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Generic Subgroups of Aut $\mathbb{B}^n$

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Abstract. We prove that for a parabolic subgroup Γ of Aut$\mathbb{B}^n$ the fixed points sets of all elements in $\Gamma \setminus \{\mathrm{id}_{\mathbb{B}^n}\}$ are the same. This result, together with a deep study of the structure of subgroups of Aut$\mathbb{B}^n$ acting freely and properly discontinuously on $\mathbb{B}^n$, entails a generalization of the so called weak Hurwitz's theorem: namely that, given a complex manifold X covered by $\mathbb{B}^n$ and such that the group of deck transformations of the covering is "sufficiently generic", then id_X is isolated in $\mathrm{Hol}(X, X)$.

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1. – Introduction

The double aim of this paper is to study subgroups of the automorphism group Aut $\mathbb{B}^n$ of the unit ball $\mathbb{B}^n = \{z \in \mathbb{C}^n : ||z|| < 1\}$ and to apply the results obtained in this way to the semigroup of holomorphic self-mappings of a complex manifold X covered by the unit ball $\mathbb{B}^n$. In particular we will be able to generalize (in an appropriate statement) the following theorem concerning non-abelian subgroups of the automorphism group of the unit disk $\Delta \subset \mathbb{C}$.

THEOREM 1.1. *Let* Γ *be a non-abelian subgroup of* AutΔ *containing no elliptic elements, then* Γ *contains a hyperbolic element.*

This result comes from the theory of Fuchsian and Kleinian groups originally developed by Poincaré, a modern introduction to the topic can be found in [11].

The interest about subgroups of the automorphism group of the unit ball in $\mathbb{C}^n$ is connected with the study of quotients of $\mathbb{B}^n$ for the action of subgroups of Aut $\mathbb{B}^n$ acting freely and properly discontinuously on it: this can be seen as a (partial) generalization of the construction of Riemann surfaces in the several complex variables setting, even if, in the lack of an uniformization theorem, no

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complete classification can be expected unless very strong conditions, both on the action and on the domain, are required (see [6]).

As a matter of fact, it is well known that, given a subgroup Γ of $\operatorname{Aut}\mathbb{B}^n$ acting freely and properly discontinuously on $\mathbb{B}^n$, the quotient $X = \mathbb{B}^n/\Gamma$ can be endowed with a complex manifold structure so that the projection $\chi : \mathbb{B}^n \to X$ is a local biholomorphism. Vice versa given a complex manifold X covered by $\mathbb{B}^n$ the group of deck transformations of the covering acts freely and properly discontinuously on $\mathbb{B}^n$. This is the reason why we are particularly interested in the structure of subgroups of $\operatorname{Aut}\mathbb{B}^n$ which contain no elliptic elements, that is acting freely on $\mathbb{B}^n$.

In the one dimensional case, the following theorem is often called a "weak version of Hurwitz's theorem":

THEOREM 1.2. *Let X be a Riemann surface whose fundamental group $\pi_1(X)$ is non abelian, then* id_X *is isolated in* $\mathrm{Hol}(X, X)$*. In particular* $\operatorname{Aut} X$ *is discrete.*

This result is due to Heins (see [9]) and when applied to compact Riemann surfaces it implies that the automorphism group $\operatorname{Aut} X$ is finite (a "strong" version would contain the estimate of the number of elements contained in $\operatorname{Aut} X$ according to the genus g of X, see [10]). Anyway, the lack of the estimate of the number of elements is in a certain sense balanced by the knowledge of the structure of $\mathrm{Hol}(X, X)$ which is much larger than $\operatorname{Aut} X$.

In the several dimensional case we generalize the notion of non-abelian subgroup by the following definition:

DEFINITON 1.3. A subgroup $\Gamma \subset \operatorname{Aut}\mathbb{B}^n$ is *generic* if there exist $\gamma_1, \gamma_2 \in \Gamma \setminus \{\mathrm{id}_{\mathbb{B}^n}\}$ such that $\mathrm{Fix}(\gamma_1) \neq \mathrm{Fix}(\gamma_2)$.

Notice that for $n = 1$ Proposition 2.5 implies that a subgroup is non-abelian if and only if is generic. For $n \geq 2$ we will be able to prove that a generic subgroup of $\operatorname{Aut}\mathbb{B}^n$ which contains no elliptic elements is not abelian (see Corollary 3.6), while Example 3.2 will show that there exists a non-abelian subgroup of $\operatorname{Aut}\mathbb{B}^n$ which is not generic (since it is parabolic and we can quote Theorem 1.2).

Anyway, as it will be shown by Example 3.8 the notion of generic subgroup is not strong enough in order to obtain a generalization of Theorem 1.2. In particular the discussion which follows this example will lead us to the following definitions.

DEFINITION 1.4. Let $\mathcal{E}$ be a subset of $\operatorname{Aut}\mathbb{B}^n$ such that $\{\mathrm{Fix}(\gamma) : \gamma \in \mathcal{E}, \gamma \neq \mathrm{id}_{\mathbb{B}^n}\}$ is not reduced to one point in $\partial\mathbb{B}^n$. We denote by $\mathcal{A}(\mathcal{E})$ the *affine subset of* $\mathbb{B}^n$ *generated by* $\{\mathrm{Fix}(\gamma) : \gamma \in \mathcal{E}, \gamma \neq \mathrm{id}_{\mathbb{B}^n}\}$, *i.e.* the least affine subset of $\mathbb{B}^n$ whose closure contains $\bigcup_{\gamma \in \mathcal{E}\setminus\{\mathrm{id}_{\mathbb{B}^n}\}} \mathrm{Fix}(\gamma)$.

This subset can be constructed as follows: consider the affine subspace A generated by $\{\mathrm{Fix}(\gamma) : \gamma \in \mathcal{E}, \gamma \neq \mathrm{id}_{\mathbb{B}^n}\}$. Since $\mathbb{B}^n$ is strictly convex and A is not one point in $\partial\mathbb{B}^n$ then the intersection $A \cap \mathbb{B}^n$ is non empty and $A \cap \overline{\mathbb{B}^n}$ is the closure of $A \cap \mathbb{B}^n$ and therefore $A \cap \mathbb{B}^n$ is equal to $\mathcal{A}(\mathcal{E})$.

Definition 1.5. A subset $\mathcal{E} \subset \mathrm{Aut}\,\mathbb{B}^n$ is said to be *completely generic* if $\mathcal{A}(\mathcal{E}) = \mathbb{B}^n$.

For $n = 1$ Theorem 1.1 entails that a non-abelian subgroup which contains no elliptic elements is completely generic, while there exists completely generic subgroups which are abelian and contain no elliptic elements (any subgroup of Aut Δ generated by a hyperbolic element is such). For $n \geq 2$ it is immediately seen from the definitions that a completely generic subgroup is generic.

Thanks to a deep study of the structure of generic and completely generic subgroups we can prove the following

Theorem 1.6. *Let X be a complex manifold covered by $\mathbb{B}^n$ with $n > 1$. If the group of deck transformations is completely generic, then* id_X *is isolated in* $\mathrm{Hol}(X, X)$. *In particular* Aut X *is discrete.*

This result can be seen as a generalization of Theorem 1.2 because of the above mentioned relations between generic, completely generic and non-abelian subgroups of Aut $\mathbb{B}^n$ acting freely on $\mathbb{B}^n$ and because of the identification between the group of deck transformations of a covering and the fundamental group of the base space, provided the covering space is simply connected.

2. – Preliminary results

We denote the unit ball for the Euclidean metric in $\mathbb{C}^n$ by $\mathbb{B}^n$ (when $n = 1$, often denoted by Δ), the following results concerning the automorphism group of $\mathbb{B}^n$ and its representation through a matrix group are well known (see *e.g.* [1]).

