assume that $\Omega = (0, 1)$, and as $\epsilon \to 0$, the least energy solution converges to a homoclinic solution of the following ODE:

$$w'' - w + f(w) = 0 \text{ in } \mathbb{R}^1, \quad w(y) \to 0 \text{ as } |y| \to +\infty. \tag{1.9}$$

Then it follows that

$$(w')^2 = w^2 - 2F(w), \quad |w'| < w. \tag{1.10}$$

As $\epsilon \to 0$, the limiting boundary condition (1.5) becomes $w'(0) - \lambda w(0) = 0$. We see from (1.10) that this is possible if and only if $\lambda < 1$. A graph of the homoclinic solution with the Robin boundary condition $w'(0) - \lambda w(0) = 0$ is depicted as follows:

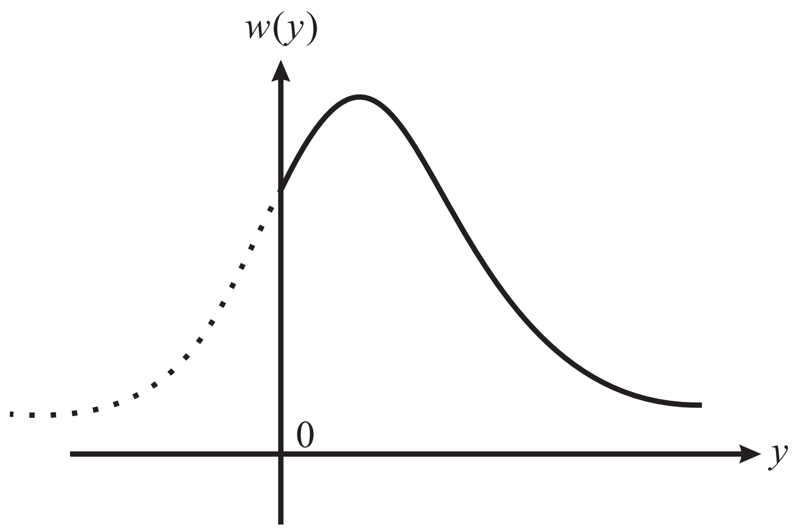

Fig. 1. The graph of $w(y)$.

So from now on, we may assume that $N \geq 2$.

To state our results, let us first put some assumptions on f. Firstly, since we look for a positive solution, by the maximum principle, we may extend f by assuming $f(t) = 0$ for $t \leq 0$. We assume the same assumptions used in [26], [27] and [30]: we suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous and satisfy the following structure conditions

(f1) $f^{'}(0) = 0$.

(f2) For $t \geq 0$, f admits the decomposition in $C^{1+\sigma}(\mathbb{R})$:

$$f(t) = f_1(t) - f_2(t)$$

where (i) $f_1(t) \geq 0$ and $f_2(t) \geq 0$ with $f_1(0) = f_1^{'}(0) = 0$, whence it follows that $f_2(0) = f_2^{'}(0) = 0$ by (f1); and (ii) there is a $q \geq 1$ such that $f_1(t)/t^q$ is nondecreasing in $t > 0$, whereas $f_2(t)/t^q$ is non-increasing in $t > 0$, and in case $q = 1$ we require further that the above monotonicity condition for $f_1(t)/t$ is strict.

(f3) $f(t) = O(t^p)$ as $t \to +\infty$ for some $1 < p < \frac{N+2}{N-2}$ and $1 < p < +\infty$ if $N = 2$.

(f4) There exists a constant $\theta \in (0, \frac{1}{2})$ such that $F(t) \leq \theta t f(t)$ for $t \geq 0$, where $F(t) = \int_0^t f(s)ds$.

Condition (f1) is first related to the fact that we already include the linear part (via $-u$) in the equation. The assumption (f2) is technical while (f3) and (f4) are classical assumptions. (f3) is an assumption of compactness (subcritical growth) and (f4) is used for Palais-Smale condition and implies a superlinear growth for f, namely, $\frac{f(t)}{t^{1+\delta}} \to +\infty$ as $t \to +\infty$ for some $\delta > 0$.

Examples of f satisfying (f1)-(f4) include:

$$f(u)=u^p-au^q \quad \text{with} \quad a \geq 0, 1 < q < p < \frac{N+2}{N-2} \quad \text{if } N \geq 3 \quad \text{and } +\infty \text{ if } N=2.$$

To understand the location of the spikes at the boundary, an essential role is played by the analogous problem in a half space with Robin boundary condition on the boundary. Thus we first consider

$$\begin{cases} \Delta u - u + f(u) = 0, \;\; u > 0 & \text{in } \mathbb{R}^N_+, \\ u \in H^1(\mathbb{R}^N_+), \; \dfrac{\partial u}{\partial \nu} + \lambda u = 0 & \text{on } \partial \mathbb{R}^N_+ \end{cases} \tag{1.11}$$

where $\mathbb{R}^N_+ = \{(y', y_N) | y_N > 0\}$ and ν is the outer normal on $\partial \mathbb{R}^N_+$.

Let

$$I_\lambda[u] = \int_{\mathbb{R}^N_+} \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{2}u^2 \right) - \int_{\mathbb{R}^N_+} F(u) + \frac{\lambda}{2} \int_{\partial \mathbb{R}^N_+} u^2. \tag{1.12}$$

As before, we define a mountain-pass vale for I_λ:

$$c_\lambda = \inf_{v \not\equiv 0, v \in H^1(\mathbb{R}^N_+)} \sup_{t>0} I_\lambda[tv]. \tag{1.13}$$

Our first result deals with the half space problem:

THEOREM 1.1.

(1) *For $\lambda \leq 1$, c_λ is achieved by some function w_λ, which is a solution of* (1.11).

(2) *For λ large enough, c_λ is never achieved.*

(3) *Set*

$$\lambda_* = \inf\{\lambda | c_\lambda \text{ is achieved}\}.$$

Then $\lambda_* > 1$ and for $\lambda \leq \lambda_*$, c_λ is achieved, and for $\lambda > \lambda_*$, c_λ is not achieved.

REMARK 1.1. We do not know if the solution to (1.11) is unique. It would be interesting to solve this question.

REMARK 1.2. It is somewhat surprising that at the borderline number $\lambda = \lambda_*$, c_λ is actually achieved.

REMARK 1.3. It is an interesting question to see how λ_* depends on N and the nonlinearity f. Note that when $N = 1$, $\lambda_* = 1$.

Now consider the problem in a bounded domain. It turns out that the critical number λ_* in Theorem (1.1) plays an essential role in the study of the asymptotic behavior of $c_{\epsilon,\lambda}$ and $u_{\epsilon,\lambda}$ defined in a bounded domain. We will show here that λ_* is the borderline between (1.2) and (1.3) in an arbitrary domain.

As before, we use $H(x_0)$ to denote the boundary mean curvature at $x_0 \in \partial\Omega$. Here is our first result for a general domain.

THEOREM 1.2. *Let $\lambda \leq \lambda_*$ and $u_{\epsilon,\lambda}$ be a least energy solution of* (1.1) *with* (1.5). *Let $x_\epsilon \in \Omega$ be a point where $u_{\epsilon,\lambda}$ reaches its maximum value. Then after passing to a subsequence, $x_\epsilon \to x_0 \in \partial\Omega$ and*

(1) $d(x_\epsilon, \partial\Omega)/\epsilon \to d_0$, *for some* $d_0 > 0$,

(2) $v_{\epsilon,\lambda}(y) = u_{\epsilon,\lambda}(x_\epsilon + \epsilon y) \to w_\lambda(y)$ *in C^1 locally, where w_λ attains c_λ of* (1.13) (*and thus is a solution of* (1.11)),

(3) *the associated critical value can be estimated as follows:*

$$c_{\epsilon,\lambda} = \epsilon^N \{c_\lambda - \epsilon \overline{H}(x_0) + o(\epsilon)\} \tag{1.15}$$

where c_λ is given by (1.13), *and $\overline{H}(x_0)$ is given by the following*

$$\overline{H}(x_0) = \max_{w_\lambda \in \mathcal{S}_\lambda} \left[- \int_{\mathbb{R}^N_+} y' \cdot \nabla' w_\lambda \frac{\partial w_\lambda}{\partial y_N} H(x_0) \right] \tag{1.16}$$

where $\mathcal{S}_\lambda$ is the set of all solutions of (1.11) *attaining c_λ, and $y' = (y_1, \ldots, y_{N-1})$, $\nabla' = (\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_{N-1}})$,*

(4) $\overline{H}(x_0) = \max_{x \in \partial\Omega} \bar{H}(x)$.

REMARK 1.4. For λ small, it is easy to see that

$$- \int_{\mathbb{R}^N_+} y' \cdot \nabla' w_\lambda \frac{\partial w_\lambda}{\partial y_N} > 0, \quad \text{for all } w_\lambda \in \mathcal{S}_\lambda \tag{1.17}$$

and hence

$$\overline{H}(x_0) = \left(\max_{w_\lambda \in \mathcal{S}_\lambda} \left[- \int_{\mathbb{R}^N_+} y' \cdot \nabla' w_\lambda \frac{\partial w_\lambda}{\partial y_N} \right] \right) H(x_0).$$

In this case, the last statement in Theorem 1.2 can be replaced by $H(x_0) = \max_{x \in \partial\Omega} H(x)$. We don't know whether or not (1.17) holds for $\lambda \leq \lambda_*$. When $\lambda = 0$, the function $\overline{H}(z)$ is called *generalized mean curvature function* in [4].

REMARK 1.5. When $\lambda \leq \lambda_*$, the maximum point x_ϵ is not on the boundary. Instead, it is in the order ϵ distance away from $\partial\Omega$. This only happens for Robin boundary problems since in the case of Neumann conditions, the maximum points lie on the boundary, while for Dirichlet condition, they are interior points which stay away from the boundary.

On the other hand, when $\lambda > \lambda_*$, a different asymptotic behavior appears.

THEOREM 1.3. *Let $\lambda > \lambda_*$ and $u_{\epsilon,\lambda}$ be a least energy solution of* (1.1) *with* (1.5). *Let $x_\epsilon \in \Omega$ be a point where $u_{\epsilon,\lambda}$ reaches its maximum value. Then after passing a subsequence, we have*

(1) $d(x_\epsilon, \partial\Omega) \to \max_{x \in \Omega} d(x, \partial\Omega)$,

(2) *$v_{\epsilon,\lambda}(y) := u_{\epsilon,\lambda}(x_\epsilon + \epsilon y) \to w(y)$ in C^1 locally, where w is the unique solution of the following problem*

$$\begin{cases} \Delta w - w + f(w) = 0 & \text{in } \mathbb{R}^N, \\ w > 0,\, w(0) = \max_{y\in\mathbb{R}^N} w(y), & \\ w(y) \to 0 & \text{as } |y| \to +\infty, \end{cases} \tag{1.18}$$

(3) *the associated critical value can be estimated as follows:*

$$c_{\epsilon,\lambda} = \epsilon^N \left[I[w] + \exp\left(-\frac{2d(x_\epsilon, \partial\Omega)}{\epsilon}(1 + o(1)) \right) \right] \tag{1.19}$$

where

$$I[w] = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla w|^2 + w^2) - \int_{\mathbb{R}^N} F(w). \tag{1.20}$$

REMARK 1.6. According to [16], w is radially symmetric and decreasing, i.e., $w = w(r), r = |y|, w'(r) < 0$ for $r > 0$. Under the conditions (f1)-(f4), it follows from the results of [18] and [3] that the solution to (1.18) is unique.

