

\mathbb{Z} -graded Lie Superalgebras of Infinite Depth and Finite Growth

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Abstract. In 1998 Victor Kac classified infinite-dimensional \mathbb{Z} -graded Lie superalgebras of finite depth. We construct new examples of infinite-dimensional Lie superalgebras with a \mathbb{Z} -gradation of infinite depth and finite growth and classify \mathbb{Z} -graded Lie superalgebras of infinite depth and finite growth under suitable hypotheses.

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Introduction

Simple finite-dimensional Lie superalgebras were classified by V. G. Kac in [K2]. In the same paper Kac classified the finite-dimensional, \mathbb{Z} -graded Lie superalgebras under the hypotheses of irreducibility and transitivity.

The classification of infinite-dimensional, \mathbb{Z} -graded Lie superalgebras of finite depth is also due to V. G. Kac [K3] and is deeply related to the classification of linearly compact Lie superalgebras. We recall that finite depth implies finite growth.

This naturally leads to investigate infinite-dimensional, \mathbb{Z} -graded Lie superalgebras of infinite depth and finite growth. The hypothesis of finite growth is central to the problem; indeed, it is well known that it is not possible to classify \mathbb{Z} -graded Lie algebras (and thus Lie superalgebras) of any growth (see [K1], [M]). The only known examples of infinite-dimensional, \mathbb{Z} -graded Lie superalgebras of finite growth and infinite depth are given by contragredient Lie superalgebras which were classified by V. G. Kac in [K2] in the case of finite dimension and by J.W. van de Leur in the general case [vdL]. Contragredient Lie superalgebras, as well as Kac-Moody Lie algebras, have a \mathbb{Z} -gradation of infinite depth and growth equal to 1, due to their periodic structure.

We construct three new examples of infinite-dimensional Lie superalgebras with a consistent \mathbb{Z} -gradation of infinite depth and finite growth, and we realize

them as covering superalgebras of finite-dimensional Lie superalgebras. It turns out that if \mathcal{G} is an irreducible, simple Lie superalgebra generated by its local part, with a consistent \mathbb{Z} -gradation, and if we assume that \mathcal{G}_0 is simple and that \mathcal{G}_1 is an irreducible \mathcal{G}_0 -module which is not contragredient to \mathcal{G}_{-1} , then \mathcal{G} is isomorphic to one of these three algebras (Theorem 3.1) and its growth is therefore equal to 1.

So far, any known example of a \mathbb{Z} -graded Lie superalgebra of infinite depth and finite growth is, up to isomorphism, either a contragredient Lie superalgebra or the covering superalgebra of a finite-dimensional Lie superalgebra. Since the aim of this paper is analyzing \mathbb{Z} -graded Lie superalgebras of infinite depth, we shall not describe the cases of finite depth which can be found in [K2], [K3].

Let \mathcal{G} be a \mathbb{Z} -graded Lie superalgebra. Suppose that \mathcal{G}_0 is a simple Lie algebra and that \mathcal{G}_{-1} and \mathcal{G}_1 are irreducible \mathcal{G}_0 -modules and are not contragredient. Let F_Λ be a highest weight vector of \mathcal{G}_{-1} of weight Λ and let E_M be a lowest weight vector of \mathcal{G}_1 of weight M . Since \mathcal{G}_{-1} and \mathcal{G}_1 are not contragredient, the sum $\Lambda + M$ is a root of \mathcal{G}_0 , and, without loss of generality, we may assume that it is a negative root, i.e. $\Lambda + M = -\alpha$ for some positive root α . The paper is based on the analysis of the relations between the \mathcal{G}_0 -modules \mathcal{G}_{-1} and \mathcal{G}_1 . It is organized in three sections: Section 1 contains some basic definitions and fundamental results in the general theory of Lie superalgebras. In Section 2 the main hypotheses on the Lie superalgebra \mathcal{G} are introduced. Section 2.1 is devoted to the case $(\Lambda, \alpha) = 0$. Since Λ is a dominant weight, in this section the rank of \mathcal{G}_0 is assumed to be greater than 1. The hypothesis $(\Lambda, \alpha) = 0$ always holds for \mathbb{Z} -graded Lie superalgebras of finite depth (see [K2], Lemma 4.1.4 and [K3], Lemma 5.3) but if the Lie superalgebra \mathcal{G} has infinite depth weaker restrictions on the weight Λ are obtained (compare, for example, Lemma 4.1.3 in [K2] with Lemma 1.14 in this paper).

In Section 2.2 we examine the case $(\Lambda, \alpha) \neq 0$. In the finite-depth case this hypothesis may not occur (cf. [K3], Lemma 5.3). It turns out that, under this hypothesis, \mathcal{G}_0 has necessarily rank one (cf. Theorem 2.17) namely it is isomorphic to $sl(2)$. Besides, a strong restriction on the possible values of (Λ, α) is obtained (cf. Corollary 2.12) so that \mathcal{G}_{-1} is necessarily isomorphic either to the adjoint module of $sl(2)$ or to the irreducible $sl(2)$ -module of dimension 2.

Finally, Section 3 is devoted to the construction of the examples and to the classification theorem.

Throughout the paper the base field is assumed to be algebraically closed and of characteristic zero.