Let $\widetilde{\mathbb{B}^n}$ be the immersion of $\mathbb{B}^n$ in $\mathbb{CP}^n$,

$$\widetilde{\mathbb{B}^n} := \{[u_1 : \cdots : u_{n+1}] \in \mathbb{CP}^n \ : \ |u_1|^2 + \cdots + |u_n|^2 < |u_{n+1}|^2\}$$

and denote by $\mathrm{U}(n, 1)$ the unitary group with respect to the standard Hermitian form $\langle \ | \ \rangle_{(n,1)}$ of signature $(n, 1)$, *i.e.*,

$$\mathrm{U}(n, 1) := \{g \in GL(n + 1, \mathbb{C}) : g^* I_{n,1} g = I_{n,1}\},$$

where $I_{n,1} = \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix}$ and I_n is the $n \times n$ identity matrix. Any $g \in \mathrm{U}(n, 1)$ can be written as a complex $(n + 1) \times (n + 1)$ matrix $g = \begin{pmatrix} G_1 & G_2 \\ G_3 & G_4 \end{pmatrix}$, with $G_4 \in \mathbb{C}$ and G_1, G_2, G_3 matrices of type $n \times n$, $n \times 1$ and $1 \times n$ respectively. Then $\widetilde{\mathbb{B}^n}$ is invariant under the action of $\mathrm{U}(n, 1)$ on $\mathbb{CP}^n$ and the map $\Psi : \mathrm{U}(n, 1) \to \mathrm{Aut}\,\mathbb{B}^n$ defined by

$$\Psi_g(z) = (G_1 z + G_2)(G_3 z + G_4)^{-1}$$

for all $z \in \mathbb{B}^n$ is a surjective group homomorphism whose kernel is given by $\{e^{i\theta} I_{n+1}, \theta \in \mathbb{R}\}$, that is the center of $\mathrm{U}(n, 1)$ (a proof can be found in [8] or [13]).

The following theorem enables us to classify holomorphic automorphisms of $\mathbb{B}^n$ according to their fixed points sets.

THEOREM 2.1. *Any holomorphic automorphism γ of $\mathbb{B}^n$ can be extended holomorphically up to a neighborhood of the closure of $\mathbb{B}^n$; if γ has no fixed points in $\mathbb{B}^n$, then its extension has at least one and at most two fixed points in $\partial\mathbb{B}^n$. Moreover the automorphism group of $\mathbb{B}^n$ acts transitively on $\mathbb{B}^n$ and doubly transitively on $\partial\mathbb{B}^n$.*

From now on we shall denote by the same symbol a holomorphic automorphism of $\mathbb{B}^n$ and its extension to the closure of $\mathbb{B}^n$, moreover for any $\gamma \in \operatorname{Aut}\mathbb{B}^n$ the symbol $\operatorname{Fix}(\gamma)$ will denote the fixed points set of γ in $\overline{\mathbb{B}^n}$, while for any $f \in \operatorname{Hol}(\mathbb{B}^n, \mathbb{B}^n)$ the symbol $\operatorname{fix}(f)$ will denote the fixed points set of f in $\mathbb{B}^n$.

DEFINITION 2.2. Let $\gamma \in \operatorname{Aut}\mathbb{B}^n$: if $\operatorname{fix}(\gamma) \neq \emptyset$, then γ is said to be *elliptic*; if $\operatorname{fix}(\gamma) = \emptyset$ and $\operatorname{Fix}(\gamma)$ contains only one point, it is said to be *parabolic*; if $\operatorname{fix}(\gamma) = \emptyset$ and $\operatorname{Fix}(\gamma)$ contains two points, it is said to be *hyperbolic*.

Analogously, a matrix $g \in \mathrm{U}(n, 1)$ is said to be hyperbolic, elliptic or parabolic, according to the fact that Ψ_g is hyperbolic, elliptic or parabolic.

DEFINITION 2.3. A subgroup $\Gamma \subset \operatorname{Aut}\mathbb{B}^n$ is said to be *parabolic* if any element of $\Gamma \setminus \{\mathrm{id}_{\mathbb{B}^n}\}$ is parabolic.

An *affine subset* of $\mathbb{B}^n$ is the intersection of $\mathbb{B}^n$ with an affine subspace of $\mathbb{C}^n$. The following proposition glues together some results on the fixed points sets of holomorphic self-maps of $\mathbb{B}^n$ (for a proof see [1]).

PROPOSITION 2.4. *The group* $\operatorname{Aut}\mathbb{B}^n$ *maps affine subsets into affine subsets. For any* $f \in \operatorname{Hol}(\mathbb{B}^n, \mathbb{B}^n)$ *the set* $\operatorname{fix}(f)$ *is either empty or is an affine subset.*

The following result gives a necessary and sufficient condition for two elements of $\operatorname{Aut}\Delta \setminus \{\mathrm{id}_\Delta\}$ to commute according to the equality of their fixed points sets (a proof can be found in [1]).

PROPOSITION 2.5. *Let* $\gamma_1, \gamma_2 \in \operatorname{Aut}\Delta \setminus \{\mathrm{id}_\Delta\}$. *Then* $\gamma_1\circ\gamma_2 = \gamma_2\circ\gamma_1$ *if and only if* $\operatorname{Fix}(\gamma_1) = \operatorname{Fix}(\gamma_2)$.

Notice that in the several dimensional case there is no connection between the fact that two elements of $\operatorname{Aut}\mathbb{B}^n \setminus \{\mathrm{id}_{\mathbb{B}^n}\}$ commute and the equality of their fixed points sets in $\overline{\mathbb{B}^n}$: the following two very simple examples show that commutation under composition in $\operatorname{Aut}\mathbb{B}^2$ does not imply equality of fixed points sets of automorphisms and equality of fixed points sets of automorphisms does not imply commutation under composition in $\operatorname{Aut}\mathbb{B}^2$.

EXAMPLE 2.6. Let $\gamma_1, \gamma_2 \in \operatorname{Aut}\mathbb{B}^2$ be given by

$$\gamma_1(z) = (-iz_1, iz_2) \quad \text{and} \quad \gamma_2(z) = (\sqrt{2}(z_1 - z_2)/2, \sqrt{2}(z_1 + z_2)/2)$$

for all $z \in \mathbb{B}^2$. It is easily seen that $\operatorname{Fix}(\gamma_1) = \operatorname{Fix}(\gamma_2) = \{(0, 0)\}$ is the same and nevertheless $\gamma_1\circ\gamma_2 \neq \gamma_2\circ\gamma_1$.

EXAMPLE 2.7. Let $\gamma_1, \gamma_2 \in \operatorname{Aut}\mathbb{B}^2$ be given by

$$\gamma_1(z) = (iz_1, z_2) \quad \text{and} \quad \gamma_2(z) = (z_1/(z_2 + \sqrt{2}), (\sqrt{2}z_2 + 1)/(z_2 + \sqrt{2}))$$

for all $z \in \mathbb{B}^2$. It is easily seen that $\{0\} \times \overline{\Delta} = \mathrm{Fix}(\gamma_1) \neq \mathrm{Fix}(\gamma_2) = \{e_2, -e_2\}$ and nevertheless $\gamma_1 \circ \gamma_2 = \gamma_2 \circ \gamma_1$.

We now study the elements of $\mathrm{U}(n, 1)$ with the aim to give necessary and sufficient conditions for them to be hyperbolic, elliptic or parabolic. Notice that the topological closure of $\widetilde{\mathbb{B}^n}$ is $\{[u] \in \mathbb{CP}^n : \langle u|u\rangle_{(n,1)} \leq 0\}$ and the eigenvectors of g of negative norm correspond to fixed points of Ψ_g in $\mathbb{B}^n$. Isotropic eigenvectors of g, *i.e.* those whose norm is equal to 0, correspond to fixed points of Ψ_g in $\partial\mathbb{B}^n$. In particular parabolic elements in $\mathrm{Aut}\,\mathbb{B}^n$ are images of elements of $\mathrm{U}(n, 1)$ with only one (up to multiples) non-zero isotropic eigenvector and no eigenvectors with negative norm.

First of all, a very simple remark gives a normal form for elliptic elements of $\mathrm{U}(n, 1)$.

Proposition 2.8. *Let g be an elliptic element in* $\mathrm{U}(n, 1)$. *Then g is conjugated in* $\mathrm{U}(n, 1)$ *to a diagonal unitary matrix.*

Proof. Let $g = \begin{pmatrix} G_1 & G_2 \\ G_3 & G_4 \end{pmatrix}$, with $G_4 \in \mathbb{C}$ and G_1, G_2, G_3 matrices of type $n \times n$, $n \times 1$ and $1 \times n$ respectively. Since $\mathrm{Aut}\,\mathbb{B}^n$ acts transitively on $\mathbb{B}^n$, we can suppose up to conjugation that the origin is a fixed point of Ψ_g, i.e. that e_{n+1} is an eigenvector of g, which implies $G_2 = 0$. As g belongs to $\mathrm{U}(n, 1)$ we obtain that G_1 is a unitary matrix, $G_3 = 0$ and $|G_4| = 1$. As the matrices of the form $\begin{pmatrix} G & 0 \\ 0 & 1 \end{pmatrix}$ with $G \in \mathrm{U}(n)$ belong to $\mathrm{U}(n, 1)$ we can apply the spectral theorem and we are done. □

The normal form for hyperbolic elements contained in the following proposition is due to de Fabritiis and Gentili (see [5]).

