

The Calderón-Zygmund Theorem and Parabolic Equations in $L_p(\mathbb{R}, C^{2+\alpha})$ -Spaces

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Abstract. A Banach-space version of the Calderón-Zygmund theorem is presented and applied to obtaining apriori estimates for solutions of second-order parabolic equations in $L_p(\mathbb{R}, C^{2+\alpha})$ -spaces.

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1. – Introduction

In [8] the author proved the solvability of second-order parabolic equations in the spaces $L_p(\mathbb{R}, C^{2+\alpha})$, $p \in (1, \infty]$, an advantage of what is that one need not assume any regularity of coefficients with respect to time and yet get $C^{2+\alpha}$ regularity in space if the data are in C^α with respect to space variables. The method of obtaining this result is quite elementary and differs from the ones commonly used to construct the L_p -theory of solvability of parabolic equations. One of those methods is based on applying an appropriate version of the Calderón-Zygmund theorem (see [2] for “constant” coefficients in a general setting and [7] for second-order parabolic equations with measurable coefficients depending only on time). The natural question arises as to if the $L_p(\mathbb{R}, C^{2+\alpha})$ -theory can also be based on the Calderón-Zygmund theorem. Here we show that the answer to this question is positive up to that one needs to use a Banach space version of this theorem. The version we have in mind is a slight extension of the one from [7], which is used there to obtain $L_q(\mathbb{R}, W_p^2)$ -estimates for solutions of second-order parabolic equations. Thus we show that the Calderón-Zygmund theorem provides a unified approach to L_p -, $L_q(\mathbb{R}, W_p^2)$ -, and $L_p(\mathbb{R}, C^{2+\alpha})$ -theories for second-order parabolic equations.

One may think that all useful kinds of Banach space versions of the Calderón-Zygmund theorem are known in the literature. Surprisingly, this is

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not the case and we refer the reader to [7] for the discussion of the issue. In particular, in the present article we need a more general version of this theorem than the one from [7]. We explain why we need such a generalization below and in Section 2, where we prove it.

For $p = \infty$, parabolic equations in $L_p(\mathbb{R}, C^{2+\alpha}(\mathbb{R}^d))$ spaces received some attention after Brandt [1] discovered that a priori estimates exist even if the coefficients of the equation are only measurable in time. Further references related to this direction can be found in [9]. Our main idea is to use a Banach space version of the Calderón-Zygmund theorem in order to interpolate between $L_\infty(\mathbb{R}, C^{2+\alpha}(\mathbb{R}^d))$ and weak-type (1,1) spaces and precisely the fact that on one end we have $p = \infty$ does not allow us to use the Calderón-Zygmund theorem from [7].

The article is organized as follows. As we have mentioned above Section 2 contains the version of the Calderón-Zygmund theorem we need with p allowed to take the infinite value. In Section 3 we present our main results on parabolic equations, Theorems 3.1 and 3.3 and their discussion. In this section we also prove Theorem 3.1. The proof of Theorem 3.3 is deferred until Section 5. In short Section 4 we collect some auxiliary results which should be considered as well known. We give them with proofs for completeness and in part in order to rehabilitate the potential based approach to Schauder estimates in the whole space, as opposed to maximum principle approach from [1] and [3] or interpolation based approach from [10] and [9]. The other approaches have considerable advantages in many other situations. However, in our case the shortest way is to use simple properties of the heat potentials. By the way, there is also Safonov's approach (see [5]), which is equally applicable to both linear and fully nonlinear equations.

Several words about notation. As usual \mathbb{R}^d stand for the d -dimensional Euclidean space of points $x = (x^1, \dots, x^d)$,

$$Du = \text{grad } u = u_x, \quad D_i u = \frac{\partial}{\partial x^i} u, \quad D_{ij\dots} = D_i D_j \dots,$$

$$D^2 u = (D_{ij} u), \quad D_t u = \frac{\partial}{\partial t} u.$$

The spaces C_p^α and $C_p^{2+\alpha}$ are introduced in Remark 3.2.

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2. – A version of the Calderón-Zygmund theorem

First we remind some well known definitions (their almost identical versions can be found, for instance, in [7]).

DEFINITION 2.1. Let $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$, $(\mathbb{Q}_n, n \in \mathbb{Z})$ be a sequence of partitions of \mathbb{R}^d each consisting of disjoint Borel subsets $Q \in \mathbb{Q}_n$ such that, for each n ,

$$R_n := \sup_{Q \in \mathbb{Q}_n} \text{diam } Q < \infty.$$

We call it a *filtration of partitions* if

- (i) the partitions become finer as n increases:

$$\inf_{Q \in \mathbb{Q}_n} |Q| \rightarrow \infty \text{ as } n \rightarrow -\infty, \quad R_n \rightarrow 0 \text{ as } n \rightarrow \infty,$$

- (ii) the partitions are nested: for each n and $Q \in \mathbb{Q}_n$ there is a (unique) $Q' \in \mathbb{Q}_{n-1}$ such that $Q \subset Q'$,
- (iii) the following regularity property holds: for Q and Q' as in (ii) we have $|Q'| \leq N_0|Q|$, where N_0 is a constant independent of n, Q, Q' .

Denote

$$\int_Q f(x) dx = \frac{1}{|Q|} \int_Q f(x) dx.$$

For any $x \in \mathbb{R}^d$ and $n \in \mathbb{Z}$, by $Q_n(x)$ we denote the (unique) $Q \in \mathbb{Q}_n$ containing x . For a function $f \in L_{1,\text{loc}}(\mathbb{R}^d, F)$ and $n \in \mathbb{Z}$, we denote

$$(2.1) \quad f_n(x) = \int_{Q_n(x)} f(y) dy.$$

If F and G are Banach spaces, by $L(F, G)$ we denote the space of bounded linear operators from F to G . Let F be a Banach space and D be a domain in \mathbb{R}^d . By $C_0^\infty(D, F)$ we mean the space of infinitely differentiable F -valued functions on D with compact support. If $p \in [1, \infty)$, by $L_p(D, F)$ we mean the closure of $C_0^\infty(D, F)$ in the norm

$$\|u\|_{L_p(D, F)} = \left(\int_D |u(x)|_F^p dx \right)^{1/p}.$$

If $F = \mathbb{R}$ we write $L_p(D)$ instead of $L_p(D, \mathbb{R})$ and $C_0^\infty(D)$ instead of $C_0^\infty(D, \mathbb{R})$. The space $L_{p,\text{loc}}(D, F)$ is defined as the set of all functions u such that $u\zeta \in L_{p,\text{loc}}(D, F)$ for any $\zeta \in C_0^\infty(D)$. Formally, each element of $L_p(D, F)$ is a sequence of elements of $C_0^\infty(D, F)$ which is a Cauchy sequence with respect to the norm $\|\cdot\|_{L_p(D, F)}$. Such sequences converge in measure to some F -valued functions, which are as usual identified with the sequences and we treat $L_p(D, F)$ as the set of functions rather than the set of sequences. Of course,

each $\|\cdot\|_{L_p(D,F)}$ -Cauchy sequence has many limit functions in the sense of convergence in measure and every two of them are equal almost everywhere. Again as usual we identify two functions in $L_p(D, F)$ if they are equal almost everywhere. Finally, $L_\infty(D, F)$ is defined as the Banach space of all functions $u \in L_{1,\text{loc}}(D, F)$ with norm

$$\|u\|_{L_\infty(D,F)} = \operatorname{ess\,sup}_{x \in D} |u(x)|_F.$$

As above we identify two functions in $L_\infty(D, F)$ if they are equal almost everywhere.

Now we state the main assumptions of this section. Let F and G be Banach spaces, $p \in (1, \infty]$.

ASSUMPTION 2.1. We are given a (nonlinear) operator

$$\mathcal{A} : L_p(\mathbb{R}^d, F) \rightarrow L_p(\mathbb{R}^d, G),$$

which is subadditive and bounded meaning that, for a constant $N < \infty$ and any $n = 1, 2, \dots$ and $u, u_i \in L_p(\mathbb{R}^d, F)$, $i = 1, \dots, n$, we have

$$(2.2) \quad \left| \mathcal{A} \left(\sum_{i=1}^n u_i(x) \right) \right|_G \leq \sum_{i=1}^n |Au_i(x)|_G \quad (\text{a.e.}),$$

$$(2.3) \quad \|\mathcal{A}u\|_{L_p(\mathbb{R}^d, G)} \leq N \|u\|_{L_p(\mathbb{R}^d, F)}.$$

The least constant satisfying (2.3) for all $u \in L_p(\mathbb{R}^d, F)$ is denoted by $M_p(\mathcal{A})$.