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1. – Basic definitions and main results

1.1. – Lie superalgebras

DEFINITION 1.1. A superalgebra is a \mathbb{Z}_2 -graded algebra $A = A_{\bar{0}} \oplus A_{\bar{1}}$; $A_{\bar{0}}$ is called the even part of A and $A_{\bar{1}}$ is called the odd part of A .

DEFINITION 1.2. A Lie superalgebra is a superalgebra $\mathcal{G} = \mathcal{G}_{\bar{0}} \oplus \mathcal{G}_{\bar{1}}$ whose product $[\cdot, \cdot]$ satisfies the following axioms:

- (i) $[a, b] = -(-1)^{\deg(a)\deg(b)}[b, a]$;
- (ii) $[a, [b, c]] = [[a, b], c] + (-1)^{\deg(a)\deg(b)}[b, [a, c]]$.

DEFINITION 1.3. A \mathbb{Z} -grading of a Lie superalgebra \mathcal{G} is a decomposition of \mathcal{G} into a direct sum of finite-dimensional \mathbb{Z}_2 -graded subspaces $\mathcal{G} = \bigoplus_{i \in \mathbb{Z}} \mathcal{G}_i$ for which $[\mathcal{G}_i, \mathcal{G}_j] \subset \mathcal{G}_{i+j}$. A \mathbb{Z} -grading is said to be consistent if $\mathcal{G}_{\bar{0}} = \bigoplus \mathcal{G}_{2i}$ and $\mathcal{G}_{\bar{1}} = \bigoplus \mathcal{G}_{2i+1}$.

REMARK 1.4. By definition, if \mathcal{G} is a \mathbb{Z} -graded Lie superalgebra, then \mathcal{G}_0 is a subalgebra of \mathcal{G} and $[\mathcal{G}_0, \mathcal{G}_i] \subset \mathcal{G}_i$; therefore the restriction of the adjoint representation to \mathcal{G}_0 induces linear representations of it on the subspaces \mathcal{G}_i .

DEFINITION 1.5. A \mathbb{Z} -graded Lie superalgebra \mathcal{G} is called irreducible if \mathcal{G}_{-1} is an irreducible \mathcal{G}_0 -module.

DEFINITION 1.6. A \mathbb{Z} -graded Lie superalgebra $\mathcal{G} = \bigoplus_{i \in \mathbb{Z}} \mathcal{G}_i$ is called transitive if for $a \in \mathcal{G}_i, i \geq 0, [a, \mathcal{G}_{-1}] = 0$ implies $a = 0$, and bitransitive if, in addition, for $a \in \mathcal{G}_i, i \leq 0, [a, \mathcal{G}_1] = 0$ implies $a = 0$.

Let $\hat{\mathcal{G}}$ be a \mathbb{Z}_2 -graded space, decomposed into the direct sum of \mathbb{Z}_2 -graded subspaces, $\hat{\mathcal{G}} = \mathcal{G}_{-1} \oplus \mathcal{G}_0 \oplus \mathcal{G}_1$. Suppose that whenever $|i + j| \leq 1$ a bilinear operation is defined: $\mathcal{G}_i \times \mathcal{G}_j \rightarrow \mathcal{G}_{i+j}, (x, y) \mapsto [x, y]$, satisfying the axiom of anticommutativity and the Jacobi identity for Lie superalgebras, provided that all the commutators in this identity are defined. Then $\hat{\mathcal{G}}$ is called a local Lie superalgebra.

If $\mathcal{G} = \bigoplus_{i \in \mathbb{Z}} \mathcal{G}_i$ is a \mathbb{Z} -graded Lie superalgebra then $\mathcal{G}_{-1} \oplus \mathcal{G}_0 \oplus \mathcal{G}_1$ is a local Lie superalgebra which is called the local part of \mathcal{G} . The following proposition holds:

PROPOSITION 1.7 [K2]. Two bitransitive \mathbb{Z} -graded Lie superalgebras are isomorphic if and only if their local parts are isomorphic.

DEFINITION 1.8. A Lie superalgebra is called simple if it contains no nontrivial ideals.

PROPOSITION 1.9 [K2]. If in a simple \mathbb{Z} -graded Lie superalgebra $\mathcal{G} = \bigoplus_{i \in \mathbb{Z}} \mathcal{G}_i$ the subspace $\mathcal{G}_{-1} \oplus \mathcal{G}_0 \oplus \mathcal{G}_1$ generates \mathcal{G} then \mathcal{G} is bitransitive.

1.2. – On the growth of \mathcal{G}

DEFINITION 1.10. Let $\mathcal{G} = \bigoplus_{i \in \mathbb{Z}} \mathcal{G}_i$ be a \mathbb{Z} -graded Lie superalgebra. The limit

$$r(\mathcal{G}) = \lim_{n \rightarrow \infty} \ln \left(\sum_{i=-n}^n \dim \mathcal{G}_i \right) / \ln(n)$$

is called the growth of \mathcal{G} . If $r(\mathcal{G})$ is finite then we say that \mathcal{G} has finite growth.