The existence and location of spikes has been studied in detail in the papers [26], [27] and [30] for Dirichlet or Neumann conditions. Here we rely on the techniques developed in these works. The main new aspect that is needed here is concerned with the problem in a half space with Robin boundary condition (Theorem 1.1). Section 2 is devoted to the proof of that theorem. Actually, the main novel aspect of this paper is to derive the existence of this separating value λ^* given by Theorem 1.1 which governs the location of spikes.

The study of spikes and their locations are carried in Section 3 for Theorem 1.2 and in Section 4 for Theorem 1.3. These two sections are somewhat more technical. We follow and use the results of the papers [26], [27] and [30]. We only carry out in detail here the new ingredients which are needed in the computations in order to deal with Robin boundary conditions. Some technical lemmas are further left to an Appendix.

Throughout the paper $C > 0$ is a generic constant which is independent of ϵ and λ and may change from line to line.

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2. – Proof of Theorem 1.1

In this section, we analyze problem (1.11). We will make use of the concentration-compactness method. To this end, it is important to introduce the problem at $+\infty$ which involves:

$$I^\infty := I[w] = \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla w|^2 + w^2) - \int_{\mathbb{R}^N} F(w), \tag{2.1}$$

where w is the unique solution to (1.18). Then, by taking $v = w(y^{'}, y_N - R)$, it is easy to see that

$$\lim_{R\to+\infty}\left[\sup_{t>0} I_\lambda[tv]\right] = I^\infty. \tag{2.2}$$

(See the proof of (2.15) below.)

Hence,

$$c_\lambda \le I^\infty, \quad \text{for } \lambda > 0. \tag{2.3}$$

Applying the well-known "concentration-compactness" principle of P.L. Lions [21], [22], we prove the following:

LEMMA 2.1. *If for some $\lambda > 0$,*

$$c_\lambda < I^\infty, \tag{2.4}$$

then c_λ is achieved.

PROOF. The proof follows from Theorem V.5 (and Remark V.2) of [22]: here by (f2), we have

$$f(t)t^{-q} \text{ is nondecreasing}$$

for some $q \ge 1$ and if $q = 1$, $\frac{f(t)}{t}$ is strictly increasing. The condition (50′) of [22] is satisfied. The main idea is the following: let u_n be a sequence such that $I_\lambda[u_n] = c_\lambda$. Of the three possibilities for this sequence, vanishing, dichotomy and compactness, we show that neither vanishing nor dichotomy can occur, leaving compactness as the only possibility. We omit the details. □

Let us first prove (1) of Theorem 1.1. To this end, we take a test function $v_R(y) := w(y^{'}, y_N - R)$. By (f2), there exists a unique t_R such that

$$\sup_{t>0} I_\lambda[tv_R] = I_\lambda[t_R v_R]. \tag{2.5}$$

(See Lemma 3.1 of [26] or Lemma 5.3 of [30].)

The asymptotic behavior of w at infinity is given by the following lemma

LEMMA 2.2. ([12]) *As $r \to +\infty$, we have*

$$w(r) = Ae^{-r}r^{-\frac{N-1}{2}}\left(1 - \frac{(N-1)(N-3)}{8}\frac{1}{r} + O\left(\frac{1}{r^2}\right)\right), \tag{2.6}$$

$$\frac{w'(r)}{w(r)} = -1 - \frac{(N-1)}{2r} - \frac{(N-1)(N-3)}{8r^2} + O\left(\frac{1}{r^3}\right), \tag{2.7}$$

where $A > 0$ is a generic constant.

We now ready to prove (1) of Theorem 1.1.

PROOF OF (1) OF THEOREM 1.1. Set

$$e_N = (0, \dots, 0, 1)^T.$$

Let us now compute $I_\lambda[t_R v_R]$ for R large. By Lemma 2.2, we deduce that

$$|f(w(y - Re_N))w(y - Re_N)| \le Ce^{-(2+\frac{\sigma}{2})R} w^{\frac{\sigma}{2}}(y), \tag{2.8}$$

and hence we have

$$\begin{aligned}
\int_{\mathbb{R}^N_+} (|\nabla v_R|^2 + v_R^2) + \int_{\partial\mathbb{R}^N_+} \lambda v_R^2 &= \int_{\mathbb{R}^N_+} f(w(y - Re_N))w(y - Re_N)dy \\
&\quad + \int_{\partial\mathbb{R}^N_+} w(y - Re_N)\frac{\partial w(y - Re_N)}{\partial \nu}dy + \lambda \int_{\partial\mathbb{R}^N_+} (w(y - Re_N))^2 dy \\
&= \int_{\mathbb{R}^N} f(w)w + O(e^{-(2+\frac{\sigma}{2})R}) \\
&\quad + \int_{\partial\mathbb{R}^N_+} \left(w(y - Re_N)w'(y - Re_N)\frac{R}{|y - Re_N|} + \lambda w^2(y - Re_N)\right) dy \\
&= \int_{\mathbb{R}^N} f(w)w + O(e^{-(2+\frac{\sigma}{2})R}) \\
&\quad + \int_{\partial\mathbb{R}^N_+} w^2(y - Re_N)\left(\frac{w'}{w}(y - Re_N)\frac{R}{|y - Re_N|} + \lambda\right) dy.
\end{aligned}$$

Put $\alpha = |y'|$. So $w(y - Re_N) = w(\sqrt{\alpha^2 + R^2})$ on $\partial\mathbb{R}^N_+$. For $R \gg 1$, we have

$$\begin{aligned}
\sqrt{\alpha^2 + R^2} = R\sqrt{1 + \frac{\alpha^2}{R^2}} &= R\left(1 + \frac{\alpha^2}{2R^2} + O\left(\frac{\alpha^4}{R^4}\right)\right) \\
&= R + \frac{\alpha^2}{2R} + O\left(\frac{\alpha^4}{R^3}\right)
\end{aligned}$$

which suggests that we take $\alpha = \sqrt{R}t$.

We consider first $\lambda < 1$. By (2.7), we obtain that

$$
\begin{aligned}
&\int_{\partial\mathbb{R}^N_+} w^2(y - Re_N)\left(\frac{w'}{w}(y - Re_N)\frac{R}{|y - Re_N|} + \lambda\right) dy \\
&= |S^{N-1}|\int_0^{+\infty} w^2(\sqrt{\alpha^2 + R^2})\left(\lambda + \frac{R}{\sqrt{\alpha^2 + R^2}}\left(-1 - \frac{N-1}{2\sqrt{\alpha^2 + R^2}}\right)\right. \\
&\qquad\qquad \left. + O\left(\frac{1}{R^2}\right)\right)\alpha^{N-2}d\alpha \qquad (2.9)\\
&= |S^{N-1}|R^{\frac{N-1}{2}}\int_0^{+\infty} w^2(\sqrt{R^2 + Rt^2})\left(\lambda - 1 + O\left(\frac{1}{R}\right)\right) t^{N-2}dt \\
&= |S^{N-1}|R^{\frac{N-1}{2}}(\lambda - 1)\int_0^{+\infty} w^2(\sqrt{R^2 + Rt^2})t^{N-2}dt\left(1 + O\left(\frac{1}{R}\right)\right) < 0
\end{aligned}
$$

where $|S^{N-1}|$ is the area of the unit sphere S^{N-1} in $\mathbb{R}^N$.

Next we consider

$$
\lambda = 1 + \frac{c_0}{R} \tag{2.10}
$$

where $c_0 > 0$ is to be determined. Then we have

$$
\begin{aligned}
&\frac{1}{|S^{N-1}|}\int_{\partial\mathbb{R}^N_+} w^2(y - Re_N)\left(\frac{w'}{w}(y - Re_N)\frac{R}{|y - Re_N|} + \lambda\right) dy \\
&= \int_0^{+\infty} w^2(\sqrt{\alpha^2 + R^2})\left(1 + \frac{c_0}{R} - \frac{R}{\sqrt{\alpha^2 + R^2}} - \frac{(N-1)R}{2(\alpha^2 + R^2)}\right. \\
&\qquad\qquad \left. + O\left(\frac{1}{R^2}\right)\right)\alpha^{N-2}d\alpha \\
&= R^{\frac{N-1}{2}}\int_0^{+\infty} w^2(\sqrt{R^2 + Rt^2})\left(\frac{c_0}{R} + \frac{t^2}{2R} - \frac{N-1}{2R} + O\left(\frac{1}{R^2}\right)\right) t^{N-2}dt \\
&= R^{-\frac{N-1}{2}}A^2\int_0^{+\infty} e^{-2R-t^2}\left(\frac{c_0}{R} + \frac{t^2}{2R} - \frac{N-1}{2R} + O\left(\frac{1}{R^2}\right)\right) t^{N-2}dt \\
&= A^2R^{-\frac{N+1}{2}}e^{-2R}\frac{1}{2}\left(\int_0^{+\infty} e^{-t^2}(2c_0 + t^2 - (N-1))t^{N-2}dt + O\left(\frac{1}{R}\right)\right)
\end{aligned}
$$

Note that

$$
(N-1)\int_0^{+\infty} t^{N-2}e^{-t^2}dt = \int_0^{+\infty} e^{-t^2}dt^{N-1} = 2\int_0^{+\infty} t^N e^{-t^2}dt.
$$

So we obtain

$$
\begin{aligned}
&\int_{\partial\mathbb{R}^N_+} w^2(y-Re_N)\left(\frac{w'}{w}(y-Re_N)\frac{R}{|y-Re_N|}+\lambda\right)dy \\
&= \frac{A^2}{2}R^{-\frac{N+1}{2}}e^{-2R}\left(2c_0\int_0^{+\infty}e^{-t^2}t^{N-2}-\int_0^{+\infty}t^Ne^{-t^2}dt+O\left(\frac{1}{R}\right)\right).
\end{aligned}
\tag{2.11}
$$

Now we choose $c_0>0$ so that

$$
4c_0\int_0^{+\infty}e^{-t^2}t^{N-2}=\int_0^{+\infty}t^Ne^{-t^2}dt \tag{2.12}
$$

We conclude that for $\lambda\le 1+\frac{c_0}{R}$ with c_0 satisfying (2.12), we have

$$
\begin{aligned}
&\int_{\mathbb{R}^N_+}(|\nabla v_R|^2+v_R^2)+\int_{\partial\mathbb{R}^N_+}\lambda v^2\le\int_{\mathbb{R}^N}f(w(y))w(y)+O(e^{-(2+\frac{\sigma}{2})R}) \\
&+\begin{cases}|S^{N-1}|R^{\frac{N-1}{2}}(\lambda-1)\int_0^{+\infty}w^2(\sqrt{R^2+Rt^2})t^{N-2}dt\left(1+O\left(\frac{1}{R}\right)\right), & \text{if } \lambda<1,\\ |S^{N-1}|\frac{A^2}{2}R^{-\frac{N+1}{2}}e^{-2R}\left(-\frac{1}{2}\int_0^{+\infty}t^Ne^{-t^2}dt+O\left(\frac{1}{R}\right)\right), & \text{if } 1\le\lambda\le 1+\frac{c_0}{R}.\end{cases}
\end{aligned}
\tag{2.13}
$$

On the other hand, one easily gets that

$$
\int_{\mathbb{R}^N_+}F(w(y-Re_N))dy=\int_{\mathbb{R}^N}F(w(y))+O(e^{-(2+\frac{\sigma}{2})R}). \tag{2.14}
$$

Now similar to the computations done in Section 5 of [30], which we omit here, we obtain immediately that for $\lambda\le 1+\frac{c_0}{R}$, and $R\gg 1$,

$$
c_\lambda\le\sup_{t>0}I_\lambda(tv_R)=I_\lambda(t_Rv_R)<\frac{1}{2}\int_{\mathbb{R}^N}f(w)w-\int_{\mathbb{R}^N}F(w)=I^\infty, \tag{2.15}
$$

which, by Lemma 2.1, proves (1) of Theorem 1.1. □

REMARK 2.1. The inequality (2.15) does not hold for $N=1$. Indeed, this can be seen to follow from the above computations and the fact that in this case $w(y)\sim e^{-|y|}$ as $|y|\to+\infty$. However, it holds for $N\ge 2$ since for $N\ge 2$, $w(y)\sim|y|^{-\frac{N-1}{2}}e^{-|y|}$ as $|y|\to+\infty$, and the previous computations can be carried through. Actually, we can see that the algebraic term $|y|^{-\frac{N-1}{2}}$ helps for this question.