Proposition 2.9. *Let g be a hyperbolic element in* $\mathrm{U}(n, 1)$. *Then there exist* $t \in \mathbb{R}^*$, $\theta \in \mathbb{R}$ *and a diagonal unitary matrix* W *of order* $n - 1$ *such that g is conjugated in* $\mathrm{U}(n, 1)$ *to*

$$g = e^{i\theta} \begin{pmatrix} W & 0 & 0 \\ 0 & \cosh t & \sinh t \\ 0 & \sinh t & \cosh t \end{pmatrix}. \tag{2.1}$$

As a consequence of this proposition we obtain a result on the structure of the fixed points set of a holomorphic map which commutes under composition with a hyperbolic automorphism of $\mathbb{B}^n$.

Proposition 2.10. *Let* $f : \mathbb{B}^n \to \mathbb{B}^n$ *be a holomorphic map which commutes with a hyperbolic automorphism* $\gamma \in \mathrm{Aut}\,\mathbb{B}^n$. *If* $\mathrm{fix}(f) \neq \emptyset$ *then it contains the affine subset of dimension* 1 *whose closure contains the fixed points of* γ.

Proof. Since the fixed points set of a holomorphic map of $\mathbb{B}^n$ into itself is an affine subset of $\mathbb{B}^n$ and the statement of the proposition is invariant by conjugation in $\mathrm{Aut}\,\mathbb{B}^n$ we can suppose that γ is given by

$$\gamma(z) = \left(\frac{Wz'}{z_n \sinh t + \cosh t}, \frac{z_n \cosh t + \sinh t}{z_n \sinh t + \cosh t} \right),$$

where $W \in \mathrm{U}(n-1)$ and $z' = (z_1, \dots, z_{n-1})$. As f and γ commute, then γ sends $\mathrm{fix}(f)$ into itself. Denote by E the affine subspace of $\mathbb{C}^n$ such that $\mathrm{fix}(f) = E \cap \mathbb{B}^n$ and choose an $m \times n$ matrix A and a vector $b \in \mathbb{C}^m$ so that $E = \{z \in \mathbb{B}^n : Az = b\}$. Then for any $z \in \mathrm{fix}(f)$ we have $A\gamma^k(z) = b$ for all $k \in \mathbb{Z}$. Set $A = |A_1\ A_2|$ where A_1 is an $m \times (n-1)$ matrix and A_2 is a vector in $\mathbb{C}^m$; a trivial computation yields

$$A_1 W^k z' + (z_n \cosh kt + \sinh kt) A_2 = (z_n \sinh kt + \cosh kt) b$$

for any $k \in \mathbb{Z}$. Since $\{A_1 W^k z' : k \in \mathbb{Z}\}$ is bounded in $\mathbb{C}^m$ we divide by $\cosh kt$ and take the limit for $k \to \pm\infty$, thus obtaining

$$(z_n + 1)A_2 = (z_n + 1)b \qquad \text{and} \qquad (z_n - 1)A_2 = (-z_n + 1)b.$$

As $z \in \mathbb{B}^n$ these equalities immediately entail $A_2 = b = 0$ and hence the complex disc Δe_n is contained in $\mathrm{fix}(f)$. Since $\mathrm{Fix}(\gamma) = \{e_n, -e_n\}$ the affine subset of dimension 1 whose closure contains $\mathrm{Fix}(\gamma)$ is given by Δe_n and then we are done. □

Now we turn to the study of parabolic elements, the following lemma is the first step towards a normal form.

LEMMA 2.11. *Let $g \in \mathrm{U}(n, 1)$ and suppose that g has a unique eigenspace of dimension 1 which consists of isotropic vectors. Then either $n = 1$ or $n = 2$. Moreover, if $n = 1$, g is conjugated to*

$$g_1 := e^{i\theta} \begin{pmatrix} 1 - it & -it \\ it & 1 + it \end{pmatrix} \tag{2.2}$$

where $\theta \in \mathbb{R}$, $t \in \mathbb{R}^*$; if $n = 2$, then g is conjugated to

$$g_2 := e^{i\theta} \begin{pmatrix} 1 & s & s \\ -s & 1 - \beta & -\beta \\ s & \beta & 1 + \beta \end{pmatrix} \tag{2.3}$$

where $\theta \in \mathbb{R}$, $\beta \in \mathbb{C}$, $\mathrm{Re}\beta > 0$ and $s = \sqrt{2\mathrm{Re}\beta}$.

The above lemma gives us the possibility of classifying parabolic elements up to conjugation, a proof can be found in [6].

THEOREM 2.12. *Let g be a parabolic element of $\mathrm{U}(n, 1)$. Then there exist $l \in \{1, 2\}$ and a diagonal unitary matrix W of order $n - l$ such that g is conjugated in $\mathrm{U}(n, 1)$ to*

$$\begin{pmatrix} W & 0 \\ 0 & g_l \end{pmatrix} \tag{2.4}$$

where g_l is given either by (2.2) or by (2.3), according to the value of l.

Then we can summarize Theorems 2.8, 2.9 and 2.12 in some criteria generalizing the ones which hold in the one-dimensional case (notice that if $n = 1$ the trace criteria is both necessary and sufficient in order to classify hyperbolic, parabolic or elliptic elements, while in the several dimensional case only a part of the sufficient condition holds true).

CRITERION 2.13. A matrix $g \in \mathrm{U}(n, 1)$ is hyperbolic if and only if its spectrum is not contained in the unit circle in $\mathbb{C}$.

CRITERION 2.14. A matrix $g \in \mathrm{U}(n, 1)$ whose trace has modulus greater than $n + 1$ is hyperbolic.

CRITERION 2.15. A matrix $g \in \mathrm{U}(n, 1)$ is parabolic iff it is not diagonalizable.

In the sequel it will be useful to consider the problem also on the Siegel half-space $\mathbb{H}^n = \{w \in \mathbb{C}^n \mid \mathrm{Im} w_n > |w_1|^2 + \cdots + |w_{n-1}|^2\}$ which is biholomorphic to $\mathbb{B}^n$ via the Cayley transform $\mathcal{C} : \mathbb{B}^n \to \mathbb{H}^n$ defined by

$$\mathcal{C}(z_1, \ldots, z_n) = \left(\frac{iz_1}{1 - z_n}, \frac{iz_2}{1 - z_n}, \ldots, i\frac{1 + z_n}{1 - z_n} \right),$$

whose inverse is given by

$$\mathcal{C}^{-1}(w_1, \ldots, w_n) = \left(\frac{2w_1}{w_n + i}, \frac{2w_2}{w_n + i}, \ldots, \frac{w_n - i}{w_n + i} \right).$$

This map gives us the possibility of translating the above results on $\mathbb{H}^n$. Let

$$\Lambda_n = \begin{pmatrix} iI_{n-1} & 0 & 0 \\ 0 & i & i \\ 0 & -1 & 1 \end{pmatrix},$$

then conjugation by Λ_n maps $\mathrm{U}(n, 1)$ to

$$\mathcal{G}_n = \{h \in \mathrm{GL}(n + 1, \mathbb{C}) : h^* K_n h = K_n\}, \tag{2.5}$$

where

$$K_n = \begin{pmatrix} I_{n-1} & 0 & 0 \\ 0 & 0 & -i/2 \\ 0 & i/2 & 0 \end{pmatrix}.$$

The factorization of Λ_n through the projection

$$(z_1, \ldots, z_{n+1}) \mapsto \left(\frac{z_1}{z_{n+1}}, \ldots, \frac{z_n}{z_{n+1}} \right)$$

on $\widetilde{\mathbb{B}^n}$ gives the Cayley transform from $\mathbb{B}^n$ onto $\mathbb{H}^n$ and hence the action induced by the group $\mathcal{G}_n$ on $\mathbb{H}^n$ via Ψ represents Aut $\mathbb{H}^n$. In order to complete the transfer to $\mathbb{H}^n$, we say that $\gamma \in \operatorname{Aut}\mathbb{H}^n$ ($\mathcal{G}_n$) is hyperbolic, parabolic or elliptic according to the fact that $\mathcal{C}^{-1}\circ\gamma\circ\mathcal{C} \in \operatorname{Aut}\mathbb{B}^n$, or equivalently $\Lambda_n^{-1}\circ\gamma\circ\Lambda_n \in \mathrm{U}(n,1)$ is such.