REMARK 2.2. One is right thinking that the main case is when \mathcal{A} is a bounded linear operator. However, in our application to $L_p(C^{2+\alpha})$ -theory we introduce $\mathcal{A}u$ as the C^α -norm of a C^α -valued function $\mathcal{A}_\lambda^{ij}u$, where \mathcal{A}_λ^{ij} is a linear operator, in which case $G = \mathbb{R}$ and $F = C^\alpha(\mathbb{R}^d)$. This is done just to avoid quite hard issue of strong measurability of C^α -valued functions and to replace it with the issue of measurability of their C^α -norm.

ASSUMPTION 2.2. The operator \mathcal{A} possesses the Calderón-Zygmund cancellation property with respect to a filtration $(\mathbb{Q}_n, n \in \mathbb{Z})$ of partitions, which is to say that there are constants $s \in [1, p)$, $N < \infty$, and for each $Q \in \cup_n \mathbb{Q}_n$ there is a Borel set Q^* such that

- (i) $|Q^*| \leq N|Q|$ for each $Q \in \cup_n \mathbb{Q}_n$ and
- (ii) for each $f \in C_0^\infty(\mathbb{R}^d, F)$, $n \in \mathbb{Z}$, and $Q \in \mathbb{Q}_n$ we have

$$(2.4) \quad \int_{\mathbb{R}^d \setminus Q^*} |\mathcal{A}(I_Q(f - f|_n))(x)|_G dx \leq N|Q| \left(\int_Q |f(x)|_F^s dx \right)^{1/s}.$$

REMARK 2.3. Let \mathcal{A} be a linear operator and the spaces F and G be not too bad, say separable and reflexive, so that the adjoint \mathcal{A}' to \mathcal{A} is an operator from $L_{p'}(\mathbb{R}^d, G')$ to $L_{p'}(\mathbb{R}^d, F')$, where $p' = p/(p - 1)$. Assume that, for any bounded function $g \in L_{p'}(\mathbb{R}^d, G')$, we have $\mathcal{A}'g \in \text{BMO}(\mathbb{R}^d, F')$ and furthermore

$$(2.5) \quad \|\mathcal{A}'g\|_{\text{BMO}(\mathbb{R}^d, F')} \leq N \|g\|_{L_\infty(\mathbb{R}^d, G')},$$

where N is independent of g . Then it turns out that Assumption 2.2 is satisfied with any $s \in (1, \infty)$.

Indeed, by denoting $\langle \cdot, \cdot \rangle$ the duality between F and F' and meaning by the sup below the supremum over all bounded $g \in L_{p'}(\mathbb{R}^d, G')$ with $\|g\|_{L_\infty(\mathbb{R}^d, G')} \leq 1$, we write the left-hand side of (2.4) as

$$\begin{aligned} \sup \int_{\mathbb{R}^d} \langle \mathcal{A}'(gI_{\mathbb{R}^d \setminus Q^*}), I_Q(f - f_n) \rangle dx &\leq \sup \int_{\mathbb{R}^d} \langle \mathcal{A}'g, I_Q(f - f_n) \rangle dx \\ &= \sup \int_Q \langle \mathcal{A}'g, f - f_n \rangle dx \\ &= \sup \int_Q \langle \mathcal{A}'g - (\mathcal{A}'g)_n, f \rangle dx, \end{aligned}$$

where in the last equality we have used the fact that $f_n|_Q$ and $(\mathcal{A}'g)_n$ are constant on $Q \in \mathbb{Q}_n$. Hence, by Hölder's inequality, the left-hand side of (2.4) is not greater than

$$|Q| \left(\int_Q |f|_F^s dx \right)^{1/s} \sup \left(\int_Q |\mathcal{A}'g - (\mathcal{A}'g)_n|_{F'}^{s'} dx \right)^{1/s'}$$

where $s' = s/(s - 1)$. It only remains to observe that the last sup is a finite constant due to (2.5) (and the fact that the BMO norm can be defined on the basis of any power of summability, see [11]).

This remark and Theorem 2.5 build the bridge between Stampacchia's interpolation theorem and the Calderón-Zygmund theorem and show that in many cases, say of separable reflexive Banach spaces, the former follows from the properly stated latter one.

We also see that instead of (2.5) we could use a weaker although more cumbersome condition:

$$\sup_{n \in \mathbb{Z}} \sup_{Q \in \mathbb{Q}_n} \sup \left(\int_Q |\mathcal{A}'(gI_{\mathbb{R}^d \setminus Q^*}) - (\mathcal{A}'(gI_{\mathbb{R}^d \setminus Q^*}))_n|_{F'}^{s'} dx \right)^{1/s'} < \infty.$$

ASSUMPTION 2.3. The operator \mathcal{A} is lower semicontinuous in the following sense: If $\alpha, \alpha_1, \alpha_2, \dots \in L_p(\mathbb{R}^d, F)$, α and all α_m vanish outside of the same ball, the norms $\|\alpha_m\|_{L_\infty(\mathbb{R}^d, F)}$ are bounded with respect to m , and $|\alpha(x) -$

$\alpha_m(x)|_F \rightarrow 0$ at each $x \in \mathbb{R}^d$, then there is a subsequence $m(n)$ such that $m(n) \rightarrow \infty$ as $n \rightarrow \infty$ and

$$|\mathcal{A}\alpha(x)|_G \leq \varliminf_{n \rightarrow \infty} |\mathcal{A}\alpha_{m(n)}(x)|_G \quad (\text{a.e.}).$$

REMARK 2.4. Assumption 2.3 is automatically satisfied if $p \neq \infty$, Assumption 2.1 holds, and $|\mathcal{A}u - \mathcal{A}v|_G \leq |\mathcal{A}(u - v)|_G$ (a.e.) for any $u, v \in L_p(\mathbb{R}^d, F)$. Indeed, due to the dominated convergence theorem, we have $\|\alpha - \alpha_m\|_{L_p(\mathbb{R}^d, F)} \rightarrow 0$, $\|\mathcal{A}\alpha - \mathcal{A}\alpha_m\|_{L_p(\mathbb{R}^d, G)} \rightarrow 0$, and $\mathcal{A}\alpha_{m(n)} \rightarrow \mathcal{A}\alpha$ (a.e.) for a subsequence $m(n)$.

THEOREM 2.5. *Under Assumptions 2.1, 2.2, and 2.3 the operator \mathcal{A} is of weak-type (s, s) on smooth functions with compact support, that is there exists a constant N such that, for any $f \in C_0^\infty(\mathbb{R}^d, F)$ and $\lambda > 0$,*

$$(2.6) \quad |\{x : |\mathcal{A}f(x)|_G > \lambda\}| \leq N\lambda^{-s} \int_{\mathbb{R}^d} |f(x)|_F^s dx.$$

Furthermore, for any $q \in (s, p)$, there is a constant $N < \infty$ such that for any $f \in C_0^\infty(\mathbb{R}^d, F)$ we have

$$(2.7) \quad \|\mathcal{A}f\|_{L_q(\mathbb{R}^d, G)} \leq N\|f\|_{L_q(\mathbb{R}^d, F)}.$$

Finally, the constants N in (2.6) and (2.7) can be chosen to depend only on q , $M_p(\mathcal{A})$, the constant N from Assumption 2.2, and N_0 associated with $(\mathbb{Q}_n, n \in \mathbb{Z})$ from Definition 2.1,

PROOF. The main idea of what follows is, of course, the same as in the classical proof (see, for instance, [11]). By passing from $\mathcal{A}f(x)$ to $|\mathcal{A}f(x)|_G$ we see that without losing generality we may assume that $G = \mathbb{R}$. Actually, this assumption only helps shorten writing. Next, owing to the Marcinkiewicz interpolation theorem, (2.3) and (2.6) imply (2.7). Therefore, it suffices to prove the first assertion of the theorem and the assertion regarding the dependence of N in (2.6) on the data.