Let λ_* be defined by (1.14). By (2.15), for $\lambda\le 1+\frac{c_0}{R}$ and $R\gg 1$, $c_\lambda<I^\infty$. Hence c_λ is achieved for $\lambda\le 1+\frac{c_0}{R}$. Thus $\lambda_*>1$. There are two cases to be considered: $\lambda_*=+\infty$, or $1<\lambda_*<+\infty$.

Let us first consider $\lambda_* = +\infty$. We will show that it is impossible. That is

LEMMA 2.3. *For λ sufficiently large, c_λ is not achieved.*

PROOF. We derive this fact by contradiction. Suppose there exists a sequence of solutions w_{λ_n} attaining c_{λ_n}, with $\lambda_n \to +\infty$. This yields that $c_{\lambda_n} \le I^\infty$ (by (2.3)). Moreover, for $\lambda < \lambda_n$, we have

$$c_\lambda \le I_\lambda[w_{\lambda_n}] < I_{\lambda_n}[w_{\lambda_n}] = c_{\lambda_n} \le I^\infty \tag{2.16}$$

which, by Lemma 2.1, yields that c_λ is achieved for $\lambda < \lambda_n$. Hence c_λ is achieved for any $\lambda < +\infty$ and $c_\lambda \le I^\infty$ (by (2.3)). Let us denote the minimizer as w_λ. We also note that as $\lambda \to +\infty$, $c_\lambda \to I^\infty$. Moreover, $w_\lambda(y) \to 0$ as $|y| \to +\infty$. By the well-known moving plane method (see [16]), $w_\lambda(y)$ is symmetric in y', i.e., $w_\lambda(y', y_N) = w_\lambda(|y'|, y_N)$.

Let $R_\lambda > 0$ be such that

$$\max_{y \in \mathbb{R}^N_+} w_\lambda(y', y_N) = w_\lambda(0, R_\lambda). \tag{2.17}$$

As $\lambda \to +\infty$, the limiting problem becomes

$$\begin{cases} \Delta u - u + f(u) = 0 \ , u \ge 0 & \text{in } \mathbb{R}^N_+, \\ u \in H^1(\mathbb{R}^N_+), u = 0 & \text{on } \partial\mathbb{R}^N_+. \end{cases} \tag{2.18}$$

By the result of Esteban and Lions [11], any solution of (2.18) must vanish identically. So $R_\lambda \to +\infty$ as $\lambda \to +\infty$. Furthermore, by concentration-compactness and usual limiting process, we immediately have that

$$\|w_\lambda(\cdot) - w(|\cdot - R_\lambda e_N|)\|_{H^1(\mathbb{R}^N_+)} \to 0, \tag{2.19}$$

$$\|w_\lambda(\cdot) - w(|\cdot - R_\lambda e_N|)\|_{L^\infty(\mathbb{R}^N_+)} \to 0, \tag{2.20}$$

as $\lambda \to +\infty$. We will show that this is impossible. This is done by expanding c_λ to the second term (Lemma 2.5 below). To this end, we need some delicate estimates of the difference $w_\lambda(\cdot) - w(|\cdot - R_\lambda e_N|)$.

We first project the limiting function w to $\mathbb{R}^N_+$ with Robin boundary condition. Let $\mathcal{P}_R w$ be the unique solution of

$$\begin{cases} \Delta \mathcal{P}_R w - \mathcal{P}_R w + f(w(\cdot - Re_N)) = 0 \ , \quad \mathcal{P}_R w > 0 & \text{in } \mathbb{R}^N_+, \\ \mathcal{P}_R w \in H^1(\mathbb{R}^N_+), \quad \mathcal{P}_R w + \lambda^{-1} \dfrac{\partial \mathcal{P}_R w}{\partial \nu} = 0 & \text{on } \partial\mathbb{R}^N_+. \end{cases} \tag{2.21}$$

Let us set

$$\eta = \lambda^{-1}, \phi_R = w(\cdot - Re_N) - \mathcal{P}_R w(\cdot). \tag{2.22}$$

Then ϕ_R satisfies the following linear equation

$$\begin{cases} \Delta \phi_R - \phi_R = 0 \text{ in } \mathbb{R}^N_+, \phi_R \in H^1(\mathbb{R}^N_+), \\ \phi_R + \eta \dfrac{\partial \phi_R}{\partial \nu} = w(\cdot - Re_N) + \eta \dfrac{\partial w(\cdot - Re_N)}{\partial \nu} \text{ on } \partial\mathbb{R}^N_+. \end{cases} \tag{2.23}$$

The following lemma on ϕ_R plays an important role. The proof of it is technical and is delayed to Appendix A.

LEMMA 2.4. (1). *As $R \to +\infty$,*

$$-R^{-1}\log\phi_R(Rx) - \Psi_0(x) \to 0, \tag{2.24}$$

where $\Psi_0(x)$ is the unique viscosity solution of the following problem

$$\begin{cases} |\nabla u|^2 = 1 & \text{in } \mathbb{R}^N_+, \\ u(x) = |x - e_N| & \text{on } \partial\mathbb{R}^N_+. \end{cases} \tag{2.25}$$

(In fact, Ψ_0 is given explicitly by the relation $\Psi_0(x) = \inf_{z\in\partial\mathbb{R}^N_+}(|z-e_N|+|z-x|)$.)

(2). *By taking a subsequence along $R \to \infty$, the renormalized ϕ_R converges in the sense that*

$$V_R(y) := \frac{\phi_R(y+Re_N)}{\phi_R(Re_N)} \to V_0 \text{ as } R \to +\infty, \tag{2.26}$$

where V_0 is a solution of the following equation

$$\begin{cases} \Delta u - u = 0, \\ u(0) = 1, u > 0 \text{ in } \mathbb{R}^N \end{cases} \tag{2.27}$$

and

$$\sup_{y\in B_{4R}(0)} (e^{(1+\sigma_1)|y|}V_R(y)) \le C, \text{ for any } \sigma_1 > 0. \tag{2.28}$$

Let us now use Lemma 2.4 to finish the proof of Lemma 2.3. Our basic idea is to obtain a lower bound for c_λ. In fact we will show that

LEMMA 2.5. *As $R_\lambda \to +\infty$, we have*

$$c_\lambda = I[w] + \exp(-2R_\lambda(1+o(1))). \tag{2.29}$$

PROOF. This is similar to Section 6 of [30]. Here we give a simplified proof. We follow the approach in Section 3 of [9].

We first obtain the following global estimates:

$$w_\lambda(y) \le Ce^{-(1-\delta)|y-R_\lambda e_N|} \tag{2.30}$$

for δ such that $(1-\delta)\lambda^{-1} < 1$, where C may depend on δ but is independent of $R_\lambda > 0$. In fact, we consider the domain $\mathbb{R}^N_+\backslash B_1(Re_N)$. Then it follows from (2.19) that $\Delta w_\lambda - (1-\delta)^2 w_\lambda \ge 0$ in $\mathbb{R}^N_+\backslash B_1(Re_N)$. Now we compare w_λ with the function $Ce^{-(1-\delta)|y-Re_N|}$. One then derives (2.30) from the Maximum Principle.

Let $v_\lambda(y) := w_\lambda(y + R_\lambda e_N)$ where

$$y \in \mathbb{R}^N_{-R_\lambda} := \{(y', y_N) \in \mathbb{R}^N | y_N > -R_\lambda\}.$$

Then, by elliptic regularity theory, we have

$$|\nabla v_\lambda| + v_\lambda \leq Ce^{-4(1-\delta)R_\lambda}, \ \text{ for } y \in \mathbb{R}^N_{-R_\lambda} \backslash B_{4R_\lambda}(0).$$

So

$$\begin{aligned} c_\lambda &= \frac{1}{2}\int_{\mathbb{R}^N_+} (|\nabla w_\lambda|^2 + w_\lambda^2) + \lambda \int_{\partial\mathbb{R}^N_+} w_\lambda^2 - \int_{\mathbb{R}^N_+} F(w_\lambda) \\ &= \frac{1}{2}\int_{\mathbb{R}^N_+} w_\lambda f(w_\lambda) - \int_{\mathbb{R}^N_+} F(w_\lambda) \\ &= \int_{\mathbb{R}^N_{-R_\lambda} \cap B_{4R}(0)} \left(\frac{1}{2} v_\lambda f(v_\lambda) - F(v_\lambda)\right) + O(e^{-3R_\lambda}) \end{aligned} \tag{2.31}$$

Consider the restriction of v_λ to the domain $\mathbb{R}^N_{-R_\lambda} \cap B_{4R_\lambda}(0)$. We decompose $v_\lambda = \mathcal{P}_{R_\lambda} w + (\phi_{R_\lambda}(R_\lambda e_N))^{1-\delta} h_\lambda(y)$, where $0 < \delta < 1$ is a small but fixed number and $y \in \mathbb{R}^N_{-R_\lambda} \cap B_{4R_\lambda}(0)$.

Then h_λ satisfies

$$\Delta h_\lambda - h_\lambda + f^{'}(\mathcal{P}_{R_\lambda} w) h_\lambda + N_\lambda + M_\lambda = 0 \text{ in } \mathbb{R}^N_{-R_\lambda} \cap B_{4R_\lambda}(0) \tag{2.32}$$

and

$$\frac{\partial h_\lambda}{\partial \nu} + \lambda h_\lambda = 0 \text{ on } \partial\mathbb{R}^N_{-R_\lambda} \cap B_{4R_\lambda}(0)$$

where

$$N_\lambda = \frac{1}{(\phi_{R_\lambda}(R_\lambda e_N))^{1-\delta}}$$

$$\times \left[f(\mathcal{P}_{R_\lambda} w + (\phi_{R_\lambda}(R_\lambda e_N))^{1-\delta} h_\lambda) - f(\mathcal{P}_{R_\lambda} w) - f^{'}(\mathcal{P}_{R_\lambda} w)(\phi_{R_\lambda}(R_\lambda e_N))^{1-\delta} h_\lambda \right],$$

and

$$M_\lambda = \frac{1}{(\phi_{R_\lambda}(R_\lambda e_N))^{1-\delta}} (f(\mathcal{P}_{R_\lambda} w) - f(w)).$$

By the mean-value property and Lemma 2.4 (see the proof of Lemma 6.1 of [30]), we see that

$$|N_\lambda| \leq C(|w(\cdot - R_\lambda e_N)| + |w_\lambda|)^\sigma |w_\lambda - w(\cdot - R_\lambda e_N)|^\sigma |h_\lambda| \tag{2.33}$$

and

$$|M_\lambda| \leq C(\phi_{R_\lambda}(R_\lambda e_N))^\delta (w + w_\lambda^R)^\sigma |V_{R_\lambda}(y)| \leq C(\phi_{R_\lambda}(R_\lambda e_N))^\delta e^{\mu|y|} \tag{2.34}$$

where μ is such that $1 - \sigma < \mu < 1$.