Let us fix some notation. Given a semisimple Lie algebra L , by $V(\omega)$ we shall denote its finite-dimensional highest weight module of highest weight ω . ω_i will be the fundamental weights. It is well known that if λ is a weight of a finite-dimensional representation of L and β is a root of L , then the set of weights of the form $\lambda + s\beta$ forms a continuous string: $\lambda - p\beta, \lambda - (p - 1)\beta, \dots, \lambda - \beta, \lambda, \lambda + \beta, \dots, \lambda + q\beta$, where p and q are nonnegative integers and $p - q = 2(\lambda, \beta) / (\beta, \beta)$. Let us put $2(\lambda, \beta) / (\beta, \beta) = \lambda(h_\beta)$. The numbers $\lambda(h_{\alpha_i})$, for a fixed basis of simple roots α_i , are called the numerical marks of the weight λ .

For any positive root β of L we shall denote by e_β a root vector of L corresponding to β .

LEMMA 1.11 [K1]. Let L be a Lie algebra containing elements $H \neq 0, E_i, F_i, i = 1, 2$, connected by the equations

$$\begin{aligned} [E_i, F_j] &= \delta_{ij} H, \\ [H, E_1] &= aE_1, & [H, E_2] &= bE_2, \\ [H, F_1] &= -aF_1, & [H, F_2] &= -bF_2, \end{aligned}$$

where $a \neq -b, b \neq -2a$, and $a \neq -2b$, then the growth of L is infinite.

LEMMA 1.12 [K1]. Let $L = \bigoplus L_i$ be a graded Lie algebra, where L_0 is semisimple. Assume that there exist weight vectors x_λ and x_μ corresponding to the weights λ and μ of the adjoint representation of L_0 on L , and a root vector e_γ , corresponding to the root γ of L_0 , which satisfy the following relations:

$$\begin{aligned} [x_\mu, x_\lambda] &= e_\gamma, \\ [x_\lambda, e_{-\gamma}] &= 0 = [x_\mu, e_\gamma], \\ \lambda(h_\gamma) &\neq -1, \quad (\lambda, \gamma) \neq 0. \end{aligned}$$

Then the growth of L is infinite.

LEMMA 1.13. Let \mathcal{G} be a consistent, \mathbb{Z} -graded Lie superalgebra and suppose that \mathcal{G}_0 is a semisimple Lie algebra. Let $E_i, F_i (i = 1, 2)$ be odd elements and H a non zero element in \mathcal{G}_0 such that:

$$(1) \quad [E_i, F_j] = \delta_{ij} H, \quad [H, E_i] = a_i E_i, \quad [H, F_i] = -a_i F_i,$$

where $a_1 \neq -a_2, a_1 \neq -2a_2$ and $a_2 \neq -2a_1$. Then the growth of \mathcal{G} is infinite.

PROOF. Suppose first that $a_1 \neq 0 \neq a_2$. Then the elements $\tilde{E}_1 = a_1^{-1/2}[E_1, E_1]$, $\tilde{E}_2 = a_2^{-1/2}[E_2, E_2]$, $\tilde{F}_1 = a_1^{-1/2}[F_1, F_1]$, $\tilde{F}_2 = a_2^{-1/2}[F_2, F_2]$, $K = -4H$ satisfy the hypotheses of Lemma 1.11 in the Lie algebra \mathcal{G}_0 . Thus, the growth of \mathcal{G}_0 is infinite and we get the thesis.

If, let us say, $a_1 \neq 0$, $a_2 = 0$ then the elements $E'_1 = [E_1, E_1]$, $E'_2 = [E_1, E_2]$, $F'_1 = -(4a_1)^{-1}[F_1, F_1]$, $F'_2 = a_1^{-1}[F_1, F_2]$, H satisfy the hypotheses of Lemma 1.11 in \mathcal{G}_0 , thus we conclude. \square

LEMMA 1.14. *Let $\mathcal{G} = \oplus \mathcal{G}_i$ be a \mathbb{Z} -graded, consistent Lie superalgebra and suppose that \mathcal{G}_0 is a semisimple Lie algebra. Assume that there exist odd elements x_λ and x_μ that are weight vectors of the adjoint representation of \mathcal{G}_0 on \mathcal{G} of weight λ and μ respectively, and a root vector $e_{-\delta}$ of \mathcal{G}_0 , connected by the relations:*

$$\begin{cases} [x_\lambda, x_\mu] = e_{-\delta} \\ [x_\lambda, e_\delta] = [x_\mu, e_{-\delta}] = 0 \end{cases}$$

with $2(\lambda, \delta) \neq (\delta, \delta)$, $(\lambda, \delta) \neq 0$ and $(\lambda, \delta) \neq (\delta, \delta)$. Then the growth of \mathcal{G} is infinite.