We end this section by giving a different presentation of the elements contained in $\operatorname{Aut}\mathbb{H}^n$. Given $t > 0$ we denote by $\delta_t \in \operatorname{Aut}\mathbb{H}^n$ the dilatation given by

$$\delta_t(w) = (tw', t^2 w_n),$$

where $w' = (w_1, \dots, w_{n-1})$. Notice that δ_t is a hyperbolic automorphism of $\mathbb{H}^n$ which fixes ∞ and 0 for any $t \neq 1$. Given $a \in \partial\mathbb{H}^n$ we denote by $h_a \in \operatorname{Aut}\mathbb{H}^n$ the translation given by

$$h_a(w) = (w' + a', w_n + a_n + 2i\langle w', a'\rangle),$$

where $w' = (w_1, \dots, w_{n-1})$ and $a' = (a_1, \dots, a_{n-1})$. Notice that h_a is a parabolic element of $\operatorname{Aut}\mathbb{H}^n$ which fixes ∞ for any $a \in \partial\mathbb{H}^n \setminus \{0\}$. Finally, given $U \in \mathrm{U}(n)$ we define $\mu_U \in \operatorname{Aut}\mathbb{H}^n$ by

$$\mu_U = \mathcal{C}\circ U\circ\mathcal{C}^{-1}.$$

Notice that μ_U is an elliptic element of $\operatorname{Aut}\mathbb{H}^n$ which fixes $(0, \dots, 0, i)$ for any $U \in \mathrm{U}(n) \setminus \{\mathrm{I}_n\}$.

PROPOSITION 2.16. *Let $h \in \operatorname{Aut}\mathbb{H}^n$. Then there exist $t > 0$, $U \in \mathrm{U}(n)$ and $a \in \partial\mathbb{H}^n$ such that $h = \delta_t\circ h_a\circ\mu_U$. Moreover $\infty \in \operatorname{Fix}(h)$ if and only if $Ue_n = e_n$, that is $\mu_U(w) = (Uw', w_n)$ for a suitable $U' \in \mathrm{U}(n-1)$; and $0, \infty \in \operatorname{Fix}(h)$ if and only if $Ue_n = e_n$ and $a = 0$.*

For a proof of the above proposition, which is obtained gluing together several results, see [1].

3. – Subgroups of $\operatorname{Aut}\mathbb{B}^n$

First of all we state and prove an appropriate generalization of Theorem 1.1. We recall that a subgroup Γ of $\operatorname{Aut}\mathbb{B}^n$ is said to be generic if there exist $\gamma_1, \gamma_2 \in \Gamma \setminus \{\mathrm{id}_{\mathbb{B}^n}\}$ such that $\operatorname{Fix}(\gamma_1) \neq \operatorname{Fix}(\gamma_2)$.

THEOREM 3.1. *Let Γ be a parabolic subgroup of $\operatorname{Aut}\mathbb{B}^n$, then Γ is not generic.*

Notice that, thanks to Proposition 2.5, in the one-dimensional case this statement is equivalent to Theorem 1.1.

PROOF. Moving the problem to $\mathbb{H}^n$ and using the map Ψ it is enough to prove the following equivalent statement:

Let H be a parabolic subgroup of $\mathcal{G}_n$ containing $e^{i\theta} I_{n+1}$ for all $\theta \in \mathbb{R}$, then the fixed points set of Ψ_h in $\overline{\mathbb{H}^n} \cup \{\infty\}$ is the same for all $h \in H$ different from a multiple of I_{n+1}.

Indeed, we can consider the subgroup $\Theta = \mathcal{C}\circ\Gamma\circ\mathcal{C}^{-1}$ of Aut $\mathbb{H}^n$ obtained by conjugating Γ with the Cayley transform and take $H = \Psi^{-1}(\Theta)$. Of course, the group Γ is parabolic if and only if any of the elements of the set $H\setminus\{e^{i\theta}I_{n+1} : \theta\in\mathbb{R}\}$ is such. Moreover $\mathcal{C}$ moves the fixed points set of any $\gamma\in\Gamma\setminus\{\mathrm{id}_{\mathbb{B}^n}\}$ to the fixed points set of $\mathcal{C}\circ\gamma\circ\mathcal{C}^{-1}$ and hence all the elements in $\Theta\setminus\{\mathrm{id}_{\mathbb{H}^n}\}$ have the same fixed points set in $\overline{\mathbb{H}^n}\cup\{\infty\}$ if and only if all the elements in $\Gamma\setminus\{\mathrm{id}_{\mathbb{B}^n}\}$ have the same fixed points set in $\overline{\mathbb{B}^n}$.

Choose any $h_0\in H$ which is not a multiple of I_{n+1}, since H contains all the multiples of the identity matrix and inner conjugation in $\mathcal{G}_n$ does not affect the hypothesis of the theorem, we can suppose that

$$h_0 = \begin{pmatrix} W & 0 & 0 & 0 \\ 0 & 1 & s & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -2is & -2i\beta & 1 \end{pmatrix}$$

where $W\in U(n-2)$ is a diagonal matrix, $\mathrm{Re}\beta\geq 0$ and $s=\sqrt{2\mathrm{Re}\beta}$. Notice that $\beta\neq 0$ because h_0 is not elliptic nor equal to the identity (h_0 is obtained by conjugating with Λ_n the parabolic element $g\in \mathrm{U}(n,1)$ given by equation (2.4)). It is easily seen by induction on k that

$$h_0^k = \begin{pmatrix} W^k & 0 & 0 & 0 \\ 0 & 1 & ks & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -2iks & -2ki\beta - ik(k-1)s^2 & 1 \end{pmatrix}.$$

The fixed points set of Ψ_{h_0} in $\overline{\mathbb{H}^n}\cup\{\infty\}$ is equal to $\{0\}$, which corresponds to the fact that the only eigenspace of h_0 containing vectors with non-positive Hermitian form induced by K_n is generated by e_{n+1}. Choose h in H which is not a multiple of the identity matrix, by the assumption on H we know that h is parabolic.

Split h in the following form $h=\left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right)$ where A is a square matrix of order $n-2$, $B=(b_1\ b_2\ b_3)$ is a $(n-2)\times 3$ matrix, C is $3\times(n-2)$ matrix and

$$D = \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{pmatrix}.$$

An easy computation shows that $hh_0^k = \left(\begin{smallmatrix} AW^k & * \\ * & T\end{smallmatrix}\right)$, where

$$T = \begin{pmatrix} d_{11} - 2id_{13}ks & * & * \\ * & d_{22} + d_{21}ks - id_{23}k((k-1)s^2 + 2\beta) & * \\ * & * & d_{33} \end{pmatrix}.$$

We now show that $d_{23}=0$. If $d_{23}\neq 0$ and $s\neq 0$, then it is easily seen that the modulus of the trace of hh_0^k tends to $+\infty$ when $k\to+\infty$. Hence there exists an element in h whose trace has modulus greater then $n+1$: by Criterion 2.14 the group H contains a hyperbolic element, which is a contradiction.

If $d_{23} \neq 0$ and $s = 0$, we obtain again—as $\beta \neq 0$— that there exists a hyperbolic element in H which is a contradiction and thus we have $d_{23} = 0$.

The fact that h belongs to $\mathcal{G}_n$ is equivalent to

$$\begin{cases} A^*A + C^*K_2C = I_{n-2}, \\ A^*B + C^*K_2D = 0, \\ B^*B + D^*K_2D = K_2. \end{cases} \tag{3.1}$$

Developing computations, we obtain from last equation in (3.1) that

$$|b_3|^2 + |d_{13}|^2 = 0 \qquad \text{and} \qquad b_3^*b_2 + \bar{d}_{13}d_{12} + i\bar{d}_{33}d_{22}/2 = i/2.$$

Then $b_3 = 0$, $d_{13} = 0$ and $\bar{d}_{33}d_{22} = 1$. Up to multiplying h by a suitable constant of modulus 1, which does not affect any of the hypothesis, we can suppose that d_{33} is real and positive. As we assumed that d_{33} is positive, then $d_{22} = 1/d_{33}$ is positive, too. Now, a straightforward computation shows that d_{33} is an eigenvalue of h and therefore by Criterion 2.13 the modulus of d_{33} has to be equal to 1. Since $d_{33} \in \mathbb{R}^+$ we obtain $d_{33} = 1$ and hence $d_{22} = 1$, too.

As h is parabolic, then Ψ_h has a unique fixed point in $\overline{\mathbb{H}^n} \cup \{\infty\}$. A straightforward computations shows that $\Psi_h(0) = 0$ and therefore $\mathrm{Fix}(\Psi_h) = \{0\}$. The proof of the theorem is complete. □

For $n \geq 2$ we give a counterexample to the very same statement of Theorem 1.1 in the several variables cases, that is for $n \geq 2$ we exhibit a non-abelian parabolic (and therefore non-generic) subgroup of $\mathrm{Aut}\,\mathbb{B}^n$. To simplify computations, we move to $\mathbb{H}^n$ and use the representation of $\mathrm{Aut}\,\mathbb{H}^n$ via the group $\mathcal{G}_n$. Moreover, since the group $\mathcal{G}_2$ can be seen as a subgroup of $\mathcal{G}_n$ via the injective homomorphism $\mathcal{G}_2 \ni h \mapsto \begin{pmatrix} I_{n-2} & 0 \\ 0 & h \end{pmatrix} \in \mathcal{G}_n$ it is enough to illustrate the example only for $n = 2$.