Hence h_λ satisfies

$$\begin{cases} \Delta h_\lambda - h_\lambda + f^{'}(\mathcal{P}_{R_\lambda} w)h_\lambda + o(1)h_\lambda + o(1)e^{\mu|y|} = 0 \text{ in } \mathbb{R}^N_{-R_\lambda} \cap B_{4R_\lambda}(0), \\ \dfrac{\partial h_\lambda}{\partial \nu} + \lambda h_\lambda = 0 \text{ on } (\partial \mathbb{R}^N_{-R_\lambda}) \cap B_{4R_\lambda}(0). \end{cases} \tag{2.35}$$

Let $G_\mu(y)$ be the unique radial solution of

$$\Delta u - \mu^2 u = 0, u(0) = 1, u > 0 \text{ in } \mathbb{R}^N. \tag{2.36}$$

Note that G_μ has the following asymptotic behavior

$$C_1|y|^{-\frac{N-1}{2}} e^{\mu|y|} \leq G_\mu(y) \leq C_2|y|^{-\frac{N-1}{2}} e^{\mu|y|}.$$

Set

$$H_\lambda = G_\mu^{-1} h_\lambda. \tag{2.37}$$

We first claim that

$$\|H_\lambda\|_{L^\infty(\mathbb{R}^N_{-R_\lambda} \cap B_{4R_\lambda}(0))} \leq C. \tag{2.38}$$

Suppose not. Without loss of generality, we may assume that

$$\|H_\lambda\|_{L^\infty(\mathbb{R}^N_{-R_\lambda} \cap B_{4R_\lambda}(0))} = H_\lambda(y_\lambda) > 0.$$

Observe that $|y_\lambda| \leq C$. Otherwise, suppose that $|y_\lambda| \to +\infty$. Then there are two possibilities:either $y_\lambda \in \partial(\mathbb{R}^N_{-R_\lambda} \cap B_{4R_\lambda}(0))$ or $y_\lambda \in \mathbb{R}^N_{-R_\lambda} \cap B_{4R_\lambda}(0)$. Note that on $\partial\mathbb{R}^N_{-R_\lambda}$,

$$H_\lambda \left(1 + \mu\lambda^{-1} \frac{\partial G_\mu}{\partial \nu} G_\mu^{-1}\right) + \lambda^{-1} \frac{\partial H_\lambda}{\partial \nu} = 0.$$

Since on $\partial\mathbb{R}^N_{-R_\lambda}$, $1 + \mu\lambda^{-1} \frac{\partial G_\mu}{\partial \nu} G_\mu^{-1} \geq 1 - \mu\lambda^{-1} > 0$, we know by the Hopf boundary lemma that $y_\lambda \notin \partial\mathbb{R}^N_{-R_\lambda}$. If $y \in \partial B_{4R_\lambda}(0)$, then we have $H_\lambda \leq e^{\mu R_\lambda} e^{-R_\lambda} \leq C$, which is a contradiction. So $y_\lambda \in \mathbb{R}^N_{-R_\lambda} \cap B_{4R_\lambda}(0)$ and then $\Delta H_\lambda(y_\lambda) \leq 0, \nabla H_\lambda(y_\lambda) = 0$. Let us compute $\Delta H_\lambda(y_\lambda)$. By (2.35), H_λ satisfies

$$\Delta H_\lambda + 2G_\mu^{-1} \nabla G_\mu \nabla H_\lambda + (-1 + \mu^2 + f^{'}(\mathcal{P}_{R_\lambda} w))H_\lambda + o(1)H_\lambda + o(1) = 0$$

Thus,

$$H_\lambda(y_\lambda) \leq o(1)H_\lambda(y_\lambda) + C,$$

which yields that $H_\lambda(y_\lambda) \le C$. This is a contradiction to our assumption that $\|H_\lambda\|_{L^\infty(\mathbb{R}^N_{-R_\lambda} \cap B_{4R_\lambda}(0))} \to +\infty$.

Now define

$$\bar{h}_\lambda = \frac{h_\lambda}{\|h_\lambda\|_{L^\infty(\mathbb{R}^N_{-R_\lambda} \cap B_{4R_\lambda}(0))}}.$$

Note that since $|y|_\lambda \le C$, one sees that

$$\|H_\lambda\|_{L^\infty(\mathbb{R}^N_{-R_\lambda} \cap B_{4R_\lambda}(0))} \sim \|h_\lambda\|_{L^\infty(\mathbb{R}^N_{-R_\lambda} \cap B_{4R_\lambda}(0))}$$

and hence $\|h_\lambda\|_{L^\infty(\mathbb{R}^N_{-R_\lambda} \cap B_{4R_\lambda}(0))} \to +\infty$. It is easy to see that by (2.35), $\bar{h}_\lambda \to h_0$ in $C^1_{\text{loc}}(\mathbb{R}^N)$, where h_0 satisfies

$$\Delta h_0 - h_0 + f'(w) h_0 = 0, \, |h_0| \le C e^{\mu |y|}. \tag{2.39}$$

By Lemma 6.5 of [30], $h_0 = \sum_{j=1}^N a_j \frac{\partial w}{\partial y_j}$ for some constants a_j. However, by definition

$$\nabla_y \bar{h}_\lambda(0) = \frac{1}{(\phi_{R_\lambda}(R_\lambda e_N))^{1-\delta}} (\nabla w_\lambda(R_\lambda e_N) - \nabla \mathcal{P}_{R_\lambda} w(R_\lambda e_N))$$

$$= \frac{1}{(\phi_{R_\lambda}(R_\lambda e_N))^{1-\delta}} (\phi_{R_\lambda}(R_\lambda e_N) \nabla V_{R_\lambda}(0)) \to 0.$$

Hence $\nabla_y h_0(0) = 0$, which implies that $a_j = 0, j = 1, ..., N, h_0 = 0$. This contradicts the fact that $\bar{h}_\lambda \to h_0$ in C^1_{loc} and $\bar{h}_\lambda(y_\lambda) \ge C, |y_\lambda| \le C$.

Hence (2.38) holds.

Now we can carry the computation as in Section 6 of [30]. By (2.31)

$$c_\lambda = \int\limits_{\mathbb{R}^N_{-R_\lambda} \cap B_{4R_\lambda}(0)} \left(\frac{1}{2} \mathcal{P}_{R_\lambda} w f(\mathcal{P}_{R_\lambda} w) - F(\mathcal{P}_{R_\lambda} w) \right)$$

$$+ (\phi_{R_\lambda}(R_\lambda e_N))^{1-\delta} \int\limits_{\mathbb{R}^N_{-R_\lambda} \cap B_{4R_\lambda}(0)} \left(\frac{1}{2} \mathcal{P}_{R_\lambda} w f'(\mathcal{P}_{R_\lambda} w) - \frac{1}{2} f(\mathcal{P}_{R_\lambda} w) \right) h_\lambda + o(\phi_{R_\lambda}(R_\lambda e_N))$$

Note that

$$\int_{\mathbb{R}^N_{-R_\lambda} \cap B_{4R_\lambda}(0)} (\mathcal{P}_{R_\lambda} w f'(\mathcal{P}_{R_\lambda} w) - f(\mathcal{P}_{R_\lambda} w)) h_\lambda$$

$$= \int_{\mathbb{R}^N_{-R_\lambda} \cap B_{4R_\lambda}(0)} [(f'(\mathcal{P}_{R_\lambda} w) h_\lambda) \mathcal{P}_{R_\lambda} w - f(\mathcal{P}_{R_\lambda} w) h_\lambda]$$

$$= \int_{\mathbb{R}^N_{-R_\lambda} \cap B_{4R_\lambda}(0)} [(-\Delta h_\lambda + h_\lambda - N_\lambda - M_\lambda) \mathcal{P}_{R_\lambda} w - (f(\mathcal{P}_{R_\lambda} w) - f(w)) h_\lambda +$$

$$- (-\Delta \mathcal{P}_{R_\lambda} w + \mathcal{P}_{R_\lambda} w) h_\lambda)]$$

$$= \int_{\mathbb{R}^N_{-R_\lambda} \cap B_{4R_\lambda}(0)} \mathcal{P}_{R_\lambda} w (-M_\lambda) + O((\phi_{R_\lambda}(R_\lambda e_N))^{1-\delta})$$

Hence by Lemma 2.4

$$
\begin{aligned}
c_\lambda &= \int_{\mathbb{R}^N_{-R_\lambda} \cap B_{4R_\lambda}(0)} \left(\frac{1}{2} \mathcal{P}_{R_\lambda} w f(w) - F(\mathcal{P}_{R_\lambda} w) \right) + o(\phi_{R_\lambda}(R_\lambda e_N)) \\
&= \int_{\mathbb{R}^N} \left(\frac{1}{2} w f(w) - F(w) \right) + c_1 \phi_{R_\lambda}(R_\lambda e_N) + o(\phi_{R_\lambda}(R_\lambda e_N))
\end{aligned} \tag{2.40}
$$

where

$$c_1 = \int_{\mathbb{R}^N} \left(\frac{1}{2} f(w) V_0 \right) > 0$$

by Lemma 4.7 of [30]. By Lemma 2.5, $\phi_{R_\lambda}(R\lambda e_N) = e^{-R_\lambda \Psi_R(e_N)(1+o(1))} = e^{-2R_\lambda(1+o(1))}$.

Lemma 2.5 is thus proved. □

This lemma shows that for λ large enough

$$c_\lambda > I[w] \tag{2.41}$$

which is impossible since we have shown that $c_\lambda \leq I[w]$.

Lemma 2.3 is therefore proved. □

Next we consider the case $1 < \lambda_* < +\infty$. We claim that

LEMMA 2.6. *For the value $\lambda = \lambda_*$, the infimum c_λ in* (1.13) *is achieved.*

PROOF. Like Lemma 2.3, we prove it by contradiction. Suppose there exists a sequence of solutions w_{λ_n} attaining c_{λ_n}, with $\lambda_n \nearrow \lambda_*, \lambda_n < \lambda_*$. Suppose that for $\lambda = \lambda_* > 1$, c_λ is not achieved. This implies that $c_{\lambda_*} = I^\infty$ and by concentration-compactness, there exists $R_\lambda \to +\infty$ such that as $\lambda \nearrow \lambda_*, \lambda < \lambda_*$

$$\|w_\lambda - w(\cdot - R_\lambda e_N)\|_{H^1(\mathbb{R}^N_+)} \to 0, \quad \|w_\lambda - w(\cdot - R_\lambda e_N)\|_{L^\infty(\mathbb{R}^N_+)} \to 0. \tag{2.42}$$

The remaining of the proof is exactly the same as that of Lemma 2.3. Note that in the proof of Lemma 2.3, we only used the property that $\eta = \frac{1}{\lambda} < 1$. □

(2) and (3) of Theorem 1.1 now follow from Lemma 2.3 and Lemma 2.6. □

3. – Proof of Theorem 1.2

Let $\lambda \leq \lambda_*$ and let $u_{\epsilon,\lambda}$ be a least-energy solution defined in Section 1. We now study the asymptotic behavior of $u_{\epsilon,\lambda}$ for $\lambda \leq \lambda_*$ as $\epsilon \to 0$. We will prove Theorem 1.2 in this section.