For simplicity of notation we may and will assume that $M_p(\mathcal{A}) \leq 1$ and first observe a general fact that if, for a particular λ , we have $|f(x)|_F \leq \lambda$ for all x , then (2.6) holds for this same λ . Indeed, if $p = \infty$, then $\|\mathcal{A}f\|_{L_\infty(\mathbb{R}^d)} \leq \lambda$ and the left-hand side of (2.6) is zero, and if $p < \infty$, then by Chebyshev's inequality

$$(2.8) \quad \begin{aligned} |\{x : |\mathcal{A}f(x)| > \lambda\}| &\leq \lambda^{-p} \int_{\mathbb{R}^d} |\mathcal{A}f(x)|^p dx \leq \lambda^{-p} \int_{\mathbb{R}^d} |f(x)|_F^p dx \\ &\leq \lambda^{-p} \int_{\mathbb{R}^d} |f(x)|_F^s \lambda^{p-s} dx = \lambda^{-s} \int_{\mathbb{R}^d} |f(x)|_F^s dx. \end{aligned}$$

Next we take $f \in C_0^\infty(\mathbb{R}^d, F)$, fix a $\lambda > 0$ and prove (2.6) for this particular λ . Let \mathbb{Q}_n be the filtration of partitions from Assumption 2.2. Let $f|_n$ be taken from (2.1), $g := |f|_F$, and

$$\tau(x) = \inf\{n : g|_n(x) \geq \lambda/(2N_0)\} \quad (\inf \emptyset := \infty).$$

Since $\inf_{Q \in \mathbb{Q}_n} |Q| \rightarrow \infty$ as $n \rightarrow -\infty$, we have that τ is bounded from below, say by m . It follows that the set $\{x : \tau(x) < \infty\}$ lies in the R_m -neighborhood of the support of f . In particular, the set $\{x : \tau(x) < \infty\}$ is bounded. By Lemma 1.6 of [7], τ is a stopping time (that is, for any $n \in \mathbb{Z}$, the set $\{x : \tau(x) = n\}$ is the disjoint union of some elements of \mathbb{Q}_n) and

$$\begin{aligned} |\{x : \tau(x) < \infty\}| &= |\{x : g|_{\tau(x)}(x) I_{\tau(x) < \infty} \geq \lambda/(2N_0)\}| \\ &\leq N\lambda^{-s} \int_{\mathbb{R}^d} g|_{\tau}^s I_{\tau < \infty} dx \leq N\lambda^{-s} \int_{\mathbb{R}^d} g^s dx = N\lambda^{-s} \int_{\mathbb{R}^d} |f|_F^s dx. \end{aligned}$$

Now, observe that $f = \alpha + \beta$, where

$$\alpha = (f - f|_{\tau}) I_{\tau < \infty}, \quad \beta = f|_{\tau} \quad (= f I_{\tau = \infty} + f|_{\tau} I_{\tau < \infty}).$$

Also notice that $f|_n \rightarrow f$ as $n \rightarrow \infty$ since $\sup_{Q \in \mathbb{Q}_n} \text{diam } Q \rightarrow 0$ and f is continuous. It follows that $|f I_{\tau = \infty}|_F \leq \lambda/(2N_0)$. Next, Lemma 1.6 of [7] implies that $\beta \in L_p(\mathbb{R}^d, F)$, $|\beta|_F \leq \lambda/2$,

$$\int_{\mathbb{R}^d} |\beta(x)|_F^s dx \leq \int_{\mathbb{R}^d} |f(x)|_F^s dx,$$

and due to (2.8)

$$|\{x : |\mathcal{A}\beta(x)| > \lambda/2\}| \leq N\lambda^{-s} \int_{\mathbb{R}^d} |\beta(x)|_F^s dx \leq N\lambda^{-s} \int_{\mathbb{R}^d} |f(x)|_F^s dx.$$

Upon combining this with $|\mathcal{A}f| \leq |\mathcal{A}\alpha| + |\mathcal{A}\beta|$ and

$$\{x : |\mathcal{A}f(x)| > \lambda\} \subset \{x : |\mathcal{A}\alpha(x)| > \lambda/2\} \cup \{x : |\mathcal{A}\beta(x)| > \lambda/2\},$$

we see that it only remains to prove

$$(2.9) \quad |\{x : |\mathcal{A}\alpha(x)| > \lambda/2\}| \leq N\lambda^{-s} \int_{\mathbb{R}^d} |f(x)|_F^s dx.$$

To prove (2.9) notice that the set $\{\tau < \infty\}$ is the disjoint union of $\{\tau = n\}$ and each of those sets is the disjoint union of elements of \mathbb{Q}_n . Therefore, there exist disjoint $Q_{in} \in \mathbb{Q}_n$, $i = 1, 2, \dots, n \in \mathbb{Z}$, such that

$$\{x : \tau(x) = n\} = \bigcup_i Q_{in}, \quad \{x : \tau(x) < \infty\} = \bigcup_{i,n} Q_{in}.$$

Define

$$S = \bigcup_{i,n} Q_{in}^*$$

and notice that

$$(2.10) \quad |S| \leq \sum_{i,n} |Q_{in}^*| \leq N \sum_{i,n} |Q_{in}| = N|\{\tau < \infty\}| \leq N\lambda^{-s} \int_{\mathbb{R}^d} |f(x)|_F^s dx.$$

Hence

$$(2.11) \quad |\{x : |\mathcal{A}\alpha(x)| > \lambda/2\}| \leq I + N\lambda^{-s} \int_{\mathbb{R}^d} |f(x)|_F^s dx,$$

where

$$(2.12) \quad I := |\{x \notin S : |\mathcal{A}\alpha(x)| > \lambda/2\}| \leq (2/\lambda) \int_{S^c} |\mathcal{A}\alpha(x)| dx := (2/\lambda)J.$$

Now notice that the function $\alpha = (f - f_{|\tau})I_{\tau < \infty}$ is bounded with bounded support. Also, for

$$D_k := \bigcup_{i,|n| \leq k} Q_{in} \quad \text{and} \quad \alpha_k := (f - f_{|\tau})I_{D_k} = \sum_{i,|n| \leq k} (f - f_n)I_{Q_{in}},$$

we have that α_k are uniformly bounded, $\alpha = \lim_{k \rightarrow \infty} \alpha_k$ pointwise, and all α_k vanish outside of the bounded set $\{x : \tau(x) < \infty\}$. By Assumption 2.3 and by Fatou's theorem we have

$$J = \int_{S^c} |\mathcal{A}\alpha(x)| dx \leq \overline{\lim}_{k \rightarrow \infty} \int_{S^c} |\mathcal{A}\alpha_k(x)| dx,$$

which by adding the subadditivity (2.2) from Assumption 2.1 yields that J is less than

$$\overline{\lim}_{k \rightarrow \infty} \int_{S^c} \sum_{i,|n| \leq k} |\mathcal{A}[(f - f_n)I_{Q_{in}}](x)| dx \leq \sum_{i,n} \int_{S^c} |\mathcal{A}[(f - f_n)I_{Q_{in}}](x)| dx.$$

The last integrals become larger if we replace S^c with $\mathbb{R}^d \setminus Q_{in}^*$ and by using the cancellation property (2.4) from Assumption 2.2 we see that

$$J \leq \sum_{i,n} |Q_{in}|^{1-1/s} \left(\int_{Q_{in}} |f(x)|_F^s dx \right)^{1/s}.$$

Next we apply Hölder’s inequality and use (2.10) again to get that

$$\begin{aligned}
 J &\leq \left(\sum_{i,n} |Q_{in}| \right)^{1-1/s} \left(\sum_{i,n} \int_{Q_{in}} |f(x)|_F^s dx \right)^{1/s} \\
 &\leq N\lambda^{-s(1-1/s)} \left(\int_{\mathbb{R}^d} |f(x)|_F^s dx \right)^{1-1/s} \left(\int_{\mathbb{R}^d} |f(x)|_F^s dx \right)^{1/s}.
 \end{aligned}$$

Upon collecting like terms and going back to (2.12) and (2.11) we see that (2.9) holds indeed and the theorem is proved.

The cancellation property (2.4) is the heart of the theorem. Usually for $s = 1$, this property is checked out on the basis of properties of kernels associated with \mathcal{A} .