Example 3.2. We consider the subgroup $H \subset \mathcal{G}_2$ generated by

$$h_1 = \begin{pmatrix} 1 & 0 & 1 \\ 2i & 1 & 1+i \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad h_2 = \begin{pmatrix} 1 & 0 & i \\ 2 & 1 & i \\ 0 & 0 & 1 \end{pmatrix};$$

it is easily seen that both h_1 and h_2 are parabolic since they are non diagonalizable.

Each element h in H has the form $h_1^{n_1}h_2^{m_1} \ldots h_1^{n_k}h_2^{m_k}$ with $n_1, \ldots, n_k, m_1, \ldots, m_k \in \mathbb{Z}$. An easy inductive argument yields that

$$h_1^n = \begin{pmatrix} 1 & 0 & n \\ 2in & 1 & (1+i)n + n(n-1)i \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad h_2^n = \begin{pmatrix} 1 & 0 & in \\ 2n & 1 & in^2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Setting $\widetilde{h}^{n,m} = h_2^m h_1^n h_2^{-m}$, then $h = h_1^{n_1}\widetilde{h}^{n_2,m_1}\dots\widetilde{h}^{n_k,m_1+\dots+m_{k-1}}h_2^{m_1+\dots m_k}$. Since

$$\widetilde{h}^{n,m} = \begin{pmatrix} 1 & 0 & n \\ 2in & 1 & 4mn + in^2 + n \\ 0 & 0 & 1 \end{pmatrix}$$

a long but straightforward computation yields

$$\begin{aligned} h &= \begin{pmatrix} 1 & 0 & n_1 \\ 2in_1 & 1 & \alpha_1 \\ 0 & 0 & 1 \end{pmatrix} \dots \begin{pmatrix} 1 & 0 & n_k \\ 2in_k & 1 & \alpha_k \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & is_k \\ 2s_k & 1 & \alpha_{k+1} \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & \sigma_k + is_k \\ 2i\sigma_k + 2s_k & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix}, \end{aligned}$$

where $\alpha_1, \dots, \alpha_{k+1}, \beta$ are in $\mathbb{C}$, $s_k = (m_1 + \dots + m_k)$ and $\sigma_k = (n_1 + \dots + n_k)$.

As $(h - I_3)^3 = 0$, h is diagonalizable iff it is equal to I_3. Thus each element of H is diagonalizable iff it is equal to I_3 and therefore Criterion 2.15 shows the group H contains only parabolic elements and the identity matrix.

Nevertheless $h_1 h_2 h_1^{-1} h_2^{-1}$ is not a multiple of the identity matrix and hence the group $\Psi(H)$ is a non-abelian parabolic subgroup of Aut $\mathbb{H}^2$.

The above example shows that Theorem 3.1 can be seen as a common fixed points result obtained without any commutation hypothesis (see [2] and [3] for a detailed discussion on the topic). In fact, if Γ is a parabolic subgroup of Aut $\mathbb{B}^n$, then all elements in $\Gamma \setminus \{\mathrm{id}_{\mathbb{B}^n}\}$ have a common fixed point in $\overline{\mathbb{B}^n}$ which is the unique fixed point of any of the parabolic elements contained in Γ.

The next results are a further step in the comprehension of the structure of generic subgroups of Aut $\mathbb{B}^n$.

Lemma 3.3. *Let $\gamma_1 \in$ Aut $\mathbb{B}^n$ be hyperbolic and $\gamma \in$ Aut $\mathbb{B}^n \setminus \{\mathrm{id}_{\mathbb{B}^n}\}$ be non elliptic. If* $\mathrm{Fix}(\gamma_1) \neq \mathrm{Fix}(\gamma)$ *then there exists $k_0 \in \mathbb{N}$ such that $\gamma_1^k\gamma$ is hyperbolic for any $k \geq k_0$ or for any $k \leq -k_0$.*

Proof. As above we transfer the problem on U$(n, 1)$ via the map Ψ, i.e. we choose $g_1 \in \Psi^{-1}(\gamma_1)$ and $g \in \Psi^{-1}(\gamma)$. Since the statement of Lemma 3.3 is invariant under conjugation by Aut $\mathbb{B}^n$ we can suppose that g_1 is given by

$$\begin{pmatrix} W & 0 & 0 \\ 0 & \cosh t & \sinh t \\ 0 & \sinh t & \cosh t \end{pmatrix},$$

where W is a diagonal unitary matrix of order $n - 1$ and $t \in \mathbb{R}^*$.

In order to prove that for $k >> 1$ or $-k >> 1$ the element $g_1^k g$ is hyperbolic we split g in the following form $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where A is a square matrix of order $n - 1$, $B = (b_1\ b_2)$ is a $(n - 1) \times 2$ matrix, C is a $2 \times (n - 1)$ matrix and $D = \begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix}$. Since

$$\mathrm{tr}(g_1^k g) = \mathrm{tr}(W^k A) + (d_1 + d_4)\cosh kt + (d_2 + d_3)\sinh kt$$

and $\operatorname{tr}(W^k A)$ is bounded by $|a_{11}| + \cdots + |a_{n-1,n-1}|$, then there exists $k_0 \in \mathbb{N}$ such that for any $k \geq k_0$ or for any $k \leq -k_0$ we have $|\operatorname{tr}(g_1^k g)| > n+1$ unless $d_1 + d_4 = d_2 + d_3 = 0$.

If $d_1 = -d_4$ and $d_2 = -d_3$, the fact that $g \in U(n, 1)$ gives

$$|b_1|^2 + |d_1|^2 - |d_3|^2 = 1 \quad \text{and} \quad |b_2|^2 + |d_2|^2 - |d_4|^2 = -1$$

and hence $b_1 = b_2 = 0$. Since $B = 0$ we then obtain that $A \in U(n-1)$, $C = 0$ and $D \in U(1, 1)$. A straightforward computation shows that there exist $\tau, \theta \in \mathbb{R}$ such that $D = e^{i\theta} \begin{pmatrix} \cosh\tau & \sinh\tau \\ -\sinh\tau & -\cosh\tau \end{pmatrix}$. It is easily seen that $-\sinh\tau e_n/(1+\cosh\tau)$ is a fixed point of γ in $\mathbb{B}^n$ which is a contradiction.

Then there exists $k_0 \in \mathbb{N}$ such that $|\operatorname{tr}(g_1^k g)| > n+1$ for any $k \geq k_0$ or for any $k \leq -k_0$ and hence Criterion 2.14 yields that $\gamma_1^k \gamma$ is hyperbolic for any $k \geq k_0$ or for any $k \leq -k_0$. This concludes the proof. □

The following proposition shows that generic subgroups of $\operatorname{Aut}\mathbb{B}^n$ acting freely on $\mathbb{B}^n$ contain a wide variety of hyperbolic elements, since we have

Proposition 3.4. *Let Γ be a generic subgroup of* $\operatorname{Aut}\mathbb{B}^n$ *which contains no elliptic elements. Then there exist $\gamma_1, \gamma_2 \in \Gamma$ which are both hyperbolic and such that* $\operatorname{Fix}(\gamma_1) \neq \operatorname{Fix}(\gamma_2)$.

Proof. As usual we transfer the problem on $U(n, 1)$ via the map Ψ. Denoting by G the subgroup $\Psi^{-1}(\Gamma)$ we can restate the assertion as follows:

Let G be a generic subgroup of $U(n, 1)$ which contains all multiples of I_{n+1} and no elliptic elements, then there exist $g_1, g_2 \in G$ which are both hyperbolic and such that $\operatorname{Fix}(\Psi_{g_1}) \neq \operatorname{Fix}(\Psi_{g_2})$.

By Theorem 3.1 there exists a hyperbolic element $g_1 \in G$. The assumption on the fixed points set of the elements of $\Psi(G)$ ensures that there exists $g \in G$ such that $\Psi_g \neq \mathrm{id}_{\mathbb{B}^n}$ and $\operatorname{Fix}(\Psi_g) \neq \operatorname{Fix}(\Psi_{g_1}) =: \{p_1, p_2\}$. Our candidate for g_2 is $g_1^k g$ for a suitable $k \in \mathbb{Z}^*$.