PROOF. We choose a root vector e_δ in \mathcal{G}_0 such that $[e_\delta, e_{-\delta}] = h_\delta$ and consider the following elements:

$$\begin{aligned} E_1 &= [e_\delta, x_\mu] \\ E_2 &= [[[x_\mu, e_\delta], e_\delta], e_\delta] \\ F_1 &= x_\lambda \\ F_2 &= -1/6\lambda(h_\delta)^{-1}(\lambda(h_\delta) - 1)^{-1}[[x_\lambda, e_{-\delta}], e_{-\delta}] \\ H &= h_\delta. \end{aligned}$$

By a direct computation it is easy to check that E_i, F_i, H satisfy the hypotheses of Lemma 1.13 with $a_1 = (\mu + \delta)(h_\delta) = -\lambda(h_\delta)$, $a_2 = (\mu + 3\delta)(h_\delta) = -\lambda(h_\delta) + 4$. By Lemma 1.13 the growth of \mathcal{G} is therefore infinite. \square

We can reformulate Lemma 1.14 as follows:

COROLLARY 1.15. *Suppose that \mathcal{G} is a Lie superalgebra of finite growth. Let $x_\lambda, x_\mu, e_{-\delta}$ be as in Lemma 1.14. Then one of the following holds:*

- (i) $(\lambda, \delta) = 0$,
- (ii) $(\lambda, \delta) = (\delta, \delta)$,
- (iii) $(\lambda, \delta) = 1/2(\delta, \delta)$.

THEOREM 1.16 [K1]. *Let $L = \oplus L_i$ be a \mathbb{Z} -graded Lie algebra with the following properties:*

- a) the Lie algebra L_0 has no center;
- b) the representations ϕ_{-1} and ϕ_1 of L_0 on L_{-1} and L_1 are irreducible;
- c) $[L_{-1}, L_1] \neq 0$;

- d) $\Lambda + M = -\alpha$ where Λ is the highest weight of ϕ_{-1} , M is the lowest weight of ϕ_1 and α is a positive root of L_0 ;
- e) the representations ϕ_{-1} and ϕ_1 are faithful;
- f) the growth of L is finite.

Then L_0 is isomorphic to one of the Lie algebras A_n or C_n , ϕ_{-1} is the corresponding standard representation and α is the highest root of L_0 .

In the following sl_n , sp_n and so_n will denote the standard representations of the corresponding Lie algebras.

COROLLARY 1.17. *Let $\mathcal{G} = \oplus \mathcal{G}_i$ be a Lie superalgebra with a consistent \mathbb{Z} -gradation. Suppose that \mathcal{G}_0 is simple. Suppose that there exist a highest weight vector x in \mathcal{G}_{-2} of weight $\lambda \neq 0$ and a lowest weight vector y in \mathcal{G}_2 of weight μ such that $[x, y] \neq 0$ and $\lambda + \mu = -\rho$ for a positive root ρ of \mathcal{G}_0 . Then, if the growth of \mathcal{G} is finite, \mathcal{G}_0 is isomorphic to one of the Lie algebras A_n or C_n , ρ is the highest root of \mathcal{G}_0 and \mathcal{G}_{-2} is the standard \mathcal{G}_0 -module.*

PROOF. It follows from Theorem 1.16. □

2. – Main results

In this section we will consider an irreducible, consistent, simple \mathbb{Z} -graded Lie superalgebra \mathcal{G} generated by its local part, and we will always suppose that \mathcal{G} has finite growth. Besides, we will assume that \mathcal{G}_0 is a simple Lie algebra and that \mathcal{G}_1 is an irreducible \mathcal{G}_0 -module which is not contragredient to \mathcal{G}_{-1} . Let us fix a Cartan subalgebra \mathcal{H} of \mathcal{G}_0 and the following notation: let F_Λ be a highest weight vector of \mathcal{G}_{-1} of weight Λ (dominant weight) and let E_M be a lowest weight vector of \mathcal{G}_1 of weight M . As shown in [K2], Proposition 1.2.10, it turns out that $[F_\Lambda, E_M] = e_{-\alpha}$, where $\alpha = -(\Lambda + M)$ is a root of \mathcal{G}_0 and $e_{-\alpha}$ is a root vector in \mathcal{G}_0 corresponding to $-\alpha$. Interchanging, if necessary, \mathcal{G}_k with \mathcal{G}_{-k} we can assume that α is a positive root. Indeed, by transitivity, $[F_\Lambda, E_M] \neq 0$ and for any $t \in \mathcal{H}$ we have:

$$[t, [F_\Lambda, E_M]] = (\Lambda + M)(t)[F_\Lambda, E_M].$$

Notice that $\Lambda + M \neq 0$ since the representations of \mathcal{G}_0 on \mathcal{G}_{-1} and \mathcal{G}_1 are not contragredient.

REMARK 2.1. Under the above assumptions, $-M = \Lambda + \alpha$ is a dominant weight. Therefore $(\Lambda + \alpha, \beta) \geq 0$ for every positive root β of \mathcal{G}_0 .

LEMMA 2.2. *Under the above hypotheses, $[E_M, E_M] = 0$ and $[E_M, [e_\rho, E_M]] = 0$ for every positive root ρ .*

PROOF. We have $[F_\Lambda, [E_M, E_M]] = 2[e_{-\alpha}, E_M] = 0$ since E_M is a lowest weight vector. Transitivity and irreducibility imply $[E_M, E_M] = 0$. Now, since E_M is odd, for every positive root ρ we have:

$$[E_M, [e_\rho, E_M]] = [[E_M, e_\rho], E_M] = -[E_M, [e_\rho, E_M]]$$

therefore $[E_M, [e_\rho, E_M]] = 0$. □

2.1. – Case $(\Lambda, \alpha) = 0$

In this paragraph we suppose $(\Lambda, \alpha) = 0$. If Λ is zero then the depth of \mathcal{G} is finite. Therefore we suppose that Λ is not zero. This implies that the rank of \mathcal{G}_0 is greater than one.