DEFINITION 2.6. Let F and H be Banach spaces and \mathbb{Q}_n be a filtration of partitions. For each $x, y \in \mathbb{R}^d, x \neq y$, let a $K(x, y) \in L(F, H)$ be defined. We say that K is an $L(F, H)$ -valued Calderón-Zygmund kernel relative to \mathbb{Q}_n if

- (i) for any x and $r > 0, K(x, \cdot) \in L_{1,\text{loc}}(B_r^c(x), L(F, H))$, where $B_r^c(x) = \{y \in \mathbb{R}^d : |x - y| \geq r\}$,
- (ii) the function $|K(x, y) - K(x, z)|_{L(F, H)}$ is measurable as a function of $(x, y, z) \in \mathbb{R}^{3d} \cap \{(x, y, z) : x \neq z, x \neq y\}$,
- (iii) there is a constant $N_0 \geq 1$ and, for each $Q \in \cup_n \mathbb{Q}_n$, there is a Borel set Q^* such that $\bar{Q} \subset Q^*, |Q^*| \leq N_0|Q|$, and

$$(2.13) \quad \int_{\mathbb{R}^d \setminus Q^*} |K(x, y) - K(x, z)|_{L(F, H)} dx \leq N_0,$$

whenever $y, z \in Q$.

REMARK 2.7. Actually, instead of (2.13) one always uses another inequality

$$(2.14) \quad \int_{\mathbb{R}^d \setminus Q^*} |K(x, y) - K|_n(x, y)|_{L(F, H)} dx \leq N_0,$$

which holds for any $n, Q \in \mathbb{Q}_n, y \in Q$, where

$$K|_n(x, y) := \int_{Q_n(y)} K(x, z) dz \quad (x \notin \bar{Q}_n(y)).$$

Inequality (2.14) follows from (2.13) because

$$K(x, y) - K|_n(x, y) = \int_{Q_n(y)} (K(x, y) - K(x, z)) dz.$$

REMARK 2.8 (see Example 3.4 of [7]). Let K satisfy conditions (i) and (ii) of Definition 2.6 and \mathbb{Q}_n be the filtration of dyadic cubes. Assume that $K(x, y)$ is weakly differentiable in y for $y \neq x$ and $|K_{y^i}(x, y)|_{L(F,H)} \leq N_1\phi(|x - y|)$ for all $x \neq y, i$ with a constant N_1 independent of x, y, i and a function ϕ satisfying

$$(2.15) \quad r \int_r^\infty s^{d-1} \phi(s) ds \leq N_1$$

for all $r > 0$. Then K is a Calderón-Zygmund kernel relative to the filtration of dyadic cubes with constant N_0 in (2.13) depending only on N_1 and d .

THEOREM 2.9. *Let Assumption 2.1 be satisfied. Also assume that for each $f \in C_0^\infty(\mathbb{R}^d, F)$, for almost any x outside of the closed support of f we have*

$$(2.16) \quad |\mathcal{A}f(x)|_G \leq \left| \int_{\mathbb{R}^d} K(x, y)f(y) dy \right|_H,$$

where $K(x, y)$ is an $L(F, H)$ -valued Calderón-Zygmund kernel relative to a filtration of partitions. Finally, if $p = \infty$, replace Assumption 2.3 with a stronger condition which is obtained by substituting the words “ $|\alpha(x) - \alpha_m(x)|_F \rightarrow 0$ at almost each $x \in \mathbb{R}^d$ ” instead of “ $|\alpha(x) - \alpha_m(x)|_F \rightarrow 0$ at each $x \in \mathbb{R}^d$ ” in Assumption 2.3. Then all the assumptions of Theorem 2.5 are satisfied and its assertions continue to hold with $s = 1$.

PROOF. We only need to check that Assumption 2.2 is satisfied. Of course we take Q^* associated with $K(x, y)$ and first assume that for each $f \in C_0^\infty(\mathbb{R}^d, F)$, $n = 1, 2, \dots$, and $Q \in \mathbb{Q}_n$ we have

$$(2.17) \quad |\mathcal{A}(I_Q(f - f_n))(x)|_G \leq \left| \int_Q K(x, y)(f(y) - f_n(y)) dy \right|_H$$

almost everywhere in $\mathbb{R}^d \setminus Q^*$.

Then by noticing that f_n is constant on Q , we get

$$\int_Q K(x, y)f_n(y) dy = \int_Q K_n(x, y)f(y) dy,$$

$$|\mathcal{A}(I_Q(f - f_n))(x)|_G \leq \int_Q |K(x, y) - K_n(x, y)|_{L(F,H)} |f(y)|_F dy,$$

which after integrating over $\mathbb{R}^d \setminus Q^*$ and applying Remark 2.7 immediately yields (2.4).

Thus it only remains to prove (2.17), that is to prove that (2.16) holds if we take there $g := I_Q(f - f_n)$ in place of f . Notice that g is easily represented

as the product of I_Q and a $C_0^\infty(\mathbb{R}^d, F)$ -function. Therefore, it suffices to prove that, for any $f \in C_0^\infty(\mathbb{R}^d, F)$ and $Q \in \cup_n Q_n$,

$$(2.18) \quad |A(fI_Q)(x)|_G \leq \left| \int_{\mathbb{R}^d} K(x, y) f(y) I_Q(y) dy \right|_H$$

almost everywhere in $\mathbb{R}^d \setminus Q^*$ ($\subset \mathbb{R}^d \setminus \bar{Q}$).

Owing to the regularity of Lebesgue measure, there are closed sets $B_m \subset B_{m+1} \subset Q$ such that $|Q \setminus B_m| \rightarrow 0$. Due to Definition 2.6 (i) and the dominated convergence theorem, for each $x \notin \bar{Q}$,

$$\int_{\mathbb{R}^d} K(x, y) f(y) I_Q(y) dy = \lim_{m \rightarrow \infty} \int_{\mathbb{R}^d} K(x, y) f(y) I_{B_m}(y) dy.$$

Moreover, by the modified Assumption 2.3, for an appropriate subsequence $m(k)$,

$$|A(fI_Q)(x)|_G \leq \varliminf_{k \rightarrow \infty} |A(fI_{B_{m(k)}})(x)|_G \quad (\text{a.e.}).$$

Hence, it suffices to prove that (2.18) holds almost everywhere in $\mathbb{R}^d \setminus Q$ assuming that Q is just arbitrary *closed bounded* set. Furthermore, in this situation it suffices to prove that, for any $\varepsilon > 0$, (2.18) holds for almost any x outside of the open ε -neighborhood Q_ε of Q . Fix an $\varepsilon > 0$. Then there is a uniformly bounded sequence $\zeta_k \in C_0^\infty(Q_\varepsilon)$ such that $\zeta_k \rightarrow I_Q$ everywhere as $k \rightarrow \infty$. For each $x \notin Q_\varepsilon$ we have

$$\int_{\mathbb{R}^d} K(x, y) f(y) I_Q(y) dy = \lim_{k \rightarrow \infty} \int_{\mathbb{R}^d} K(x, y) f(y) \zeta_k(y) dy$$

by the dominated convergence theorem and by virtue of Definition 2.6 (i). Also, for a subsequence $k(j)$ by the modified Assumption 2.3

$$|A(fI_Q)|_G \leq \varliminf_{j \rightarrow \infty} |A(f\zeta_{k(j)})|_G \quad (\text{a.e.}).$$

Finally using (2.16) with $f\zeta_k$ in place of f yields (2.18) almost everywhere in $\mathbb{R}^d \setminus Q_\varepsilon$ thus proving the theorem.

3. – $L_p(C^{2+\delta})$ -estimates for parabolic equations

Let $a(t) = (a^{ij}(t))$ be a symmetric $d \times d$ matrix valued function defined for $t \in \mathbb{R}$. Assume that it is measurable and, for some constants $\Gamma \geq \gamma > 0$,

$$(3.1) \quad \Gamma |\xi|^2 \geq a^{ij}(t) \xi^i \xi^j \geq \gamma |\xi|^2$$

for all $t \in \mathbb{R}$ and $\xi \in \mathbb{R}^d$. Define

$$L = a^{ij} D_{ij} - D_t$$

and for $\alpha \in (0, 1]$ let

$$[u]_\alpha = \sup_{\substack{x, y \in \mathbb{R}^d \\ x \neq y}} \frac{|u(x) - u(y)|}{|x - y|^\alpha}, \quad |u|_0 = [u]_0 = \sup_{x \in \mathbb{R}^d} |u(x)|.$$

The following apriori estimate is our first main result. Notice that in (3.2) and in similar formulas elsewhere naturally, if $p = \infty$, the L_p -norms are understood as ess sup 's of integrands.