If for some $k \in \mathbb{Z}^*$ we had $\operatorname{Fix}(\Psi_{g_1^k g}) = \{p_1, p_2\}$ then $\operatorname{Fix}(\Psi_g)$ would contain $\{p_1, p_2\}$ and therefore $\operatorname{Fix}(\Psi_g) = \{p_1, p_2\}$ because there are no elliptic elements in G. This is a contradiction to the choice of g. Hence we are left to prove that for some $k \in \mathbb{Z}^*$ the element $g_1^k g$ is hyperbolic and Lemma 3.3 entails the proof. □

We quote from [5] a result concerning the fixed points sets of two hyperbolic automorphism which commute under composition which enables us to obtain a better knowledge of generic abelian subgroups of $\operatorname{Aut}\mathbb{B}^n$.

Proposition 3.5. *Let $\gamma_1, \gamma_2 \in$* $\operatorname{Aut}\mathbb{B}^n \setminus \{\mathrm{id}_{\mathbb{B}^n}\}$. *If γ_1 is hyperbolic and γ_1, γ_2 commute under composition, then either γ_2 is hyperbolic and* $\operatorname{Fix}(\gamma_1) = \operatorname{Fix}(\gamma_2)$ *or γ_2 is elliptic and* $\operatorname{Fix}(\gamma_1) \subset \operatorname{Fix}(\gamma_2)$.

As an immediate consequence of this result we obtain the following

Corollary 3.6. *A generic subgroup of* $\operatorname{Aut}\mathbb{B}^n$ *which contains no elliptic elements is not abelian.*

Notice that we cannot drop the hypothesis concerning elliptic elements: in fact the subgroup $\Gamma \subset \operatorname{Aut}\mathbb{B}^2$ generated by

$$g : \mathbb{B}^2 \ni z \mapsto (z_1, -z_2) \in \mathbb{B}^2 \quad \text{and} \quad h : \mathbb{B}^2 \ni z \mapsto (-z_1, z_2) \in \mathbb{B}^2$$

is abelian but is completely generic (and hence generic) since $\operatorname{Fix}(g) = \overline{\Delta} \times \{0\}$ and $\operatorname{Fix}(h) = \{0\} \times \overline{\Delta}$ and therefore $\mathbb{C}^2 = \mathcal{A}(g, h) \subseteq \mathcal{A}(\Gamma) \subseteq \mathbb{C}^2$.

As we already said in the Introduction, we are particularly interested in subgroups of $\operatorname{Aut}\mathbb{B}^n$ which act freely and properly discontinuously on $\mathbb{B}^n$. The following result shows that a generic group which acts freely and properly discontinuously on $\mathbb{B}^n$ is really "generic", i.e. there is no common fixed point of the elements of such a subgroup.

Proposition 3.7. *Let Γ be a generic subgroup of* $\operatorname{Aut}\mathbb{B}^n$ *acting freely and properly discontinuously on $\mathbb{B}^n$. Then there exist $\gamma_1, \gamma_2 \in \Gamma$ which are both hyperbolic and such that* $\operatorname{Fix}(\gamma_1) \cap \operatorname{Fix}(\gamma_2) = \emptyset$. *In particular* $\bigcap_{\gamma\in\Gamma} \operatorname{Fix}(\gamma) = \emptyset$.

Proof. As Γ acts freely and and properly discontinuously on $\mathbb{B}^n$ then it is discrete. Proposition 3.4 yields that there exist $\gamma_1, \gamma_2 \in \Gamma$ which are both hyperbolic and such that $\operatorname{Fix}(\gamma_1) \neq \operatorname{Fix}(\gamma_2)$. Suppose by contradiction that $\operatorname{Fix}(\gamma_1) \cap \operatorname{Fix}(\gamma_2) \neq \emptyset$. By conjugating the elements of Γ with the Cayley transform we can move the problem to $\mathbb{H}^n$. Denote $\mathcal{C}\circ\Gamma\circ\mathcal{C}^{-1}$ by H and $\mathcal{C}\circ\gamma_j\circ\mathcal{C}^{-1}$ by h_j for $j = 1, 2$. Since $\operatorname{Fix}(h_1) \cap \operatorname{Fix}(h_2) \neq \emptyset$ and $\operatorname{Aut}\mathbb{H}^n$ acts doubly transitively on $\partial\mathbb{H}^n \cup \{\infty\}$ we can suppose that $\operatorname{Fix}(h_1) = \{0, \infty\}$ and $\infty \in \operatorname{Fix}(h_2)$.

By Proposition 2.16 there exist $t_1, t_2 > 0$, $U_1, U_2 \in \mathrm{U}(n-1)$, $a \in \partial\mathbb{H}^n$ such that

$$h_1 = \delta_{t_1}\circ\mu_{U_1} \qquad \text{and} \qquad h_2 = \delta_{t_2}\circ h_a\circ\mu_{U_2}.$$

It is easily seen that $\delta_{t_1}, \delta_{t_2}, \mu_{U_1}$ and μ_{U_2} commute; moreover $t_1 \neq 1$ because h_1 is hyperbolic and $a \neq 0$ because $\operatorname{Fix}(h_1) \neq \operatorname{Fix}(h_2)$. Moreover up to replacing h_1 with h_1^{-1} we can suppose that $t_1 > 1$.

Now notice that the map $k : \mathbb{N} \ni m \mapsto k_m = h_1^{-m} h_2 h_1^m \in H$ is one-to-one. In fact if $k_m = k_l$ then $h_1^{l-m} h_2 = h_2 h_1^{l-m}$ and therefore h_2 and h_1^{l-m} should commute. By Proposition 3.5 we obtain that if $m \neq l$ then $\operatorname{Fix}(h_1) = \operatorname{Fix}(h_1^{l-m}) = \operatorname{Fix}(h_2)$ which contradicts the choice of h_1 and h_2.

Now we have

$$\begin{aligned} k_m = h_1^{-m} h_2 h_1^m &= \mu_{U_1^{-m}}\circ\delta_{t_1^{-m}}\delta_{t_2}\circ h_a\circ\mu_{U_2}\circ\delta_{t_1^m}\circ\mu_{U_1^m} \\ &= \delta_{t_2}\circ\mu_{U_1^{-m}}\circ\delta_{t_1^{-m}}\circ h_a\circ\delta_{t_1^m}\circ\mu_{U_2}\circ\mu_{U_1^m} = \delta_{t_2}\circ\mu_{U_1^{-m}}\circ h_{\delta_{t_1}^{-m}(a)}\circ\mu_{U_2}\circ\mu_{U_1^m}. \end{aligned}$$

As $\mathrm{U}(n-1)$ is compact we can choose a subsequence $l \mapsto m_l$ such that $l \mapsto U_1^{m_l}$ converges to $U_0 \in \mathrm{U}(n-1)$. Then $\lim_{l\to+\infty} k_{m_l}$ exists in $\operatorname{Aut}\mathbb{H}^n$ and is equal to $\delta_{t_2}\circ\mu_{U_0^{-1}}\circ\mu_{U_2}\circ\mu_{U_0}$ because $\delta_{t_1}^{-m}(a) \to 0$ when $m \to +\infty$ and $h_0 = \mathrm{id}_{\mathbb{H}^n}$.

As Γ is discrete, then H is discrete and therefore closed in $\operatorname{Aut}\mathbb{H}^n$; thus the existence of the sequence $l \mapsto k_{m_l}$ in H which converges (in $\operatorname{Aut}\mathbb{H}^n$ and therefore in H) gives the required contradiction. □

We now give an example of a generic subgroup $\Gamma \subset \operatorname{Aut}\mathbb{B}^2$ acting freely and properly discontinuously on $\mathbb{B}^2$ such that the automorphism group of the quotient $X = \mathbb{B}^2/\Gamma$ is not discrete. Since the fundamental group of X is isomorphic to Γ, this yields that the several dimensional situation is quite different from the one dimensional case where it is well known that if X is a Riemann surface with non-abelian fundamental group then id_X is isolated in $\operatorname{Hol}(X, X)$.

EXAMPLE 3.8. Let $\widehat{G}$ be a non-abelian subgroup of $\mathrm{U}(1,1)$ which does not contain multiples of I_2 different from I_2 and such that $\widehat{\Gamma} = \Psi(\widehat{G})$ acts freely and properly discontinuously on Δ (the existence of such a subgroup can be seen via hyperbolic geometrical tools). Notice that as a consequence of Proposition 2.5 the subgroup $\widehat{\Gamma}$ is generic.

Embed $\widehat{G}$ in $\mathrm{U}(2,1)$ via $\varepsilon : \widehat{G} \ni \widehat{g} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & \widehat{g} \end{pmatrix} \in \mathrm{U}(2,1)$ and denote by G the image $\varepsilon(\widehat{G})$ and by Γ the subgroup $\Psi(G) \subset \operatorname{Aut}\mathbb{B}^2$. We claim that Γ is generic and acts freely and properly discontinuously on $\mathbb{B}^2$.