We first derive an upper bound for $c_{\epsilon,\lambda}$.

LEMMA 3.1. *Let $\lambda \leq \lambda_*$. Then for ϵ sufficiently small, we have*

$$c_{\epsilon,\lambda} \leq \epsilon^N \{c_\lambda - \epsilon \bar{H}(P) + o(\epsilon)\}\,, \tag{3.1}$$

for any $P \in \partial\Omega$, where c_λ is given in (1.13) *and $\overline{H}(P)$ is the generalized mean curvature defined by* (1.16).

PROOF. Fix any $P \in \partial\Omega$.

By Theorem 1.1 (3), c_λ is achieved by some function $w_\lambda \in H^1(\mathbb{R}^N_+)$, which solves (1.11). Let

$$\mathcal{S}_\lambda = \{w_\lambda | w_\lambda \text{ solves (1.11) and achieves } c_\lambda\}\,. \tag{3.2}$$

Since $c_\lambda \leq I^\infty$, we see that for any $w_\lambda \in \mathcal{S}_\lambda$,

$$\int_{\mathbb{R}^N_+} (|\nabla w_\lambda|^2 + w_\lambda^2) + \lambda \int_{\partial\mathbb{R}^N_+} w_\lambda^2 - 2\int_{\mathbb{R}^N_+} F(w_\lambda) \leq 2c_\lambda = 2I^\infty\,. \tag{3.3}$$

On the other hand, since w_λ solves (1.11), we derive the following identity

$$\int_{\mathbb{R}^N_+} (|\nabla w_\lambda|^2 + w_\lambda^2) + \lambda \int_{\partial\mathbb{R}^N_+} w_\lambda^2 = \int_{\mathbb{R}^N_+} f(w_\lambda) w_\lambda\,. \tag{3.4}$$

Using the assumption (f4), from (3.3) and (3.4), we see that there exists some constant C independent of λ such that

$$\int_{\mathbb{R}^N_+} (|\nabla w_\lambda|^2 + w_\lambda^2) \leq C, \;\; \forall w_\lambda \in \mathcal{S}_\lambda, \forall \lambda > 0\,.$$

This shows that the set $\mathcal{S}_\lambda$ is a compact set. So for each fixed $P \in \partial\Omega$, there exists a $w_\lambda \in \mathcal{S}_\lambda$ such that

$$\overline{H}(P) = -\int_{\mathbb{R}^N_+} y' \nabla' w_\lambda \frac{\partial w_\lambda}{\partial y_N} H(P) = \max_{w_\lambda \in \mathcal{S}_\lambda} \left[-\int_{\mathbb{R}^N_+} y' \nabla' w_\lambda \frac{\partial w_\lambda}{\partial y_N} H(P) \right]. \tag{3.5}$$

Now we choose the w_λ which achieves the maximum at (3.5). Multiplying (1.11) by $|y'|^2 \frac{\partial w}{\partial y_N}$ and integrating by parts, we obtain that

$$-\int_{\mathbb{R}^N_+} y' \nabla' w_\lambda \frac{\partial w_\lambda}{\partial y_N} = \frac{1}{2} \int_{\partial\mathbb{R}^N_+} |y'|^2 \left(\frac{1}{2} |\nabla w_\lambda|^2 + \frac{1}{2} w_\lambda^2 - F(w_\lambda) + \lambda w_\lambda \frac{\partial w_\lambda}{\partial y_N} \right)$$

So we have another expression for $\overline{H}(P)$

$$\overline{H}(P) = \frac{1}{2} \int_{\partial\mathbb{R}^N_+} |y'|^2 \left(\frac{1}{2} |\nabla w_\lambda|^2 + \frac{1}{2} w_\lambda^2 - F(w_\lambda) + \lambda w_\lambda \frac{\partial w_\lambda}{\partial y_N} \right) H(P) \tag{3.6}$$

Let

$$v_\epsilon(x) = w_\lambda\left(\frac{x-P}{\epsilon}\right) \tag{3.7}$$

be our test function. We now compute $J[v_\epsilon]$.

Without loss of generality, we may assume that $P = 0$ and that there is a smooth transformation $\rho : \Omega \cap B_{r_0}(0) \to \mathbb{R}^N_+$ such that $\rho(0) = 0, \nabla\rho(0) = 0$, where $r_0 > 0$ is a small number. Moreover, $\partial\Omega \cap B_{r_0}(0) = \{(x', x_N) | x_N > \rho(x')\}$. Now define

$$\epsilon y' = x', \epsilon y_N = x_N - \rho(x') \tag{3.8}$$

Then we have

$$\begin{aligned} &\int_\Omega (\epsilon^2|\nabla v_\epsilon|^2 + v_\epsilon^2) \\ &= \epsilon^N\left(\int_{\mathbb{R}^N_+}\left[(|\nabla w_\lambda|^2 + w_\lambda^2) + \frac{\partial}{\partial y_N}(|\nabla w_\lambda|^2 + w_\lambda^2)\frac{1}{2}\sum_{i,j=1}^{N-1}\epsilon\rho_{ij}(0)y_i'y_j'\right]\right) \\ &\quad + O(e^{-\frac{c}{\epsilon}}) \\ &= \epsilon^N\left(\int_{\mathbb{R}^N_+}(|\nabla w_\lambda|^2 + w_\lambda^2) - \frac{\epsilon}{2}H(P)\int_{\partial\mathbb{R}^N_+}(|\nabla w_\lambda|^2 + w_\lambda^2)|y'|^2 + o(\epsilon)\right) \end{aligned} \tag{3.9}$$

Similarly we have

$$\int_\Omega F(v_\epsilon) = \epsilon^N\left(\int_{\mathbb{R}^N_+} F(w_\lambda) - \frac{1}{2}\epsilon H(P)\int_{\partial\mathbb{R}^N_+} F(w_\lambda)|y'|^2 + o(\epsilon)\right) \tag{3.10}$$

On the other hand

$$\begin{aligned} \lambda\epsilon\int_{\partial\Omega} v_\epsilon^2 &= \lambda\epsilon\int_{\partial\Omega} v_\epsilon^2\left(y', \frac{\rho(\epsilon y')}{\epsilon}\right)dx' = \lambda\epsilon^N\int_{\partial\mathbb{R}^N_+} v_\epsilon^2\left(y', \frac{\rho(\epsilon y')}{\epsilon}\right)dy' \\ &= \lambda\epsilon^N\left(\int_{\partial\mathbb{R}^N_+} w_\lambda^2 - \epsilon H(P)\int_{\partial\mathbb{R}^N_+} w_\lambda\frac{\partial w_\lambda}{\partial y_N}|y'|^2 + o(\epsilon)\right). \end{aligned} \tag{3.11}$$

So

$$\begin{aligned} J[v_\epsilon] = \epsilon^N\Bigg(J[w_\lambda] - \epsilon H(P)\frac{1}{2}\int_{\partial\mathbb{R}^N_+}|y'|^2\Bigg(&\frac{1}{2}|\nabla w_\lambda|^2 \\ &+\frac{1}{2}w_\lambda^2 - F(w_\lambda) + \lambda w_\lambda\frac{\partial w_\lambda}{\partial y_N}\Bigg) + o(\epsilon)\Bigg) \end{aligned}$$

By (3.6), this finishes the proof of (3.1). □

Next, we shall obtain a lower bound for $c_{\epsilon,\lambda}$. Since we do not know if the solution to (1.11) is unique, we use an idea of [4], where a simplified proof of the results of [27] and [30] is obtained.

Let $u_{\epsilon,\lambda}$ be a least energy solution of (1.1) with (1.5) and $u_{\epsilon,\lambda}(x_\epsilon) = \max_{x\in\bar{\Omega}} u_{\epsilon,\lambda}(x)$. Then, by the Maximum Principle, we know that $x_\epsilon \in \Omega$. Moreover, if $\frac{d(x_\epsilon,\partial\Omega)}{\epsilon}$ were not bounded, then, we would have $u_{\epsilon,\lambda} \to w(\frac{x-x_\epsilon}{\epsilon})$ and $\liminf_{\epsilon\to 0} \epsilon^{-N} c_{\epsilon,\lambda} \geq I[w]$, by the same argument as in [26]. Thus $\frac{d(x_\epsilon,\partial\Omega)}{\epsilon}$ is bounded. Let

$$v_{\epsilon,\lambda}(y) = u_{\epsilon,\lambda}(x_\epsilon + \epsilon y),\, y \in \Omega_\epsilon = \{y | \epsilon y + x_\epsilon \in \Omega\}\,. \tag{3.12}$$

Then $v_{\epsilon,\lambda}$ converges in the H^1-sense to w_λ, a least energy solution of (1.11). Moreover, as in [26], for certain positive constants a and b, we have

$$v_{\epsilon,\lambda}(y) \leq a e^{-b|y|}\,. \tag{3.13}$$

Let $\tilde{x}_\epsilon$ be the closest point to x_ϵ on $\partial\Omega$. After a rotation and a translation, we may also assume that $\tilde{x}_\epsilon = 0$ and that a fixed neighborhood $\Omega \cap B_{r_0}(0)$ of the set Ω can be represented as the set $\{(x', x_N) | x_N > \rho_\epsilon(x')\}$ with $\rho_\epsilon(0) = 0, \nabla\rho_\epsilon(0) = 0$. For an open set Λ, we define

$$J_{\epsilon,\Lambda}[v] = \frac{1}{2}\int_\Lambda (|\nabla v|^2 + v^2) + \frac{\lambda}{2}\int_{\partial\Lambda} v^2 - \int_\Lambda F(v),\, v \in H^1(\Lambda) \tag{3.14}$$

From the variational characterization of $c_{\epsilon,\lambda} = J_\epsilon[u_{\epsilon,\lambda}]$, we infer that

$$\epsilon^{-N} J_\epsilon[u_{\epsilon,\lambda}] \geq \epsilon^{-N} J_\epsilon[t u_{\epsilon,\lambda}] = J_{\epsilon,\Omega_\epsilon}[t v_{\epsilon,\lambda}]\,. \tag{3.15}$$

for all $t > 0$.