REMARK 2.3. Let \mathcal{G} be a bitransitive, irreducible \mathbb{Z} -graded Lie superalgebra. If $(\Lambda, \alpha) = 0$ then the vectors $[F_\Lambda, F_\Lambda]$ and $[[F_\Lambda, e_{-\rho}], F_\Lambda]$ are zero for every positive root ρ .

PROOF. Once we have shown that $[F_\Lambda, F_\Lambda] = 0$, we proceed as in Lemma 2.2 and conclude that $[[F_\Lambda, e_{-\rho}], F_\Lambda] = 0$ for every positive root ρ . Since $[[F_\Lambda, F_\Lambda], E_M] = 2[F_\Lambda, e_{-\alpha}] = 0$, we conclude by bitransitivity. □

LEMMA 2.4. α is the highest root of one of the parts of the Dynkin diagram of \mathcal{G}_0 into which it is divided by the numerical marks of Λ .

PROOF. Suppose by contradiction that α is not the highest root of one of the parts of the Dynkin diagram of \mathcal{G}_0 into which it is divided by the numerical marks of Λ . Then there exists a simple root β such that $(\Lambda, \beta) = 0$ and $\alpha + \beta$ is a root. This gives a contradiction because: $0 = [[e_{-\beta}, F_\Lambda], E_M] = [e_{-\beta}, [F_\Lambda, E_M]] = e_{-\beta-\alpha} \neq 0$. □

LEMMA 2.5. If Λ has at least two numerical marks then, for every numerical mark γ , we have:

$$(\Lambda + \alpha, \gamma) = 0.$$

PROOF. From Lemma 2.4 we know that α is the highest root of one of the parts of the Dynkin diagram of \mathcal{G}_0 into which it is divided by the numerical marks of Λ . Therefore we can choose a numerical mark β such that $\alpha + \beta$ is a root. Now suppose that γ is a numerical mark, $\gamma \neq \beta$, such that $(\Lambda + \alpha, \gamma) \neq 0$.

Notice that γ and β are not subroots of α , since $(\Lambda, \gamma) \neq 0$ and $(\Lambda, \beta) \neq 0$, therefore $\gamma(h_\alpha) \leq 0$, $\beta(h_\alpha) < 0$.

Consider the following vectors:

$$x := [[[F_\Lambda, e_{-\beta}], e_{-\gamma}], F_\Lambda]$$

$$y := [[[E_M, e_\alpha], e_\gamma], E_M].$$

First of all we want to show that x is a highest weight vector in \mathcal{G}_{-2} . By Remark 2.3, since β and γ are simple roots, it is sufficient to show that $x \neq 0$. In fact, $[e_\gamma, [x, E_M]] = (\Lambda + \alpha)(h_\gamma)[F_\Lambda, [e_{-\alpha}, e_{-\beta}]] \neq 0$.

Now let us prove that y is a lowest weight vector in \mathcal{G}_2 . First $y \neq 0$, indeed:

$$[y, F_\Lambda] = (2 - \gamma(h_\alpha))[E_M, e_\gamma]$$

which is different from 0 since $\gamma(h_\alpha) \leq 0$ and by the assumption $(\Lambda + \alpha, \gamma) \neq 0$.

We now compute the commutators $[y, e_{-\alpha_k}]$ for any simple root α_k . If $\alpha_k = \gamma$ then, by Lemma 2.2, $[y, e_{-\alpha_k}] = 0$, since $\alpha - \gamma$ is not a root. If $\alpha_k \neq \gamma$, $[y, e_{-\alpha_k}] = [[E_M, e_{\alpha - \alpha_k}], e_\gamma, E_M]$, and this can be shown to be zero using the transitivity of \mathcal{G} .

Notice that $[x, y] = (2 - \gamma(h_\alpha))(\Lambda + \alpha)(h_\gamma)e_{-\alpha - \beta}$. By Theorem 1.16 we get a contradiction since $\alpha + \beta$ cannot be the highest root of \mathcal{G}_0 . As a consequence, $(\Lambda + \alpha, \gamma) = 0$. In particular, $\alpha + \gamma$ is a root and we can repeat the same argument interchanging β and γ in order to get $(\Lambda + \alpha, \beta) = 0$. \square

COROLLARY 2.6. *If \mathcal{G}_0 is of type A_n, B_n, C_n, F_4, G_2 then Λ has at most two numerical marks; if \mathcal{G}_0 is of type D_n, E_6, E_7, E_8 then Λ has at most three numerical marks.*

PROOF. Immediate from Lemma 2.5. \square

LEMMA 2.7. *If Λ has only one numerical mark β then either $(\Lambda + \alpha, \beta) = 0$ or $\Lambda(h_\beta) = 1$.*

PROOF. Suppose both $(\Lambda + \alpha, \beta) \neq 0$ and $\Lambda(h_\beta) > 1$, and define

$$x := [[[F_\Lambda, e_{-\beta}], e_{-\beta}], F_\Lambda]$$

$$y := [[[E_M, e_\alpha], e_\beta], E_M].$$

Then x is a highest weight vector in \mathcal{G}_{-2} and y is a lowest weight vector in \mathcal{G}_2 . Besides, $[x, y] = 2(2 - \beta(h_\alpha))(\Lambda + \alpha)(h_\beta)e_{-\alpha - \beta}$. By Theorem 1.16, \mathcal{G}_0 is either of type A_n or of type C_n , $\alpha + \beta$ is the highest root of \mathcal{G}_0 and \mathcal{G}_{-2} is its elementary representation. It is easy to show that these conditions cannot hold. \square

PROPOSITION 2.8. *Let β be a positive root such that:*

- $\alpha + \beta$ is a root;
- $\alpha - \beta$ is not a root;
- $2\alpha + \beta$ is not a root.