Indeed for any $\widehat{g} \in \widehat{G} \setminus \{I_2\}$ the eigenvectors of $\varepsilon(\widehat{g})$ are necessarily of the form $v = \binom{v_1}{v'}$ where $v' \in \mathbb{C}^2$ is an eigenvector of $\widehat{g}$ (or is equal to zero). Since $\widehat{G}$ contains no elliptic elements, then $\langle v'|v'\rangle_{(1,1)} = 0$ and hence $\langle v|v\rangle_{(2,1)} = |v_1|^2 \geq 0$. This implies that there are no eigenvectors of $\varepsilon(\widehat{g})$ with negative form (and hence Γ contains no elliptic elements) and that the fixed points set of $\Psi_{\varepsilon(\widehat{g})}$ in $\overline{\mathbb{B}^2}$ is given by $\{0\} \times \operatorname{Fix}(\Psi_{\widehat{g}})$, which implies that also Γ is generic. In particular Corollary 3.6 implies that Γ is not abelian.

Now we show that Γ acts properly discontinuously on $\mathbb{B}^2$. Fix $p = (p_1, p_2) \in \mathbb{B}^2$, since $\widehat{\Gamma}$ acts properly discontinuously on Δ we can find a compact neighborhood $\widehat{V}$ of p_2 in Δ such that $\widehat{\gamma}(V) \cap V = \emptyset$ for any $\widehat{\gamma} \in \widehat{\Gamma} \setminus \{\mathrm{id}_\Delta\}$. Choose a compact neighborhood $V \subset \mathbb{C} \times \widehat{V}$ of p in $\mathbb{B}^2$. If $w = (w_1, w_2) \in \Psi_g(V) \cap V$ for some $g = \varepsilon(\widehat{g}) \in G$, then as the second component of $\Psi_g(w)$ is equal to $\Psi_{\widehat{g}}(w_2)$ and $V \subset \mathbb{C} \times \widehat{V}$ we obtain that $\Psi_{\widehat{g}} = \mathrm{id}_\Delta$. Then $\widehat{g} = I_2$ and therefore $\Psi_g = \mathrm{id}_{\mathbb{B}^2}$.

Now let $\theta : \mathbb{N} \to (0, 2\pi)$ be a sequence converging to 0: since the elliptic automorphisms $\gamma_n : \mathbb{B}^2 \ni z = (z_1, z_2) \mapsto (e^{i\theta_n} z_1, z_2) \in \mathbb{B}^2$ all belong to the normalizer of Γ in $\operatorname{Aut}\mathbb{B}^2$ and give raise to a sequence in $\operatorname{Aut} X \setminus \{\mathrm{id}_X\}$ converging to id_X, then we proved that the automorphism group of $X = \mathbb{B}^2/\Gamma$ is not discrete.

The above example motivates Definition 1.5: in fact the subgroup $\Gamma \subset \mathrm{U}(2,1)$ given in Example 3.8 is generic but is not completely generic, since $\mathcal{A}(\Gamma)$ is contained in $\{0\} \times \Delta$. (To be more precise $\mathcal{A}(\Gamma) = \{0\} \times \Delta$ since there exists a hyperbolic element $\widehat{g} \in \widehat{G}$ and the fixed points set of $\Psi_{\varepsilon(\widehat{g})}$ is equal to $\{0\} \times \operatorname{Fix}(\Psi_{\widehat{g}})$. Then the affine subset $\mathcal{A}(\Gamma)$ cannot be reduced to one point and therefore must have dimension at least 1. As $\mathcal{A}(\Gamma) \subset \{0\} \times \Delta$ we obtain the equality between $\mathcal{A}(\Gamma)$ and $\{0\} \times \Delta$.)

The following proposition will be used in the proof of the generalization of Hurwitz's theorem and says that a completely generic subgroup Γ which acts freely on $\mathbb{B}^n$ contains "a lot" of hyperbolic elements. In fact we prove that there exist a finite subset $\mathcal{E} \subset \Gamma$ which contains only hyperbolic elements and such that $\mathcal{A}(\mathcal{E}) = \mathbb{B}^n$.

PROPOSITION 3.9. *Let $\Gamma \subset \operatorname{Aut}\mathbb{B}^n$ be a completely generic subgroup acting freely on $\mathbb{B}^n$. Then there exist $k \le n$ and $\gamma_1, \dots, \gamma_k \in \Gamma$ hyperbolic such that*

$$\mathcal{A}(\gamma_1, \dots, \gamma_k) = \mathbb{B}^n.$$

PROOF. We prove by induction on d the following assertion:

(†) if $\dim \mathcal{A}(\Gamma) \ge d$ then there exist $k(d) \le d$ and $\gamma_1, \dots, \gamma_{k(d)} \in \Gamma$ hyperbolic and such that $\dim \mathcal{A}(\gamma_1, \dots, \gamma_{k(d)}) \ge d$.

The case $d = n$ gives the proof of the proposition.

If $d = 1$, Theorem 3.1 ensures that the group Γ cannot be parabolic since in this case all elements in Γ would have the same fixed point and therefore Γ would not be completely generic. Then there exist a hyperbolic element $\gamma \in \Gamma$ and setting $k(1) = 1$ and $\gamma_1 = \gamma$ this proves the assertion in the case $d = 1$.

Now suppose $\dim \mathcal{A}(\Gamma) \ge d+1$. The inductive hypothesis entails that there exists $\gamma_1, \dots, \gamma_{k(d)} \in \Gamma$ hyperbolic and such that $\dim \mathcal{A}(\gamma_1, \dots, \gamma_{k(d)}) \ge d$. If $\dim \mathcal{A}(\gamma_1, \dots, \gamma_{k(d)}) \ge d + 1$ setting $k(d + 1) = k(d)$ we are done.

If $\dim \mathcal{A}(\gamma_1, \dots, \gamma_{k(d)}) = d$ acting by conjugation on Γ with $\operatorname{Aut}\mathbb{B}^n$ we act on the fixed points sets of elements of Γ: since $\operatorname{Aut}\mathbb{B}^n$ maps affine subsets into affine subsets, we can suppose that $\mathcal{A}(\gamma_1, \dots, \gamma_{k(d)})$ contains the origin and, acting again with $\mathrm{U}(n)$ we can also suppose that $\mathcal{A}(\gamma_1, \dots, \gamma_{k(d)}) = \{0\} \times \mathbb{B}^d$.

If there exists an element $\gamma \in \Gamma$ hyperbolic and such that $\mathrm{Fix}(\gamma) \not\subset \{0\} \times \overline{\mathbb{B}^d}$ setting $k(d + 1) = k(d) + 1$ and $\gamma_{k(d+1)} = \gamma$ gives the proof of the inductive step.

Then we are left to prove the inductive step in the case when $\mathrm{Fix}(\gamma) \subset \{0\} \times \overline{\mathbb{B}^d}$ for any $\gamma \in \Gamma$ hyperbolic. Since $\dim \mathcal{A}(\Gamma) \ge d + 1$ there exists $\sigma \in \Gamma$ parabolic such that $\mathrm{Fix}(\sigma) \notin \{0\} \times \overline{\mathbb{B}^d}$. Now for any $j \in \{1, \dots, k(d)\}$ set $\widetilde{\gamma}_j = \sigma \circ \gamma_j \circ \sigma^{-1}$. It is easily seen that $\mathrm{Fix}(\widetilde{\gamma}_j) = \sigma(\mathrm{Fix}(\gamma_j))$. Since $\widetilde{\gamma}_j$ is hyperbolic for any $j = 1, \dots, k(d)$ then $\sigma(\mathrm{Fix}(\gamma_j)) \subset \{0\} \times \overline{\mathbb{B}^d}$ for any $j = 1, \dots, k(d)$. As $\mathcal{A}(\gamma_1, \dots, \gamma_{k(d)}) = \{0\} \times \mathbb{B}^d$ and $\operatorname{Aut}\mathbb{B}^n$ maps affine subsets into affine subsets, we then obtain that $\sigma(\{0\} \times \mathbb{B}^d) \subseteq \{0\} \times \mathbb{B}^d$ and hence, by continuity, we have that $\sigma(\{0\} \times \overline{\mathbb{B}^d}) \subseteq \{0\} \times \overline{\mathbb{B}^d}$. Then Brouwer's theorem implies that there exists a fixed point of σ in $\{0\} \times \overline{\mathbb{B}^d}$ and this contradicts the choice of σ, thus completing the proof of the assertion. □

The concept of completely generic subgroup is the one we need to prove the following generalization of Hurwitz's theorem for $n > 1$.