THEOREM 3.1. *Let $p \in (1, \infty]$ and $\alpha \in (0, 1)$. Then for any $u \in C_0^\infty(\mathbb{R}^{d+1})$ and constant $\lambda \geq 0$ we have*

$$(3.2) \quad \left(\int_{\mathbb{R}} [D^2 u(t, \cdot)]_\alpha^p dt \right)^{1/p} \leq N \left(\int_{\mathbb{R}} [(L - \lambda)u(t, \cdot)]_\alpha^p dt \right)^{1/p},$$

$$(3.3) \quad \lambda \left(\int_{\mathbb{R}} |u(t, \cdot)|_0^p dt \right)^{1/p} \leq \left(\int_{\mathbb{R}} |(L - \lambda)u(t, \cdot)|_0^p dt \right)^{1/p}$$

with a constant N in (3.2) depending only on α, p, d, γ .

REMARK 3.2 (cf. Remark 2.2 of [8]). For $p \in (1, \infty]$, by C_p^α we mean $L_p(\mathbb{R}, C^\alpha)$. Define $C_p^{2+\alpha}$ as the space of all continuous functions $u(t, x)$ given on \mathbb{R}^{d+1} such that $u, Du, D^2u \in C_p^\alpha$ and there exists an $f \in C_p^\alpha$ satisfying

$$u(t, x) - u(s, x) = \int_s^t f(r, x) dr$$

for any x, s , and t . For $u \in C_p^{2+\alpha}$ we define $D_t u = f$ and

$$\|u\|_{C_p^{2+\alpha}} = \|u\|_{C_p^\alpha} + \|D^2 u\|_{C_p^\alpha} + \|D_t u\|_{C_p^\alpha}.$$

Theorem 3.1 and standard methods of the theory of parabolic differential equations (see, for instance, [5]) allow us to conclude that if $a^{ij}(t, x)$, $b^i(t, x)$, $c(t, x)$, $i, j = 1, \dots, d$, are real-valued C_∞^α -functions and a^{ij} satisfies (3.1) for any (t, x) , then for any $p \in (1, \infty]$ there exists a $\lambda_1 < \infty$ such that, for any $\lambda > \lambda_1$ and $f \in C_p^\alpha$, in $C_p^{2+\alpha}$ there exists a unique solution of

$$(3.4) \quad D_t u = a^{ij} D_{ij} u + b^i D_i u + cu + f - \lambda u.$$

In addition,

$$(3.5) \quad \|u\|_{C_p^{2+\alpha}} \leq N \|f\|_{C_p^\alpha},$$

where N is independent of f .

Naturally, if $f(t, \cdot) = 0$ for $t \leq 0$, then $u(t, \cdot) = 0$ for $t \leq 0$ and the solution of (3.4) also solves the Cauchy problem with zero initial condition. Replacing u with $ve^{-\lambda t}$ allows one to get rid of λ in (3.4) and convert (3.5) to an estimate of the restriction of u on $[0, T]$, $T \in (0, \infty)$, through the restriction of f to the same interval provided that $f(t, \cdot) = 0$ for $t \leq 0$. For $p = \infty$ this yields part of results in [9] (also see the references therein).

The author of [9] also considers some weighted spaces. Part of his results can be obtained in the following way. Let $w(t, x) > 0$ be any smooth function on \mathbb{R}^{d+1} satisfying

$$\|w^{-1}Dw\|_{C^\infty} + \|w^{-1}D^2w\|_{C^\infty} + \|w^{-1}D_t w\|_{C^\infty} < \infty.$$

Then by denoting $u = wv$ we transform (3.4) into

$$(3.6) \quad D_t v = a^{ij} D_{ij} v + \tilde{b}^i D_i v + \tilde{c} v + g - \lambda v,$$

where $g = f/w$ and \tilde{b}, \tilde{c} are certain easily found functions of class C^∞ . Since the above solvability result is valid for (3.6), we conclude that for sufficiently large λ and any f such that $g = f/w \in C_p^\alpha$ equation (3.4) has a unique solution u in the class of functions such that $v = u/w \in C_p^{2+\alpha}$. Furthermore, for this solution estimate (3.5), valid for v and g , yields

$$\|u/w\|_{C_p^{2+\alpha}} \leq N \|f/w\|_{C_p^\alpha}.$$

For $p = \infty$ and w independent of t and having a particular form this result is obtained in [9] by a different method which uses some interesting results on weighted spaces estimates of fundamental solutions.

We derive Theorem 3.1 from our second main result before which we introduce some notation. For $t > s$ let

$$(3.7) \quad A_{st} := \int_s^t a(r) dr, \quad B_{st} := A_{st}^{-1}, \quad \sigma_{st} := A_{st}^{1/2}.$$

Observe that the matrices A_{st} are nondegenerate so that B_{st} is well defined and

$$\Gamma(t-s)|\xi|^2 \geq A_{st}^{ij} \xi^i \xi^j \geq \gamma(t-s)|\xi|^2,$$

$$(3.8) \quad \gamma^{-1}(t-s)^{-1}|\xi|^2 \geq B_{st}^{ij} \xi^i \xi^j \geq \kappa(t-s)^{-1}|\xi|^2, \quad \kappa := \Gamma^{-1}.$$

Finally let

$$\begin{aligned} p(s, t, x) &= I_{t>s} (4\pi)^{-d/2} (\det B_{st})^{1/2} \exp(-(B_{st}x, x)/4), \\ p_{ij}(s, t, x) &= D_{ij} p(s, t, x), \quad p_{ijk}(s, t, x) = D_{ijk} p(s, t, x), \\ G_\lambda f(t, x) &= \int_{-\infty}^t \int_{\mathbb{R}^d} p(s, t, x-y) f(s, y) e^{-\lambda(t-s)} dy ds. \end{aligned}$$

Notice that, for any $f \in C_0^\infty(\mathbb{R}^{d+1})$ and $(t, x) \in \mathbb{R}^{d+1}$, the expression

$$(3.9) \quad D_{ij}G_\lambda f(t, x) = \int_{-\infty}^t \int_{\mathbb{R}^d} p(s, t, y) f_{x^i x^j}(s, x - y) e^{-\lambda(t-s)} dy ds$$

is well defined.

THEOREM 3.3. *Let $p \in (1, \infty]$ and $\alpha \in (0, 1)$. Then for any $f \in C_0^\infty(\mathbb{R}^{d+1})$ and constant $\lambda \geq 0$ we have*

$$(3.10) \quad \left(\int_{\mathbb{R}} [D^2 G_\lambda f(t, \cdot)]_\alpha^p dt \right)^{1/p} \leq N \left(\int_{\mathbb{R}} [f(t, \cdot)]_\alpha^p dt \right)^{1/p},$$

where N depends only on α, p, d, γ .

The proof of this theorem is given in Section 5.

PROOF OF THEOREM 3.1. If L' is an operator with coefficients a' of the same type as L , then

$$[(L - L')u(t, \cdot)]_\alpha \leq N|a(t) - a'(t)|,$$

where N depends only on u . In addition the left-hand side has compact support. This allows us to use approximations and therefore without losing generality we assume that a is infinitely differentiable.

Define $f = \lambda u - Lu$. Then $f \in C_0^\infty(\mathbb{R}^{d+1})$. Furthermore, it is well known (see, for instance [7]) that $u = G_\lambda f$. Now (3.2) becomes (3.10). Estimate (3.3) follows from Minkowski's inequality since, for $t > s$, we have $\int p(s, t, x) dx = 1$, so that

$$\sup_x |u(t, x)| \leq \int_{-\infty}^t e^{-\lambda(t-s)} |f(s, \cdot)|_0 ds = \phi(s) * |f(s, \cdot)|_0,$$

where $\phi(s) = I_{s>0} \exp(-\lambda s)$ with $\|\phi\|_{L_1} = \lambda^{-1}$. The theorem is proved.