Then either $(\Lambda + \alpha, \beta) = 0$ or $\Lambda(h_\beta) = 1$.

PROOF. Let us first make some remarks:

- (a) Since $\beta + \alpha$ is a root but $\beta + 2\alpha$ and $\beta - \alpha$ are not, we have $\beta(h_\alpha) = -1$. It follows that $\alpha + \beta$ and β are roots of the same length.
- (b) Since $\beta - (\alpha + \beta)$ is a root and $\beta - 2(\alpha + \beta)$ is not, then $\beta(h_{\alpha + \beta}) \leq 1$.

Now suppose that $\Lambda(h_\beta) > 1$, which implies $[F_\Lambda, e_{-\beta}] \neq 0$.

Let $x_\mu = E_M$ and $x_\lambda = [F_\Lambda, e_{-\beta}]$. We have:

$$[x_\lambda, x_\mu] = e_{-\alpha - \beta}$$

$$[e_{-\alpha - \beta}, x_\mu] = 0 = [x_\lambda, e_{\alpha + \beta}].$$

Therefore, by Lemma 1.14, we deduce that the difference $\Lambda(h_\beta) - \beta(h_{\alpha+\beta})$ is equal to 0, 1, or 2. In particular, $2 \leq \Lambda(h_\beta) \leq 3$ and $0 \leq \beta(h_{\alpha+\beta}) \leq 1$. We therefore distinguish the following two cases:

CASE A: $\beta(h_{\alpha+\beta}) = 0$, i.e. $\alpha + 2\beta$ is a root, $2\alpha + 3\beta$ is not, and $\Lambda(h_\beta)=2$. In this case $(\beta, \beta) = -(\beta, \alpha)$ and $(\Lambda, \beta) = (\beta, \beta)$ therefore $(\Lambda + \alpha, \beta) = 0$ which concludes the proof in this case.

CASE B: $\beta(h_{\alpha+\beta}) = 1$, i.e. $\alpha + 2\beta$ is not a root, and $\Lambda(h_\beta)$ is either 2 or 3. In this case $\beta(h_\alpha) = -1 = \alpha(h_\beta)$, therefore $(\Lambda + \alpha, \beta) \neq 0$. The two cases $\Lambda(h_\beta) = 2$ and $\Lambda(h_\beta) = 3$ need to be analyzed separately.

(i) $\Lambda(h_\beta) = 2$

Let us define the following elements:

$$x_\lambda = [[[F_\Lambda, e_{-\beta}], e_{-\beta}], F_\Lambda]$$

$$x_\mu = [[[E_M, e_{\alpha+\beta}], e_\beta], E_M].$$

Then $[x_\lambda, x_\mu] = 6e_{-\alpha}$, $[x_\lambda, e_\alpha] = 0$ since $\alpha - \beta$ is not a root, and $[x_\mu, e_{-\alpha}] = 0$ since $(\Lambda + \alpha)(h_\beta) = 1$, thus $[[E_M, e_\beta], e_\beta] = 0$. Then we find a contradiction to Lemma 1.12 applied to the Lie algebra $\mathcal{G}_{\bar{0}}$, since \mathcal{G} was assumed to have finite growth. Indeed, using the same notation as in Lemma 1.12, we have: $\lambda(h_\gamma) = -\lambda(h_\alpha) = -(2\Lambda - 2\beta)(h_\alpha) = 2\beta(h_\alpha) = -2$.

(ii) $\Lambda(h_\beta) = 3$

Let us define the following elements:

$$E_1 = 1/8[[E_M, e_{\alpha+\beta}], [E_M, e_{\alpha+\beta}]]$$

$$F_1 = [[F_\Lambda, e_{-\beta}], [F_\Lambda, e_{-\beta}]]$$

$$E_2 = 1/64[[[E_M, e_{\alpha+\beta}], e_\beta], [[E_M, e_{\alpha+\beta}], e_\beta]]$$

$$F_2 = [[[F_\Lambda, e_{-\beta}], e_{-\beta}], [[F_\Lambda, e_{-\beta}], e_{-\beta}]]$$

$$H = h_{\alpha+\beta} = h_\alpha + h_\beta$$

Then the hypotheses of Lemma 1.11 are satisfied with $a_1 = -4$ and $a_2 = -2$, and this leads to a contradiction. □

In the following, for what concerns simple Lie algebras, we will use the same notation as in [H, §11, §12]. In particular we shall adopt the same enumeration of the vertices in the Dynkin diagrams and refer to the bases of simple roots described by Humphreys [H].