Let $V = B_{\frac{r_0}{\epsilon}}(0)$. We now define $\tilde{v}_\epsilon$ on $\mathbb{R}^N_+ \cap V$ as follows

$$\tilde{v}_\epsilon(y', y_N) = \begin{cases} v_{\epsilon,\lambda}(y', y_N), \text{ if } \rho(\epsilon y') > 0\,, \\ v_{\epsilon,\lambda}\left(y', \dfrac{\rho(\epsilon y')}{\epsilon}\right) + \dfrac{\partial v_{\epsilon,\lambda}}{\partial y_N}\left(y_N - \dfrac{\rho(\epsilon y')}{\epsilon}\right), \text{ if } \rho(\epsilon y') < 0\,. \end{cases} \tag{3.16}$$

Then $\tilde{v}_\epsilon(y', y_N)$ is a function defined on $\mathbb{R}^N_+ \cap V$. By (3.13), we may extend $\tilde{v}_\epsilon(y', y_N)$ to the whole $\mathbb{R}^N_+$. We still denote such extension as $\tilde{v}_\epsilon(y', y_N)$. Then

$$J_{\epsilon,\Omega_\epsilon}[t v_{\epsilon,\lambda}] \geq J_{\mathbb{R}^N_+ \cap V}[t\tilde{v}_\epsilon] + J_{\Omega_\epsilon \cap V \setminus \mathbb{R}^N_+}[t v_{\epsilon,\lambda}] - J_{\mathbb{R}^N_+ \cap V \setminus \Omega_\epsilon}[t\tilde{v}_\epsilon] = I_1 + I_2 + I_3 \tag{3.17}$$

where $I_i, i = 1, 2, 3$ are defined at the last equality.

Let us choose $t = t_\epsilon$ so that $J_{\epsilon,\mathbb{R}^N_+ \cap V}[t\tilde{v}_\epsilon]$ maximizes in t. Then by the definition of c_λ and the exponential decay of $v_{\epsilon,\lambda}$, we get that

$$J_{\epsilon,\mathbb{R}^N_+ \cap V}[t_\epsilon \tilde{v}_\epsilon] \geq c_\lambda + O(e^{-\frac{c}{\epsilon}}) \tag{3.18}$$

Moreover

$$t_\epsilon = 1 + o(1)$$

Next,

$$\begin{aligned}
I_2 &= J_{\Omega_\epsilon \cap V \setminus \mathbb{R}^N_+}[t_\epsilon v_{\epsilon,\lambda}] \\
&= \int_{B_{\frac{r_0}{\epsilon}}(0)} dy' \int_{\frac{\rho_\epsilon^-(\epsilon y')}{\epsilon}}^0 \left(\frac{1}{2}|\nabla t_\epsilon v_{\epsilon,\lambda}|^2 + \frac{1}{2}(t_\epsilon v_{\epsilon,\lambda})^2 - F(t_\epsilon v_{\epsilon,\lambda})\right)(y', y_N)dy_N + O(e^{-\frac{c}{\epsilon}}) \\
&\quad + \int_{\partial\Omega_\epsilon \cap \{\rho_\epsilon(\epsilon y')<0\}} \lambda t_\epsilon^2 v_{\epsilon,\lambda}^2 \left(y', \frac{\rho_\epsilon(\epsilon y')}{\epsilon}\right) - \int_{\partial\Omega_\epsilon \cap \{\rho_\epsilon(\epsilon y')<0\}} \lambda t_\epsilon^2 v_{\epsilon,\lambda}^2(x', 0)d\sigma
\end{aligned}$$

and

$$\begin{aligned}
I_3 &= \int_{B_{\frac{r_0}{\epsilon}}(0)} dy' \int_0^{\rho^+(\epsilon y')/\epsilon} \left(\frac{1}{2}|\nabla t_\epsilon v_{\epsilon,\lambda}|^2 + \frac{1}{2}(t_\epsilon v_{\epsilon,\lambda})^2 - F(t_\epsilon v_{\epsilon,\lambda})\right)(y', y_N)dy_N \\
&\quad + O(e^{-\frac{c}{\epsilon}}) + \int_{\partial\Omega_\epsilon \cap \{\rho_\epsilon(\epsilon y')>0\}} \lambda t_\epsilon^2 v_{\epsilon,\lambda}^2 \left(y', \frac{\rho_\epsilon(\epsilon y')}{\epsilon}\right) \\
&\quad - \int_{\partial\Omega_\epsilon \cap \{\rho_\epsilon(\epsilon y')>0\}} \lambda t_\epsilon^2 v_{\epsilon,\lambda}^2(x', 0)d\sigma
\end{aligned}$$

Note that

$$\sqrt{1 + |\nabla \rho_\epsilon(\epsilon y')|^2} = 1 + O(\epsilon^2 |y'|^2)$$

So $d\sigma = (1 + O(\epsilon^2|y'|^2))dx'$.

Since $v_\lambda \to w_\lambda$ in C^1 locally with uniform exponential decay, by the Lebesgue's convergence theorem,

$$\frac{I_2 + I_3}{\epsilon} \to -\frac{1}{2} H(\tilde{x}_\epsilon) \int_{\partial\mathbb{R}^N_+} |y'|^2 \left(\frac{1}{2}|\nabla w_\lambda|^2 + \frac{1}{2} w_\lambda^2 - F(w_\lambda) + \lambda w_\lambda \frac{\partial w_\lambda}{\partial y_N}\right) dy' \tag{3.19}$$

Thus we conclude that

$$\epsilon^{-N} c_{\epsilon,\lambda} \geq c_\lambda - \epsilon \bar{H}(x_\epsilon) + o(\epsilon)\,. \tag{3.20}$$

Now comparing (3.1) and (3.20), we have proved Theorem 1.2. $\square$

4. – Proof of Theorem 1.3

Let $\lambda > \lambda_*$ and $u_{\epsilon,\lambda}$ be a least energy solution. Let $u_{\epsilon,\lambda}(x_\epsilon) = \max_{x\in\Omega} u_{\epsilon,\lambda}(x)$. We first observe that

$$\frac{d(x_\epsilon, \partial\Omega)}{\epsilon} \to +\infty\,. \tag{4.1}$$

In fact, suppose not. Then $\frac{d(x_\epsilon,\partial\Omega)}{\epsilon} \le C$ and after taking a subsequence, $x_\epsilon \to x_0 \in \partial\Omega$, $\frac{d(x_\epsilon,\partial\Omega)}{\epsilon} \to d_0 \ge 0$. Set $v_{\epsilon,\lambda}(y) = u_{\epsilon,\lambda}(x_\epsilon + \epsilon y)$, $y \in \Omega_\epsilon = \{y | \epsilon y + x_\epsilon \in \Omega\}$. Let $v_{\epsilon,\lambda}(y) \to w_\lambda$ in H^1-sense, as $\epsilon \to 0$, where w_λ is also a solution of (1.11). By a simple test function $w(\frac{x-P}{\epsilon})$ with $P \in \Omega$, we see that

$$\lim_{\epsilon\to 0} \epsilon^{-N} c_{\epsilon,\lambda} \le I[w]$$

and whence

$$I_\lambda[w_\lambda] \le I[w] \tag{4.2}$$

On the other hand, Theorem 1.1 (3) yields that for $\lambda > \lambda_*$, $c_\lambda = I[w]$. Hence w_λ attains c_λ, which is impossible by Theorem 1.1 (3).

Thus (4.1) holds. By the same argument as in [26], [27] or [30], we have that

$$\|u_{\epsilon,\lambda}(x_\epsilon + \epsilon y) - w(y)\|_{L^\infty(\Omega_\epsilon)} \to 0\,. \tag{4.3}$$

As in Section 3, we first obtain an upper bound for $c_{\epsilon,\lambda}$.

Lemma 4.1. *For ϵ sufficiently small, we have*

$$c_{\epsilon,\lambda} \le \epsilon^N [I[w] + e^{-\frac{2d(P,\partial\Omega)}{\epsilon}(1+o(1))}] \tag{4.4}$$

for any $P \in \Omega$.

Proof. This follows immediately from Proposition 5.1 of [30] if we take the test function $\mathcal{P}_{\Omega_{\epsilon,P}} w$ (defined in [30]) and note that

$$\int_{\partial\Omega} (\mathcal{P}_{\Omega_{\epsilon,P}} w)^2 = 0\,. \qquad \square$$

Next we shall obtain a lower bound for $c_{\epsilon,\lambda}$:

LEMMA 4.2. *For ϵ sufficiently small, we have*

$$c_{\epsilon,\lambda} = \epsilon^N \left[I[w] + e^{-\frac{2d(x_\epsilon,\partial\Omega)}{\epsilon}(1+o(1))} \right]. \tag{4.5}$$

Now Theorem 1.3 follows from Lemma 4.1 and Lemma 4.2. □

The remaining of the paper is devoted to the proof of Lemma 4.2. Let $x_\epsilon \to x_0 \in \bar{\Omega}$. There are two cases to be considered: Case I, $x_0 \in \partial\Omega$ and Case II, $x_0 \in \Omega$.

Let us fist consider Case II. That is, $x_0 \in \Omega$. In the end, we will show that Case I can be transformed to Case II.

Like in [30], fix any $P \in \Omega$ and define $w_{\epsilon,P}$ to be the unique solution of the following problem with Robin boundary conditions:

$$\begin{cases} \epsilon^2 \Delta w_{\epsilon,P} - w_{\epsilon,P} + f\left(w\left(\dfrac{x-P}{\epsilon}\right)\right) = 0 \text{ in } \Omega\,, \\ \lambda w_{\epsilon,P} + \epsilon \dfrac{\partial w_{\epsilon,P}}{\partial \nu} = 0 \text{ on } \partial\Omega\,. \end{cases} \tag{4.6}$$

Put

$$\varphi_{\epsilon,P}(x) = w\left(\frac{x-P}{\epsilon}\right) - w_{\epsilon,P}\,. \tag{4.7}$$

Then $\varphi_{\epsilon,P}$ satisfies

$$\begin{cases} \epsilon^2 \Delta \varphi_{\epsilon,P} - \varphi_{\epsilon,P} = 0 \text{ in } \Omega \\ \lambda \varphi_{\epsilon,P} + \epsilon \dfrac{\partial \varphi_{\epsilon,P}}{\partial \nu} = \lambda w\left(\dfrac{x-P}{\epsilon}\right) + \epsilon \dfrac{\partial w\left(\frac{x-P}{\epsilon}\right)}{\partial \nu} \text{ on } \partial\Omega\,. \end{cases} \tag{4.8}$$

On $\partial\Omega$, we have

$$\begin{aligned} \lambda w\left(\frac{x-P}{\epsilon}\right) + \epsilon \frac{\partial w\left(\frac{x-P}{\epsilon}\right)}{\partial \nu} &= \lambda w\left(\frac{x-P}{\epsilon}\right) + w'\left(\frac{x-P}{\epsilon}\right) \frac{\langle x-P, \nu\rangle}{|x-P|} \\ &= w\left(\frac{x-P}{\epsilon}\right)\left(\lambda - \frac{\langle x-P,\nu\rangle}{|x-P|} + O\left(\frac{\epsilon}{d(P,\partial\Omega)}\right)\right) \ge (\lambda - 1 - \delta) w\left(\frac{x-P}{\epsilon}\right) \end{aligned}$$

where $\lambda - \delta - 1 > 0$. Therefore, there exist two positive constants C_1 and C_2 such that

$$C_1 \varphi_{\epsilon,P,1} \le \varphi_{\epsilon,P} \le C_2 \varphi_{\epsilon,P,1} \tag{4.9}$$

where $\varphi_{\epsilon,P,1}$ satisfies

$$\begin{cases} \epsilon^2 \Delta \varphi_{\epsilon,P,1} - \varphi_{\epsilon,P,1} = 0 \text{ in } \Omega\,, \\ \varphi_{\epsilon,P,1} + \lambda^{-1} \epsilon \dfrac{\partial \varphi_{\epsilon,P,1}}{\partial \nu} = w\left(\dfrac{x-P}{\epsilon}\right) \text{ on } \partial\Omega\,. \end{cases} \tag{4.10}$$

The study of (4.10) depends on the following lemma.