REMARK 3.4. It only follows from our proofs which we present in the subsequent sections that the constant N in Theorem 3.3 depends only on α, p, d, γ , and Γ . The fact that it is actually independent of Γ follows from a general Theorem 2.2 of [4] (see also similar Theorem 5.1 in [6]) saying, roughly speaking that, whatever estimate in translation invariant spaces is true for the heat equation, it is also true with *the same constant* for parabolic equations of main type with coefficients depending only on time provided that the matrix of second order coefficients is greater than the unit matrix in the matrix sense.

This general result allows us to concentrate on $a^{ij} \equiv \delta^{ij}$. However, we do not do this for the sake of making the presentation more traditional.

4. – $L_\infty(C^{2+\delta})$ -estimates for parabolic equations

In this section we assume that the function $a(t)$ is infinitely differentiable and each derivative of it is bounded.

For each $s, t \in \mathbb{R}$, $\lambda \geq 0$, and $i, j = 1, \dots, d$, define an operator \mathcal{A}_λ^{ij} mapping functions on \mathbb{R}^{d+1} into functions on \mathbb{R}^{d+1} by the formula

$$\begin{aligned} \mathcal{A}_\lambda^{ij} f(t, x) &= \int_{-\infty}^t \left(\int_{\mathbb{R}^d} p_{ij}(s, t, x - y) f(s, y) dy \right) e^{-\lambda(t-s)} ds \\ (4.1) \quad &:= \lim_{n \rightarrow \infty} \int_{-n}^{t-1/n} \left(\int_{\mathbb{R}^d} p_{ij}(s, t, x - y) f(s, y) dy \right) e^{-\lambda(t-s)} ds. \end{aligned}$$

Notice that integrating by parts shows that

$$(4.2) \quad \mathcal{A}_\lambda^{ij} f(t, x) = D_{ij} G_\lambda f(t, x) \quad \forall f \in C_0^\infty(\mathbb{R}^{d+1}),$$

so that estimating $D_{ij} G_\lambda f$ is equivalent to estimating $\mathcal{A}_\lambda^{ij} f$. The operator \mathcal{A}_λ^{ij} has the advantage with respect to $D_{ij} G_\lambda$ of being defined on a larger set.

We are going to use the facts that for $s < t$

$$(4.3) \quad \frac{\partial}{\partial t} p(s, t, x) = a^{ij}(t) D_{ij} p(s, t, x), \quad \frac{\partial}{\partial s} p(s, t, x) = -a^{ij}(s) D_{ij} p(s, t, x),$$

$$(4.4) \quad |D_t^n D_x^m p(s, t, x)| + |D_s^n D_x^m p(s, t, x)| \leq N(t - s)^{-n-(d+m)/2} e^{-\kappa|x|^2/(8t-8s)},$$

where N is independent of s, t, x and, if $n = 0$, then N depends only on m, d, γ , and Γ .

LEMMA 4.1. *If $\lambda > 0$, $\alpha \in (0, 1]$, and $f \in L_\infty(\mathbb{R}, C^\alpha(\mathbb{R}^d))$, then $\mathcal{A}_\lambda^{ij} f(t, x)$ is well defined for any $(t, x) \in \mathbb{R}^{d+1}$. Furthermore,*

$$(4.5) \quad |\mathcal{A}_\lambda^{ij} f(t, x)| \leq N \int_{-\infty}^t [f(s, \cdot)]_\alpha (t - s)^{\alpha/2-1} e^{-\lambda(t-s)} ds,$$

where N depends only on d, γ , and Γ .

To prove the lemma it suffices to observe that, by virtue of (4.4),

$$\begin{aligned} & \left| \int_{\mathbb{R}^d} p_{ij}(s, t, x - y) f(s, y) dy \right| \\ (4.6) \quad &= \left| \int_{\mathbb{R}^d} p_{ij}(s, t, x - y) [f(s, y) - f(s, x)] dy \right| \\ &\leq N [f(s, \cdot)]_\alpha (t - s)^{-1-d/2} \int_{\mathbb{R}^d} |x - y|^\alpha e^{-\kappa|x-y|^2/(8t-8s)} dy \\ &= N [f(s, \cdot)]_\alpha (t - s)^{\alpha/2-1}. \end{aligned}$$

The following lemma is part of basic Schauder estimates for parabolic equations. To be closer to the classical approach we state it for $C_0^\infty(\mathbb{R}^{d+1})$ functions.

LEMMA 4.2. *Let $f \in C_0^\infty(\mathbb{R}^{d+1})$, $\lambda \geq 0$, and $\alpha \in (0, 1)$. Then there exists a constant $N = N(d, \alpha, \gamma, \Gamma)$ such that*

(i) *if $f(s, \cdot) = 0$ for $s \leq -2$, then*

$$(4.7) \quad |D^2 G_\lambda f(0, x)| \leq N \sup_{s \leq 0} [f(s, \cdot)]_\alpha;$$

(ii) *if $f(s, \cdot) = 0$ for $s \geq -1$, then*

$$(4.8) \quad |D^3 G_\lambda f(0, x)| \leq N \sup_{s \leq 0} [f(s, \cdot)]_\alpha;$$

(iii) *for any $x, y \in \mathbb{R}^d$*

$$(4.9) \quad |D^2 G_\lambda f(0, x) - D^2 G_\lambda f(0, y)| \leq N \sup_{s \leq 0} [f(s, \cdot)]_\alpha |x - y|^\alpha.$$

PROOF. Assertion (i) follows at once from (4.2) and (4.5) since

$$\int_{-2}^0 (-s)^{\alpha/2-1} ds < \infty.$$

The proof of (ii) is basically the same. Indeed,

$$D_{ijk} G f(0, x) = \int_{-\infty}^0 e^{\lambda s} \left(\int_{\mathbb{R}^d} p_{ijk}(s, 0, y) [f(s, y) - f(s, x)] dy \right) ds,$$

$$\begin{aligned} |D_{ijk} G f(0, x)| &\leq N \sup_{s \leq -1} [f(s, \cdot)]_\alpha \int_{-\infty}^{-1} \int_{\mathbb{R}^d} (-s)^{-3/2-d/2} |y|^\alpha e^{-\kappa|y|^2/(8s)} dy ds, \\ &= N \sup_{s \leq -1} [f(s, \cdot)]_\alpha \int_{-\infty}^{-1} (-s)^{\alpha/2-3/2} ds = N \sup_{s \leq 0} [f(s, \cdot)]_\alpha. \end{aligned}$$

To prove (iii) first notice that parabolic dilations easily show that, if (4.9) holds with a constant N for all $\lambda \geq 0$, smooth a satisfying (3.1), and x, y such that $|x - y| = 1$, then (4.9) holds for all $\lambda \geq 0$, x, y , and smooth a satisfying (3.1). Therefore, we assume $|x - y| = 1$ and take a $\zeta \in C_0^\infty(\mathbb{R})$ such that $\zeta(s) = 1$ for $s \in [-1, 0]$ and $\zeta(s) = 0$ for $s \leq -2$. Write $\eta = 1 - \zeta$,

$$\begin{aligned} G_\lambda f &= G_\lambda(\zeta f) + G_\lambda(\eta f), \quad |D^2 G_\lambda f(0, x) - D^2 G_\lambda f(0, y)| \\ &\leq |D^2 G_\lambda(\zeta f)(0, x)| + |D^2 G_\lambda(\zeta f)(0, y)| \\ &\quad + |D^2 G_\lambda(\eta f)(0, x) - D^2 G_\lambda(\eta f)(0, y)|. \end{aligned}$$

Here the last term on the right is less than $\sup_x |D^3 G_\lambda(\eta f)(0, x)|$. Therefore, by applying (i) and (ii) we get (4.9). The lemma is proved.