LEMMA 2.9. *Let M be the lowest weight of the \mathcal{G}_0 -module \mathcal{G}_1 .*

- (i) *Let $z := [[E_M, e_{\alpha+\beta}], [e_\gamma, E_M]]$, where β and γ are positive roots of \mathcal{G}_0 such that $[E_M, e_\beta] = 0$, $\alpha + \beta + \gamma$ is not a root, $\beta + \gamma$ is not a root and $\gamma - \alpha$ is a negative root. Then $[z, F_\Lambda] = 0$.*
- (ii) *Let β and ρ be positive roots such that $\alpha + \beta$ and $\beta + \rho$ are positive roots, $\alpha + \beta + \rho$ is not a root, $\rho - \alpha$ is a negative root. If $(M, \beta) = 0$ and $(M, \rho) \neq 0$, then the vector $[[E_M, e_{\alpha+\beta}], [e_\rho, E_M]]$ is non-zero.*
- (iii) *Let β and ρ be as in (ii) and let α_k be a simple root of \mathcal{G}_0 . Suppose, in addition, that either $\rho + \beta - \alpha_k$ is not a root or $(M, \rho + \beta - \alpha_k) = 0$. Then $[[[E_M, e_{\alpha+\beta-\alpha_k}], [e_\rho, E_M]], F_\Lambda] = 0$.*
- (iv) *If ρ is a positive root such that $\alpha + \rho$ is not a root, $\rho - \alpha$ is a negative root, $(M, \rho) \neq 0$ and $\rho(h_\alpha) = 1$, then $[[[E_M, e_\alpha], [e_\rho, E_M]], F_\Lambda] = 0$.*

PROOF. The proof consists of simple direct computations. \square

THEOREM 2.10. *Let \mathcal{G} be an irreducible, simple, \mathbb{Z} -graded Lie superalgebra of finite growth, generated by its local part. Suppose that \mathcal{G}_0 is simple, that the \mathbb{Z} -gradation of \mathcal{G} is consistent and that $(\Lambda, \alpha) = 0$. If \mathcal{G} has infinite depth then one of the following holds:*

- \mathcal{G}_0 is of type A_3 , \mathcal{G}_{-1} is its adjoint module, $\mathcal{G}_1 = V(2\omega_2)$;
- \mathcal{G}_0 is of type B_n ($n \geq 2$), \mathcal{G}_{-1} is its adjoint module, $\mathcal{G}_1 = V(2\omega_1)$;
- \mathcal{G}_0 is of type C_n ($n \geq 3$), $\mathcal{G}_{-1} \cong \Lambda_0^2 sp_{2n}$, \mathcal{G}_1 is its adjoint module;
- \mathcal{G}_0 is of type D_n ($n \geq 4$), \mathcal{G}_{-1} is its adjoint module, $\mathcal{G}_1 = V(2\omega_1)$.

PROOF. Let us analyze all the possible cases. Corollary 2.6 states that if \mathcal{G}_0 is of type A_n , B_n , C_n , F_4 or G_2 then Λ might have one or two numerical marks while if \mathcal{G}_0 is of type D_n , E_6 , E_7 or E_8 then Λ might also have three numerical marks. Using Lemma 2.5 one can easily see that if \mathcal{G}_0 is not of type A_n then the hypothesis that Λ has at least two numerical marks contradicts Proposition 2.8. It follows that if \mathcal{G}_0 is not of type A_n then Λ has exactly one numerical mark and this numerical mark satisfies Lemma 2.7.

Using Remark 2.1 we immediately exclude the following possibilities, for which the weight M is not antidominant:

- \mathcal{G}_0 of type B_n ($n \geq 2$), $\mathcal{G}_{-1} = V(\omega_n)$;
- \mathcal{G}_0 of type C_n ($n \geq 3$), $\mathcal{G}_{-1} = V(\omega_1)$;
- \mathcal{G}_0 of type C_n ($n \geq 3$), $\mathcal{G}_{-1} = V(\omega_i)$ with $2 \leq i \leq n - 1$, $\alpha = 2\alpha_{i+1} + \dots + 2\alpha_{n-1} + \alpha_n$;
- \mathcal{G}_0 of type F_4 , $\mathcal{G}_{-1} = V(\omega_3)$, $\alpha = \alpha_1 + \alpha_2$;
- \mathcal{G}_0 of type F_4 , $\mathcal{G}_{-1} = V(\omega_4)$;
- \mathcal{G}_0 of type G_2 , $\mathcal{G}_{-1} = V(2\omega_1)$;
- \mathcal{G}_0 of type G_2 , $\mathcal{G}_{-1} = V(\omega_1)$ (simplest representation).

Proposition 2.8 allows us to rule out the cases summarized in Table 1, where we describe the irreducible modules \mathcal{G}_{-1} and \mathcal{G}_1 through their highest weights and indicate the positive root β used in Proposition 2.8.

On the other hand, Corollary 1.17 allows us to rule out the cases summarized in Table 2, where the vectors x and y used in Corollary 1.17 are indicated, and where the columns denoted by \mathcal{G}_{-1} and \mathcal{G}_1 contain the highest weights of these \mathcal{G}_0 -modules. In order to show that the vectors x and y in Table 2 are highest and lowest weight vectors in the \mathcal{G}_0 -modules \mathcal{G}_{-2} and \mathcal{G}_2 respectively, one can use the bitransitivity of \mathcal{G} and, where needed, Lemma 2.9.