LEMMA 4.3 (Lemma 3.8 of [34]). *Suppose that* $d(P, \partial\Omega) > d_0$ *for some* $d_0 > 0$. *Let* $\varphi^D_{\epsilon,P}$ *be the unique solution of*

$$\begin{cases} \epsilon^2 \Delta\varphi^D_{\epsilon,P} - \varphi^D_{\epsilon,P} = 0 & \text{in } \Omega\,, \\ \varphi^D_{\epsilon,P} = w\left(\dfrac{x-P}{\epsilon}\right) & \text{on } \partial\Omega\,. \end{cases} \tag{4.11}$$

Then for any arbitrarily small $\delta > 0$, *the following holds for* ϵ *sufficiently small*

$$\left| \epsilon \frac{\partial \varphi^D_{\epsilon,P}}{\partial \nu} \right| \leq (1+\delta)\varphi^D_{\epsilon,P}\,. \tag{4.12}$$

From Lemma 4.3, we infer that on $\partial\Omega$,

$$\varphi^D_{\epsilon,P} + \lambda^{-1}\epsilon \frac{\partial \varphi^D_{\epsilon,P}}{\partial \nu} \leq \varphi^D_{\epsilon,P}(1+\lambda^{-1}(1+\delta)) \leq (1+\lambda^{-1}(1+\delta))w\left(\frac{x-P}{\epsilon}\right)$$

and

$$\varphi^D_{\epsilon,P} + \lambda^{-1}\epsilon \frac{\partial \varphi^D_{\epsilon,P}}{\partial \nu} \geq \varphi^D_{\epsilon,P}(1-\lambda^{-1}(1+\delta)) \geq (1-\lambda^{-1}(1+\delta))w\left(\frac{x-P}{\epsilon}\right).$$

By comparison principle, it is straightforward to derive the following lemma.

LEMMA 4.4. *There exist two positive constants* C_1 *and* C_2 *such that*

$$C_1\varphi^D_{\epsilon,P} \leq \varphi_{\epsilon,P} \leq C_2\varphi^D_{\epsilon,P} \tag{4.13}$$

where $\varphi^D_{\epsilon,P}$ *satisfies* (4.11).

The study of (4.11) is contained in Section 4 of [30]. By Lemma 4.6 of [30] and Lemma 4.4, we derive the following convergence results.

LEMMA 4.5.

(i) $\varphi_{\epsilon,x_\epsilon}(x_\epsilon + \epsilon y)/\varphi_{\epsilon,x_\epsilon}(x_\epsilon) \to V_0(y)$ *locally, where* $V_0(y)$ *is a solution of* (2.27). *Moreover, for any* $\sigma > 0$,

$$\sup_{y\in\Omega_\epsilon} e^{-(1+\sigma)|y|}|V_\epsilon(y) - V_0(y)| \to 0 \tag{4.14}$$

(ii) *As* $\epsilon \to 0$,

$$-\epsilon \log(\varphi_{\epsilon,x_\epsilon}(x_\epsilon)) \to 2d(x_0, \partial\Omega)\,. \tag{4.15}$$

From Lemma 4.5, we can now prove Theorem 1.3. This is similar to the proof of Lemma 2.5.

We first obtain the following global estimates:

$$u_{\epsilon,\lambda} \le Ce^{-\frac{(1-\delta)|x-x_\epsilon|}{\epsilon}} \tag{4.16}$$

for δ such that $(1-\delta)\lambda^{-1} < 1$, where C may depend on δ but is independent of $\epsilon > 0$. In fact, we consider the domain $\Omega^1 := \Omega \backslash B_{R\epsilon}(x_\epsilon)$ where R is large. Then we have $\epsilon^2 \Delta u_{\epsilon,\lambda} - (1-\delta)^2 u_{\epsilon,\lambda} \ge 0$ in Ω^1. Now we compare $u_{\epsilon,\lambda}$ with the function $Ce^{-\frac{(1-\delta)|x-x_\epsilon|}{\epsilon}}$. The estimate (4.16) follows from the Maximum Principle.

Define $v_{\epsilon,\lambda}(y) := u_{\epsilon,\lambda}(x_\epsilon + \epsilon y) = w_{\epsilon,x_\epsilon} + (\varphi_{\epsilon,x_\epsilon}(x_\epsilon))^{1-\delta}\phi_\epsilon(y)$. Substituting the expression into the equation for $u_{\epsilon,\lambda}$ yields that ϕ_ϵ satisfies

$$\Delta\phi_\epsilon - \phi_\epsilon + f'(w_{\epsilon,x_\epsilon})\phi_\epsilon + N_\epsilon + M_\epsilon = 0 \text{ in } \Omega_\epsilon \tag{4.17}$$

and

$$\frac{\partial \phi_\epsilon}{\partial \nu_\epsilon} + \lambda\phi_\epsilon = 0 \text{ on } \partial\Omega_\epsilon$$

where ν_ϵ is the outerward normal on $\partial\Omega_\epsilon$,

$$\begin{aligned} N_\epsilon = \frac{1}{(\varphi_{\epsilon,x_\epsilon}(x_\epsilon))^{1-\delta}}\Big[& f(w_{\epsilon,x_\epsilon} + (\varphi_{\epsilon,x_\epsilon}(x_\epsilon))^{1-\delta}\phi_\epsilon) - f(w_{\epsilon,x_\epsilon}) \\ & - f'(w_{\epsilon,x_\epsilon})(\varphi_{\epsilon,x_\epsilon}(x_\epsilon))^{1-\delta}\phi_\epsilon\Big], \end{aligned} \tag{4.18}$$

and

$$M_\epsilon = \frac{1}{(\varphi_{\epsilon,x_\epsilon}(x_\epsilon))^{1-\delta}}(f(w_{\epsilon,x_\epsilon}) - f(w)). \tag{4.19}$$

By the mean-value theorem and Lemma 4.5, we have that

$$|N_\epsilon| \le C(|w_{\epsilon,x_\epsilon}| + |v_{\epsilon,\lambda}|)^\sigma |v_{\epsilon,\lambda} - w_{\epsilon,x_\epsilon}|^\sigma |\phi_\epsilon| \tag{4.20}$$

and

$$|M_\epsilon| \le C(\varphi_{\epsilon,x_\epsilon}(x_\epsilon))^\delta (|w| + |w_{\epsilon,x_\epsilon}|)^\sigma |V_\epsilon(y)| \le C(\varphi_{\epsilon,x_\epsilon}(x_\epsilon))^\delta e^{\mu|y|} \tag{4.21}$$

for any $1-\sigma < \mu < 1$.

Hence ϕ_ϵ satisfies

$$\begin{cases} \Delta\phi_\epsilon - \phi_\epsilon + f'(w_{\epsilon,x_\epsilon})\phi_\epsilon + o(1)\phi_\epsilon + o(1)e^{\mu|y|} = 0 \text{ in } \Omega_\epsilon, \\ \dfrac{\partial\phi_\epsilon}{\partial\nu_\epsilon} + \lambda\phi_\epsilon = 0 \text{ on } \partial\Omega_\epsilon. \end{cases} \tag{4.22}$$

As in the proof of Lemma 2.5 of Section 2, we set

$$\Phi_\epsilon = G_\mu^{-1}\phi_\epsilon, \tag{4.23}$$

where G_μ satisfies (2.36).

Similar to the proof of (2.38), we obtain the following bound

$$\|\Phi_\epsilon\|_{L^\infty(\Omega_\epsilon)} \leq C\,. \tag{4.24}$$

Now we can compute in the same manner as in Section 6 of [30]

$$\begin{aligned}
\epsilon^{-N} c_{\epsilon,\lambda} &= \epsilon^{-N}\left[\frac{1}{2}\int_\Omega (\epsilon^2|\nabla v_{\epsilon,\lambda}|^2 + v_{\epsilon,\lambda}^2) + \lambda\epsilon\int_{\partial\Omega} v_{\epsilon,\lambda}^2 - \int_\Omega F(v_{\epsilon,\lambda})\right]\\
&= \frac{1}{2}\int_{\Omega_\epsilon} v_{\epsilon,\lambda} f(v_{\epsilon,\lambda}) - \int_{\Omega_\epsilon} F(v_{\epsilon,\lambda})\\
&= \int_{\Omega_\epsilon}\left(\frac{1}{2} w_{\epsilon,x_\epsilon} f(w_{\epsilon,x_\epsilon}\right) - F(w_{\epsilon,x_\epsilon}))\\
&\quad + (\varphi_{\epsilon,x_\epsilon}(x_\epsilon))^{1-\delta}\int_{\Omega_\epsilon}\left(\frac{1}{2} w_{\epsilon,x_\epsilon} f'(w_{\epsilon,x_\epsilon}) - \frac{1}{2} f(w_{\epsilon,x_\epsilon})\right)\phi_\epsilon + o(\varphi_{\epsilon,x_\epsilon}(x_\epsilon)).
\end{aligned}$$

Note that

$$\begin{aligned}
\int_{\Omega_\epsilon} (w_{\epsilon,x_\epsilon} f'(w_{\epsilon,x_\epsilon}) - f(w_{\epsilon,x_\epsilon}))\phi_\epsilon &= \int_{\Omega_\epsilon} [(f'(w_{\epsilon,x_\epsilon})\phi_\epsilon) w_{\epsilon,x_\epsilon} - f(w_{\epsilon,x_\epsilon})\phi_\epsilon]\\
&= \int_{\Omega_\epsilon}\Big[(-\Delta\phi_\epsilon + \phi_\epsilon - N_\epsilon - M_\epsilon) w_{\epsilon,x_\epsilon} - (f(w_{\epsilon,x_\epsilon})\\
&\qquad - f(w))\phi_\epsilon - (-\Delta w_{\epsilon,x_\epsilon} + w_{\epsilon,x_\epsilon})\phi_\epsilon)\Big]\\
&= \int_{\Omega_\epsilon} w_{\epsilon,x_\epsilon}(-M_\epsilon) + O((\varphi_{\epsilon,x_\epsilon}(x_\epsilon))^{1-\delta}).
\end{aligned}$$

Hence

$$\begin{aligned}
c_{\epsilon,\lambda} &= \epsilon^N\left[\int_{\Omega_\epsilon}\left(\frac{1}{2} w_{\epsilon,x_\epsilon} f(w) - F(w_{\epsilon,x_\epsilon})\right) + o(\varphi_{\epsilon,x_\epsilon}(x_\epsilon))\right]\\
&= \epsilon^N[\int_{\mathbb{R}^N}\left(\frac{1}{2} w f(w) - F(w)\right) + c_1\varphi_{\epsilon,x_\epsilon}(x_\epsilon) + o(\varphi_{\epsilon,x_\epsilon}(x_\epsilon))]
\end{aligned}$$

where

$$c_1 = \int_{\mathbb{R}^N} (\frac{1}{2} f(w) V_0) > 0$$

by Lemma 4.7 of [30].

This proves (4.5) in Case II. □

Finally, let us consider the first case: $x_0 \in \partial\Omega$. That is, we assume that $d_\epsilon := d(x_\epsilon, \partial\Omega) \to 0, x_\epsilon \to x_0 \in \partial\Omega$.

We now show that this case can be transformed to Case II by a suitable change of variables. Let $R > 3\lambda$ be a large but fixed number.