In order to be able to check out condition (2.13) we need some properties of the kernel in (4.1). For each $s, t \in \mathbb{R}$ and $i, j = 1, \dots, d$, define an operator $K_{ij}(t, s)$ mapping functions of $x \in \mathbb{R}^d$ into functions of x by the formula

$$K_{ij}(t, s)f(x) = \int_{\mathbb{R}^d} p_{ij}(s, t, x - y)f(y) dy.$$

LEMMA 4.3. (i) *The operators $K_{ij}(t, s)$ are bounded operators from $L_\infty(\mathbb{R}^d)$ into $C^1(\mathbb{R}^d)$ and are also bounded operators acting in $C^\alpha(\mathbb{R}^d)$ for any $\alpha \in [0, 1]$.*

(ii) *For any $\alpha \in [0, 1]$, $s < t$, and $f \in C^\alpha(\mathbb{R}^d)$, we have*

$$(4.10) \quad [K_{ij}(t, s)f]_\alpha \leq N[f]_\alpha(t - s)^{-1}, \quad [D_s K_{ij}(t, s)f]_\alpha \leq N[f]_\alpha(t - s)^{-2},$$

where N depends only on d, γ , and Γ .

PROOF. The first part of assertion (i) is trivial due to the fact that, for each fixed $s < t$, all derivatives of $p_{ij}(s, t, x)$ in x are bounded by constants times $\exp(-\kappa|x|^2/(8t - 8s))$. Since, $C^1 \subset C^\alpha \subset L_\infty$, the first part of assertion (i) implies its second part.

The first estimate in (4.10) for $\alpha = 0$ follows from (4.6). If $\alpha \in (0, 1]$ we observe

$$\begin{aligned} K_{ij}(t, s)f(x) &= \int_{\mathbb{R}^d} p_{ij}(s, t, y)f(x - y) dy, \quad |K_{ij}(t, s)f(x) - K_{ij}(t, s)f(z)| \\ &\leq N \sup_y |f(x - y) - f(z - y)|(t - s)^{-1-d/2} \int_{\mathbb{R}^d} e^{-\kappa|y|^2/(8t-8s)} dy \\ &\leq N[f]_\alpha|x - z|^\alpha(t - s)^{-1}. \end{aligned}$$

In the proof of the second estimate in (4.10) we use (4.13) to conclude that

$$\begin{aligned} D_s K_{ij}(t, s)f(x) &= \int_{\mathbb{R}^d} D_s p_{ij}(s, t, y)f(x - y) dy \\ &\quad - a^{kr}(s) \int_{\mathbb{R}^d} D_{krij} p(s, t, y)f(x - y) dy, \end{aligned}$$

which allows us to repeat the above argument almost literally. For instance, owing to (4.4), for $s < t$, we get

$$\begin{aligned} |D_s K_{ij}(t, s)f(x)| &\leq N \sup_z |f(z)|(t - s)^{-d/2-2} \int_{\mathbb{R}^d} e^{-\kappa|y|^2/(8t-8s)} dy \\ &= N \sup_z |f(z)|(t - s)^{-2}. \end{aligned}$$

This proves the second estimate in (4.10) for $\alpha = 0$. In the same way we can consider $\alpha \in (0, 1]$. The lemma is proved.

5. – Proof of Theorem 3.3

First we loosely describe the idea of proof of Theorem 3.3 for smooth $a(t)$ without caring about the truthfulness of our assertions. Take the operator \mathcal{A}_λ^{ij} introduced in (4.1):

$$(5.1) \quad \mathcal{A}_\lambda^{ij} f(t, \cdot) = \int_{\mathbb{R}} e^{-\lambda(t-s)} K_{ij}(t, s) f(s, \cdot) ds.$$

By Lemma 4.2

$$(5.2) \quad \sup_t [\mathcal{A}_\lambda^{ij} f(t, \cdot)]_\alpha \leq N \operatorname{ess\,sup}_t [f(t, \cdot)]_\alpha.$$

Furthermore,

$$(5.3) \quad D_s \{e^{-\lambda(t-s)} K_{ij}(t, s)\} = e^{-\lambda(t-s)} D_s K_{ij}(t, s) + \lambda e^{-\lambda(t-s)} K_{ij}(t, s)$$

and due to Lemma 4.3

$$[D_s \{e^{-\lambda(t-s)} K_{ij}(t, s)\} f]_\alpha \leq N [f]_\alpha \phi(|t-s|),$$

where the function $\phi(r) = r^{-2} + \lambda r^{-1} e^{-\lambda r}$ satisfies condition (2.15) for $d = 1$ (with N_1 independent of λ). This along with Remark 2.8 seem to imply the right estimate on the kernel $K_{ij}(t, s)$ needed in Theorem 2.9 to deal with the case $F = G = H = C^\alpha(\mathbb{R}^d)$. Then Theorem 2.9 should yield, for any $q \in (1, \infty)$,

$$\left(\int_{\mathbb{R}} [\mathcal{A}_\lambda^{ij} f(t, \cdot)]_\alpha^q dt \right)^{1/q} \leq N \left(\int_{\mathbb{R}} [f(t, \cdot)]_\alpha^q dt \right)^{1/q},$$

which is (3.10).

The main trouble in the above line of reasoning is that $[\cdot]_\alpha$ is not a norm in $C^\alpha(\mathbb{R}^d)$ and we did not prove that $\mathcal{A}_\lambda^{ij} f \in C_\infty^\alpha$ if $f \in C_\infty^\alpha$ and the issue is that not only should we have an estimate of $\mathcal{A}_\lambda^{ij} f$ but also prove the *measurability* of $\mathcal{A}_\lambda^{ij} f(t)$ as a $C^\alpha(\mathbb{R}^d)$ -valued function. The latter is particularly unpleasant since the space $C^\alpha(\mathbb{R}^d)$ is not separable.

Now we come back to rigorous arguments. We will be constantly using the simple fact that, if $f_n \in C^\alpha(\mathbb{R}^d)$ and $f_n(x) \rightarrow f(x)$ at any x , then

$$(5.4) \quad [f]_\alpha \leq \varliminf_{n \rightarrow \infty} [f_n]_\alpha.$$

It follows from (3.9) that if $f \in C_0^\infty(\mathbb{R}^{d+1})$, then $D^2 G_\lambda f(t, x) \rightarrow D^2 G_0 f(t, x)$ at any (t, x) as $\lambda \downarrow 0$. By using (5.4) and Fatou's theorem we see that we need only prove (3.10) for $\lambda > 0$. The same argument based on approximating a shows that it suffices to prove (3.10) when $a(t)$ is infinitely differentiable and has bounded derivatives. Therefore, below in this section we suppose that $a(t)$ is such a function and $\lambda > 0$.

We fix an $\varepsilon \in (0, 1]$ and define a new (equivalent) norm in $C^\alpha(\mathbb{R}^d)$ by

$$|u|_{\alpha\varepsilon} = \varepsilon |u|_0 + [u]_\alpha.$$

LEMMA 5.1. *For any $\alpha \in [0, 1]$, the operators $K_{ij}(t, s)$ as operators acting in $C^\alpha(\mathbb{R}^d)$ are infinitely differentiable in (s, t) for $s < t$ in the sense of the operator norm and for each derivative its norm is bounded on the sets $\{(s, t) : t - s > \beta\}$ whenever the constant $\beta > 0$.*

PROOF. We first prove that $K_{ij}(t, s)$ is once differentiable in s and, for $f \in C^\alpha(\mathbb{R}^d)$, we have

$$(5.5) \quad D_s K_{ij}(t, s) f(x) = \int_{\mathbb{R}^d} D_s p_{ij}(s, t, x - y) f(y) dy =: K_{ijs}(t, s) f(x).$$

Observe that, for $s, s + \delta < t$ and $f \in C^\alpha(\mathbb{R}^d)$, we have

$$(5.6) \quad \begin{aligned} & \delta^{-1} [K_{ij}(t, s + \delta) f(x) - K_{ij}(t, s) f(x)] - K_{ijs}(t, s) f(x) \\ &= \int_0^1 [K_{ijs}(t, s + r\delta) f(x) - K_{ijs}(t, s) f(x)] dr \\ &= \delta \int_0^1 \int_0^1 \int_{\mathbb{R}^d} \frac{\partial^2}{(\partial s)^2} p_{ij}(s + rq\delta, t, x - y) f(y) r dy dq dr. \end{aligned}$$

The cofactor of δ can be regarded as the value at x of the result of applying certain operator to f . Owing to (4.4) the norm of this operator as an operator from $L_\infty(\mathbb{R}^d)$ to $C^1(\mathbb{R}^d)$ is bounded by a constant independent of δ as long as $s + \delta$ is separated away from t . The same holds for the norm of this operator as an operator acting in $C^\alpha(\mathbb{R}^d)$ (cf. the proof of Lemma 4.3 (i)). Now (5.6) implies that $K_{ij}(t, s)$ is once differentiable with respect to s (in the operator norm), its derivative is given by $K_{ijs}(t, s)$, and its norm is bounded on $\{(s, t) : t - s > \beta\}$.