THEOREM 3.10. *Let X be a complex manifold covered by $\mathbb{B}^n$ and suppose $n > 1$. If the group of deck transformations of the covering is completely generic, then* id_X *is isolated in* $\mathrm{Hol}(X, X)$. *In particular* $\operatorname{Aut} X$ *is discrete.*

PROOF. We denote by $\Gamma \subset \operatorname{Aut}\mathbb{B}^n$ the group of deck transformations of the covering ($\mathbb{B}^n \xrightarrow{\chi} X$). Assume, by contradiction, that there exists a sequence $\{\varphi_\nu\} \subset \operatorname{Hol}(X, X)\setminus\{\operatorname{id}_X\}$ converging to id_X. We may choose $\{f_\nu\} \subset \operatorname{Hol}(\mathbb{B}^n, \mathbb{B}^n)$ to be a lifting of $\{\varphi_\nu\}$ so that $f_\nu \to \operatorname{id}_{\mathbb{B}^n}$. In fact choose $x_0 \in X$ and fix $z_0 \in \mathbb{B}^n$ such that $\chi(z_0) = x_0$. Since $\varphi_\nu(x_0) \to x_0$ we can choose f_ν such that $f_\nu(z_0) \to z_0$. In particular the sequence $\{f_\nu\}$ cannot have diverging subsequences. Let f be a holomorphic map from $\mathbb{B}^n$ to $\mathbb{C}^n$ which is a limit point of $\{f_\nu\}$; of course $f(z_0) = z_0$ and hence $f(\mathbb{B}^n) \subset \mathbb{B}^n$ because $\mathbb{B}^n$ is strictly convex. Moreover f is easily seen to be a lifting of id_X and therefore has to coincide with $\operatorname{id}_{\mathbb{B}^n}$ which is another lifting of id_X which agrees with f on z_0. Then the unique limit point of the sequence $\{f_\nu\}$ is $\operatorname{id}_{\mathbb{B}^n}$ and therefore it converges to $\operatorname{id}_{\mathbb{B}^n}$.

For any $\gamma \in \Gamma$ and $\nu \in \mathbb{N}$ we have $\chi\circ f_\nu\circ\gamma = \varphi_\nu\circ\chi\circ\gamma = \varphi_\nu\circ\chi = \chi\circ f_\nu$ and hence there exists $\alpha(\gamma, \nu) \in \Gamma$ such that $f_\nu\circ\gamma = \alpha(\gamma, \nu)\circ f_\nu$. Since f_ν converges to $\operatorname{id}_{\mathbb{B}^n}$ and Γ acts properly discontinuously on $\mathbb{B}^n$ then eventually $\alpha(\gamma, \nu) = \gamma$ for any $\gamma \in \Gamma$ and hence

$$f_\nu\circ\gamma = \gamma\circ f_\nu \tag{3.2}$$

for any $\gamma \in \Gamma$ and $\nu \geq \nu(\gamma)$. Choose $\gamma_1, \dots, \gamma_k$ given by Proposition 3.9 and set $\nu_0 = \max\{\nu(\gamma_j) : j = 1, \dots, k\}$. Up to a subsequence we can suppose that equation (3.2) holds for any $\nu \in \mathbb{N}$ and $\gamma \in \{\gamma_j : j = 1, \dots, k\}$. Now if f_ν has a fixed point in $\mathbb{B}^n$, then by Proposition 2.10 the fixed points set of f_ν is an affine subset of $\mathbb{B}^n$ which is obtained intersecting with $\mathbb{B}^n$ the complex affine subset $\mathcal{A}(\gamma_1, \dots, \gamma_k)$ and therefore $f_\nu = \operatorname{id}_{\mathbb{B}^n}$. This is a contradiction to the fact that $\varphi_\nu \neq \operatorname{id}_X$ for all $\nu \in \mathbb{N}$ and hence f_ν cannot have a fixed point for any $\nu \in \mathbb{N}$.

Since the subgroup Γ is completely generic and $n \geq 2$, then Γ is generic and therefore Proposition 3.7 implies that there exists $\widetilde{\gamma}_1, \widetilde{\gamma}_2 \in \Gamma$ which are both hyperbolic and such that $\operatorname{Fix}(\widetilde{\gamma}_1) \cap \operatorname{Fix}(\widetilde{\gamma}_2) = \emptyset$. As above, up to a subsequence, we can suppose that equation (3.2) holds for any $\nu \in \mathbb{N}$ and $\gamma \in \{\widetilde{\gamma}_j : j = 1, 2\}$.

Now choose $\nu \in \mathbb{N}$. Since f_ν has no fixed points in $\mathbb{B}^n$, then the iterates of f_ν converge to the Wolff point ω_ν of f_ν (for a definition of Wolff point and related topics, see *e.g.* [12]). This yields that ω_ν is a fixed point of $\widetilde{\gamma}_j$ for any $= 1, 2$ (in fact the iterates of f_ν converge to ω_ν and commute with $\widetilde{\gamma}_j$ and hence $\widetilde{\gamma}_j(\omega_\nu) = \omega_\nu$ for $j = 1, 2$) and this gives the required contradiction because $\omega_\nu \in \operatorname{Fix}(\widetilde{\gamma}_j)$ for any $j = 1, 2$. □

The above result has several corollaries which generalize the ones already obtained by several authors in the one-dimensional situation. Before we can state and prove them, we need a better knowledge of the structure of the automorphisms group of a complex manifold covered by $\mathbb{B}^n$.

THEOREM 3.11. *Let X be a complex manifold covered by $\mathbb{B}^n$. Then* $\operatorname{Aut} X$ *is closed in* $\operatorname{Hol}(X, X)$ *and the isotropy group of any point in X is compact.*

The assertion is immediately obtained since X is complete hyperbolic (because it is covered by $\mathbb{B}^n$ which is complete hyperbolic), therefore taut and applying the well known results on taut manifolds (see *e.g.* [1] for a wide review on taut manifolds).

The first consequence of Theorem 3.10 is a generalization of the so called Klein-Poincaré theorem.

COROLLARY 3.12. *Let X be a complex manifold covered by $\mathbb{B}^n$ and suppose $n > 1$. If the group of deck transformations of the covering is completely generic, then* Aut X *acts properly discontinuously on X.*

PROOF. If Aut X does not act properly discontinuously on $\mathbb{B}^n$ then there exists a point $x_0 \in X$, an infinite sequence $\{\gamma_\nu\}$ of distinct elements in Aut X and a sequence $\{x_\nu\}$ in X such that $\gamma_\nu(x_\nu)$ converges to x_0. This implies that there exist no compactly divergent subsequences in $\{\gamma_\nu\}$ and therefore we can suppose, up to a subsequence, that γ_ν converges to $f \in \mathrm{Hol}(X, X)$. Theorem 3.11 entails that $f \in \mathrm{Aut}\, X$ and then Aut X is not discrete against Theorem 3.10. □

Theorem 3.10 also implies that the group Aut X is at most countable.

COROLLARY 3.13. *Let X be a complex manifold covered by $\mathbb{B}^n$ and suppose $n > 1$. If the group of deck transformations of the covering is completely generic, then* Aut X *is at most countable.*

PROOF. We denote by $\omega_X : X \times X \to \mathbb{R}^+$ the Kobayashi distance on X and fix a point $x_0 \in X$. Assume by contradiction Aut X is uncountable and consider the function

$$\mu : \mathrm{Aut}\, X \ni \gamma \mapsto \omega_X(x_0, \gamma(x_0)) \in \mathbb{R}^+.$$

Since Aut X is uncountable we can find a sequence $\{\gamma_\nu\}$ of distinct elements in Aut X such that $\{\mu(\gamma_\nu)\}$ is bounded in $\mathbb{R}$. Then the sequence $\{\gamma_\nu\}$ cannot have compactly divergent subsequences because X is complete hyperbolic. Hence Montel's theorem entails that a subsequence $\{\gamma_{\nu_l}\}$ converges to $f \in \mathrm{Hol}(X, X)$ and Theorem 3.11 implies that $f \in \mathrm{Aut}\, X$, which contradicts Theorem 3.10 and ends the proof of the assertion. □

At last we consider the case of a compact complex manifold which can be seen as the announced generalization of Hurwitz's theorem.

COROLLARY 3.14. *Let X be a compact manifold covered by $\mathbb{B}^n$ and suppose $n > 1$. If the group of deck transformations of the covering is completely generic, then* Aut X *is finite.*

PROOF. The hypothesis implies that Aut X is discrete. Since X is compact there are no compactly divergent sequences in $\mathrm{Hol}(X, X)$ and therefore Montel's theorem and Theorem 3.11 entail that Aut X is compact, and hence finite. □

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