Let $J_{\epsilon,\Lambda}$ be defined at (3.14) and

$$c_{\epsilon,\lambda,\Lambda} = \inf_{v\not\equiv 0, v\in H^1(\Lambda)} \max_{t>0} J_{\epsilon,\Lambda}[tv].$$

Let

$$\rho_\epsilon = \frac{d_\epsilon}{\epsilon}. \tag{4.26}$$

By (4.16) and elliptic regularity theory, it is easy to see that

$$|v_{\epsilon,\lambda}| \le Ce^{-R(1-\delta)\rho_\epsilon}, x \in \Omega\backslash\Omega_R \tag{4.27}$$

where $\Omega_R = \Omega \cap B_{R\epsilon}(x_\epsilon)$.

Now let $\bar{u}_\epsilon$ be the function defined as the unique solution of the following problem

$$\begin{cases} \epsilon^2\Delta\bar{u}_\epsilon - \bar{u}_\epsilon + f(u_\epsilon) = 0 & \text{in } \Omega_R, \\ \epsilon\dfrac{\partial\bar{u}_\epsilon}{\partial\nu} + \lambda\bar{u}_\epsilon = 0 & \text{on } \partial\Omega_R. \end{cases} \tag{4.28}$$

From (4.27), it follows that

$$|u_\epsilon - \bar{u}_\epsilon| \le Ce^{-R\rho_\epsilon}$$

Hence $\bar{u}_\epsilon$ satisfies the equation on $\bar{\Omega}_R$:

$$\begin{cases} \epsilon^2\Delta\bar{u}_\epsilon - \bar{u}_\epsilon + f(\bar{u}_\epsilon) + O(e^{-R\rho_\epsilon}) = 0 & \text{in } \Omega_R, \\ \epsilon\dfrac{\partial\bar{u}_\epsilon}{\partial\nu} + \lambda\bar{u}_\epsilon = 0 & \text{on } \partial\Omega_R. \end{cases} \tag{4.29}$$

Now we rescale Ω_R as follows:

$$x = x_\epsilon + d_\epsilon\bar{x}, \quad \bar{x} \in \Lambda_R = (\Omega_R - x_\epsilon)/d_\epsilon\,. \tag{4.30}$$

Then we have

$$c_{\epsilon,\lambda,\Omega_R} = d_\epsilon^N c_{\bar{\epsilon},\lambda,\Lambda_R} \tag{4.31}$$

where $\bar{\epsilon} = \frac{\epsilon}{d_\epsilon}$. Note that by (4.1) $\bar{\epsilon} \to 0$ and

$$c_{\bar{\epsilon},\lambda,\Lambda_R} = J_{\bar{\epsilon},\Lambda_R}(u_\epsilon) = J_{\bar{\epsilon},\Lambda_R}(\bar{u}_\epsilon) + O(\bar{\epsilon}^N e^{-\frac{R}{2\bar{\epsilon}}})$$

In the new domain Λ_R, $\bar{u}_\epsilon$ attain its global maximum at $\bar{x}_\epsilon$ where $d(\bar{x}_\epsilon, \partial\Lambda_R) \to 1$. Moreover, $\bar{u}_\epsilon$ satisfies the following equation

$$\begin{cases} \bar{\epsilon}^2\Delta\bar{u}_\epsilon - \bar{u}_\epsilon + f(\bar{u}_\epsilon) + O(e^{-\frac{R}{\bar{\epsilon}}}) = 0 & \text{in } \Lambda_R, \\ \bar{\epsilon}\dfrac{\partial\bar{u}_\epsilon}{\partial\nu} + \lambda\bar{u}_\epsilon = 0 & \text{on } \partial\Lambda_R. \end{cases} \tag{4.32}$$

We are now in the Case II with the new domain Λ_R and the new function $\bar{u}_\epsilon$. Now following the same proof as in Case II, we obtain that

$$c_{\bar{\epsilon},\lambda,\Lambda_R} = \bar{\epsilon}^N(I[w] + e^{-\frac{2}{\bar{\epsilon}}(1+o(1))}) \tag{4.32}$$

Substituting (4.33) into (4.4), we obtain the lower bound (4.5) in Case I. □

Now comparing (4.5) and (4.4) allows us to conclude and derive the proof of Theorem 1.3. □

5. – Appendix A: Proof of Lemma 2.4

We devote this appendix to the proof of Lemma 2.4. Our main idea is to use vanishing viscosity method as in Section 4 of [30] and Section 3 of [34]. (For vanishing viscosity method, we refer to [23].) Since our domain is $\mathbb{R}^N_+$ which is unbounded, we have to work with a sufficiently large domain and then take a limit.

We begin with the following observation: for $\eta < 1$ and $y \in \partial\mathbb{R}^N_+$,

$$C_1 w(y - Re_N) \le w(y - Re_N) + \eta \frac{\partial w(y - Re_N)}{\partial \nu} \le C_2 w(y - Re_N) \tag{5.1}$$

for some constants $C_1, C_2 > 0$. (Here we have used the asymptotic behavior of w stated in Lemma 2.2.) Comparison principle yields that

$$C_1 \phi_R^1 \le \phi_R \le C_2 \phi_R^1$$

where ϕ_R^1 satisfies

$$\begin{cases} \Delta \phi_R^1 - \phi_R^1 = 0 & \text{in } \mathbb{R}^N_+, \phi_R^1 \in H^1(\mathbb{R}^N_+) \\ \phi_R^1 + \eta \dfrac{\partial \phi_R^1}{\partial \nu} = w(\cdot - Re_N) & \text{on } \partial\mathbb{R}^N_+. \end{cases} \tag{5.2}$$

This implies that in order to prove Lemma 2.4, it is enough to consider ϕ_R^1. To study ϕ_R^1, we introduce another problem: fix a large number $M > 4$, let ϕ_R^2 be the unique solution of

$$\begin{cases} \Delta \phi_R^2 - \phi_R^2 = 0 & \text{in } \mathbb{R}^N_+ \cap B_{MR}(Re_N), \\ \phi_R^2 + \eta \dfrac{\partial \phi_R^2}{\partial \nu} = w(\cdot - Re_N) & \text{on } \partial(\mathbb{R}^N_+ \cap B_{MR}(Re_N)). \end{cases} \tag{5.3}$$

Since $\phi_R^1 \le w(y - Re_N)$, comparison principle (see similar arguments leading to (2.30)) gives

$$|\phi_R^1 - \phi_R^2| \le C e^{-MR}. \tag{5.4}$$

We have reduced our problem to consider ϕ_R^2 only. To study ϕ_R^2, we compare ϕ_R^2 with the following function: let ϕ_R^3 be the unique solution of the following problem

$$\begin{cases} \Delta \phi_R^3 - \phi_R^3 = 0 & \text{in } \mathbb{R}^N_+ \cap B_{MR}(Re_N), \\ \phi_R^3 = w(\cdot - Re_N) & \text{on } \partial(\mathbb{R}^N_+ \cap B_{MR}(Re_N)). \end{cases} \tag{5.5}$$

Put

$$y = Rx,\ R^{-1} = \alpha,\ \Psi_\alpha(x) = -\alpha \log \phi_R^3(y). \tag{5.6}$$

Then $\Psi_\alpha(x)$ satisfies

$$\begin{cases} \alpha \Delta \Psi_\alpha - |\nabla \Psi_\alpha|^2 + 1 = 0 & \text{in } \mathbb{R}^N_+ \cap B_M(e_N), \\ \Psi_\alpha = -\alpha \log w\left(\dfrac{\cdot - e_N}{\alpha}\right) & \text{on } \partial(\mathbb{R}^N_+ \cap B_M(e_N)). \end{cases} \tag{5.7}$$

We shall prove

LEMMA A.

(1) *As* $R \to +\infty$, $\Psi_\alpha(x) \to \Psi_0^M(x)$, *where* $\Psi_0^M(x)$ *is the unique viscosity solution of the following problem*

$$\begin{cases} |\nabla u|^2 = 1 & \text{in } \mathbb{R}_+^N \cap B_M(e_N), \\ u = |x - e_N| & \text{on } \partial(\mathbb{R}_+^N \cap B_M(e_N)) \end{cases} \tag{5.8}$$

In fact, Ψ_0^M *can be written explicitly*

$$\Psi_0^M(x) = \inf_{z \in \partial(\mathbb{R}_+^N \cap B_M(e_N))} (|z - e_N| + |z - x|).$$

(2) *There exists a positive constant* $C > 0$ *such that*

$$\|\nabla \Psi_\alpha\|_{L^\infty(\mathbb{R}_+^N \cap B_M(e_N))} \leq C.$$

(3) *On* $\partial(\mathbb{R}_+^N \cap B_M(e_N))$*, we have as* $R \to +\infty$*,*

$$\frac{\partial \Psi_\alpha}{\partial \nu} \to \frac{\partial \Psi_0^M}{\partial \nu}. \tag{5.9}$$

Lemma 2.4 now follows from Lemma A: in fact, by (5.9), we have that on $\partial(\mathbb{R}_+^N \cap B_{MR}(Re_N))$,

$$\left| \frac{\partial \phi_R^3}{\partial \nu} \right| \leq (1 + \delta) \phi_R^3$$

for any δ small, which implies that

$$C_1 \phi_R^3 \leq \phi_R^3 + \eta \frac{\partial \phi_R^3}{\partial \nu} \leq C_2 \phi_R^3 .$$

Therefore by comparison principle, we derive

$$C_2 \phi_R^3 \leq \phi_R^2 \leq C_2 \phi_R^3, \quad y \in \mathbb{R}_+^N \cap B_{MR}(Re_N). \tag{5.10}$$

Observing that for M large enough, we have

$$\Psi_0^M(x) = \Psi_0(x) = \inf_{z \in \partial \mathbb{R}_+^N} (|z - e_N| + |z - x|), x \in B_4(e_N). \tag{5.11}$$

The rest of the proof of Lemma 2.4 is similar to that of Lemma 4.6 of [30]. □

It remains to prove Lemma A. To prove Lemma A, it is enough to prove (3) of Lemma A: in fact, suppose (3) is true, then we have $\|\nabla\Psi_\alpha\|_{L^\infty(\partial(\mathbb{R}^N_+\cap B_M(e_N)))}\leq C$ and simple computations shows that

$$\Delta(|\nabla\Psi_\alpha|^2)-\frac{2}{\alpha}\nabla\Psi_\alpha\cdot\nabla(|\nabla\Psi_\alpha|^2)\geq 0 \text{ in } \mathbb{R}^N_+\cap B_M(e_N). \tag{5.12}$$

So by the Maximum Principle, (2) of Lemma A holds. From (2) and by taking a limiting process as in Appendix A of [30], we obtain (1) of Lemma A.

To prove (3) of Lemma A, we follow the proof of Lemma 3.7 of [34]. The key fact is that for any $M>0$, there exists a constant $0<l_M<1$ such that

$$\left|-\alpha\log w\left(\frac{z_1-e_N}{\alpha}\right)+\alpha\log w\left(\frac{z_2-e_N}{\alpha}\right)\right|<l_M|z_1-z_2| \tag{5.13}$$

for all $z_1,z_2\in\partial\mathbb{R}^N_+$, $|z_1|,|z_2|\leq M$, where l_M is independent of α. (This corresponds to Lemma 3.5 of [34].) Then we follow the proof of Lemma 3.7 of [34]. (See similar arguments in Section 8.3 of [23].) We omit the details. □

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