The first derivative of $K_{ij}(t, s)$ with respect to t is considered similarly and higher derivatives in (s, t) are investigated in the same way starting with formulas like (5.5). The lemma is proved.

In the rest of the section we take

$$\alpha \in (0, 1), \quad p = \infty, \quad (F, |\cdot|_F) = (H, |\cdot|_H) = (C^\alpha(\mathbb{R}^d), |\cdot|_{\alpha\epsilon}), \quad G = \mathbb{R},$$

$$(5.7) \quad i, j \in \{1, \dots, d\}, \quad \mathcal{A}f(t) = [\mathcal{A}_\lambda^{ij} f(t, \cdot)]_\alpha, \quad K(t, s) = K_{ij}(t, s) e^{-\lambda(t-s)}.$$

Remember that $C_\infty^\alpha = L_\infty(\mathbb{R}, C^\alpha(\mathbb{R}^d))$ and $\lambda > 0$.

LEMMA 5.2. For any $f \in C_\infty^\alpha$ and $t \in \mathbb{R}$, expression $\mathcal{A}f(t)$ is well defined, finite, and there is a constant N depending only on α, d, γ , and Γ such that

$$\sup_t |\mathcal{A}f(t)| \leq N \operatorname{ess\,sup}_t [f(t, \cdot)]_\alpha.$$

PROOF. First observe that for $f \in C_\infty^\alpha$ by Lemma 4.1 we have

$$(5.8) \quad \sup_t |\mathcal{A}_\lambda^{ij} f(t, \cdot)|_0 \leq N \operatorname{ess\,sup}_t [f(t, \cdot)]_\alpha$$

By Lemma 4.2 we have (5.2) if $f \in C_0^\infty(\mathbb{R}^{d+1})$. However, by using (4.5) and (5.4) one easily carries over (5.2) to all $f \in C_\infty^\alpha$. The lemma is proved.

LEMMA 5.3. *Assumption (2.16) and the modified Assumption 2.3 introduced in Theorem 2.9 are satisfied for the objects from (5.7). Moreover, K is an $L(F, H)$ -valued Calderón-Zygmund kernel relative to the dyadic filtration on \mathbb{R} with the corresponding constant N_0 depending only on d, γ , and Γ .*

PROOF. We start with the last assertion. Observe that the measurability and integrability of K follow from the fact that it is a smooth operator-valued function. The strong (and hence weak) derivative of $K_{ij}(t, s)$ with respect to s is given by (5.5). This along with (5.3) and Lemma 4.3 shows that

$$\begin{aligned} |D_s K(t, s)f|_0 &\leq N|f|_0\phi(|t-s|), \\ [D_s\{e^{-\lambda(t-s)}K_{ij}(t, s)\}f]_\alpha &\leq N[f]_\alpha\phi(|t-s|), \\ |D_s K(t, s)f|_{\alpha\varepsilon} &\leq N|f|_{\alpha\varepsilon}\phi(|t-s|), \end{aligned}$$

with ϕ as above and N depending only on d, γ , and Γ . By Remark 2.8 we conclude that indeed K is a Calderón-Zygmund kernel relative to the dyadic filtration.

Lemma 4.3 also shows that

$$|K(t, s)f|_{\alpha\varepsilon} \leq N|f|_{\alpha\varepsilon}|t-s|^{-1},$$

so that, if t is not in the closed support of a $C^\alpha(\mathbb{R}^d)$ -valued function $f(s, \cdot)$ of class $C_0^\infty(\mathbb{R}, C^\alpha(\mathbb{R}^d))$, then the (Bochner) integral in (5.1) exists. By applying to it the continuous linear function $f \rightarrow f(x_0)$, where x_0 is an arbitrary fixed point, and remembering definition (4.1) we see that (5.1) holds. Thus, assumption (2.16) is satisfied.

As long as the modified Assumptions 2.3 is concerned, Lemma 4.1 and the dominated convergence theorem show that, if $f_n \in L_\infty(C^\alpha(\mathbb{R}^d))$ and $|f_n(t, \cdot)|_{\alpha\varepsilon}$ are uniformly bounded and vanish outside of a fixed interval and $|(f - f_n)(t, \cdot)|_{\alpha\varepsilon} \rightarrow 0$ for almost any t , then $|\mathcal{A}_\lambda^{ij}f(t, x) - \mathcal{A}_\lambda^{ij}f_n(t, x)| \rightarrow 0$ at any point (t, x) (actually, even uniformly). Therefore, (5.4) implies that

$$|\mathcal{A}f(t)| \leq \varliminf_{n \rightarrow \infty} |\mathcal{A}f_n(t)|$$

for any t . The lemma is proved.

Now we are ready to finish proving Theorem 3.3. The subadditivity of \mathcal{A} is obvious. Therefore, if we knew that $\mathcal{A}f(t)$ is a measurable function of t for any $f \in C_\infty^\alpha$, then Lemmas 5.2 and 5.3 would allow us to apply Theorem 2.9 and conclude that, for any $\varepsilon \in (0, 1]$, $q \in (1, \infty)$, and $f \in C_q^\alpha$,

$$\int_{\mathbb{R}} [\mathcal{A}_\lambda^{ij}f(t, \cdot)]_\alpha^q dt = \|\mathcal{A}f\|_{L_q(\mathbb{R})}^q \leq N \int_{\mathbb{R}} (\varepsilon|f(t, \cdot)|_0 + [f(t, \cdot)]_\alpha)^q dt,$$

where N depends only on d, q, α, γ , and Γ . By letting $\varepsilon \downarrow 0$ and remembering (4.2) we would conclude the proof of Theorem 3.3.

Thus, it only remains to prove that $\mathcal{A}f(t)$ is measurable. We will see that this function is lower semicontinuous, so that it is even Borel measurable.

Notice that, due to Lemma 5.1 and the fact that $\lambda > 0$, for $r < t$ and $f \in C_\infty^\alpha$, the integral

$$I(r, t)f := \int_{-\infty}^r K(t, s)f(s, \cdot) ds$$

is well defined as a $C^\alpha(\mathbb{R}^d)$ integral and is continuous as a $C^\alpha(\mathbb{R}^d)$ -valued function of (r, t) for $r < t$. Furthermore, as in (4.6), for $\varepsilon > 0$ we have

$$\begin{aligned} |I(t-\varepsilon, t)f(x) - \mathcal{A}_\lambda^{ij}f(t, x)| &= \left| \int_{-\infty}^{t-\varepsilon} \int_{\mathbb{R}^d} p_{ij}(s, t, x-y)e^{-\lambda(t-s)}f(s, y) dy ds \right. \\ &\quad \left. - \int_{-\infty}^t \int_{\mathbb{R}^d} p_{ij}(s, t, x-y)e^{-\lambda(t-s)}f(s, y) dy ds \right| \\ &\leq \int_{t-\varepsilon}^t \int_{\mathbb{R}^d} |p_{ij}(s, t, y)|e^{-\lambda(t-s)}|f(s, x-y) - f(x)| dy ds \\ &\leq \operatorname{ess\,sup}_s [f(s, \cdot)]_\alpha \int_{t-\varepsilon}^t (t-s)^{-1-d/2} \int_{\mathbb{R}^d} |y|^\alpha e^{-\kappa|y|^2/(8t-8s)} dy ds \\ &= N \operatorname{ess\,sup}_s [f(s, \cdot)]_\alpha \varepsilon^{\alpha/2}. \end{aligned}$$

We see that continuous functions $I(t-\varepsilon, t)f(x)$ converge to $\mathcal{A}_\lambda^{ij}f(t, x)$ uniformly in (t, x) as $\varepsilon \downarrow 0$. Hence, $\mathcal{A}_\lambda^{ij}f(t, x)$ is continuous and, by (5.4), its $[\cdot]_\alpha$ seminorm is lower semicontinuous:

$$\mathcal{A}f(t) \leq \liminf_{s \rightarrow t} \mathcal{A}f(s).$$

This finally brings the proof of Theorem 3.3 to an end.

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