For the remaining cases let us point out what follows: suppose that \mathcal{G}_{-2} contains a highest weight vector x of weight λ and that \mathcal{G}_2 contains a lowest weight vector y of weight $-\lambda$ such that $[x, y] \neq 0$. Then the irreducible submodules $\bar{\mathcal{G}}_{-2}$ and $\bar{\mathcal{G}}_2$ generated respectively by x and y are dual \mathcal{G}_0 -modules and the Lie subalgebra of $\bar{\mathcal{G}}_0$ with local part $\bar{\mathcal{G}}_{-2} \oplus \bar{\mathcal{G}}_0 \oplus \bar{\mathcal{G}}_2$ is an affine Kac-Moody algebra which will be denoted by \mathcal{A} .

Using the classification of affine Kac-Moody algebras we therefore exclude the cases in Table 3, where we indicate the highest weight vector x of \mathcal{G}_{-2} , the lowest weight vector y of \mathcal{G}_2 , and the highest weights of the \mathcal{G}_0 -modules \mathcal{G}_{-1} and \mathcal{G}_1 .

In the same way the classification of affine Kac-Moody algebras shows that the following cases are allowed:

- 1) \mathcal{G}_0 of type A_3 , $\mathcal{G}_{-1} = V(\omega_1 + \omega_3)$, $\mathcal{G}_1 = V(2\omega_2)$, $\alpha = \alpha_2$: under these hypotheses \mathcal{G}_{-2} contains the highest weight vector $x = [[F_\Lambda, e_{-\alpha_1}], [e_{-\alpha_3}, F_\Lambda]]$ and \mathcal{G}_2 contains the lowest weight vector $y = [[E_M, e_{\alpha_1+\alpha_2}], [e_{\alpha_2+\alpha_3}, E_M]]$. The algebra \mathcal{A} is an affine Kac-Moody algebra of type $A_5^{(2)}$.
- 2) \mathcal{G}_0 of type B_n ($n \geq 3$), $\mathcal{G}_{-1} = V(\omega_2)$, $\mathcal{G}_1 = V(2\omega_1)$, $\alpha = \alpha_1$: \mathcal{G}_{-2} contains the highest weight vector $x = [[F_\Lambda, e_{-\alpha_2}], [e_{-\alpha_2-2\alpha_3-\dots-2\alpha_n}, F_\Lambda]]$ and \mathcal{G}_2 contains the lowest weight vector $y = [[E_M, e_{\alpha_1+\alpha_2}], [e_{\alpha_1+\alpha_2+2\alpha_3+\dots+2\alpha_n}, E_M]]$. The algebra \mathcal{A} is an affine Kac-Moody algebra of type $A_{2n}^{(2)}$.
- 3) \mathcal{G}_0 of type B_2 , $\mathcal{G}_{-1} = V(2\omega_2)$, $\mathcal{G}_1 = V(2\omega_1)$, $\alpha = \alpha_1$: \mathcal{G}_{-2} contains the highest weight vector $x = [[F_\Lambda, e_{-\alpha_2}], [e_{-\alpha_2}, F_\Lambda]]$ and \mathcal{G}_2 contains the lowest weight vector $y = [[E_M, e_{\alpha_1+2\alpha_2}], [e_{\alpha_1}, E_M]]$. The algebra \mathcal{A} is an affine Kac-Moody algebra of type $A_4^{(2)}$.
- 4) \mathcal{G}_0 of type C_n ($n \geq 3$), $\mathcal{G}_{-1} = V(\omega_2)$, $\mathcal{G}_1 = V(2\omega_1)$, $\alpha = \alpha_1$: \mathcal{G}_{-2} contains the highest weight vector $x = [[F_\Lambda, e_{-\alpha_2-\dots-\alpha_n}], [e_{-\alpha_1-\dots-\alpha_{n-1}}, F_\Lambda]]$ and \mathcal{G}_2 contains the lowest weight vector $y = [[E_M, e_{\alpha_1}], [e_{2\alpha_1+\dots+2\alpha_{n-1}+\alpha_n}, E_M]]$. The algebra \mathcal{A} is an affine Kac-Moody algebra of type $A_{2n-1}^{(2)}$.
- 5) \mathcal{G}_0 of type D_n ($n \geq 4$), $\mathcal{G}_{-1} = V(\omega_2)$, $\mathcal{G}_1 = V(2\omega_1)$, $\alpha = \alpha_1$: in this case $x = [[F_\Lambda, e_{-\alpha_2}], [e_{-\alpha_2-2\alpha_3-\dots-2\alpha_{n-2}-\alpha_{n-1}-\alpha_n}, F_\Lambda]]$ and $y = [[E_M, e_{\alpha_1+\alpha_2}], [e_{\alpha_1+\alpha_2+2\alpha_3+\dots+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n}, E_M]]$. The algebra \mathcal{A} is an affine Kac-Moody algebra of type $A_{2n-1}^{(2)}$.

TABLE I

\mathcal{G}_0	\mathcal{G}_{-1}	\mathcal{G}_1	α	β	
B_n	ω_j	$\omega_1 + \omega_{j-1}$	$\alpha_1 + \dots + \alpha_{j-1}$	$\alpha_{i-1} + 2\alpha_i + \dots + 2\alpha_n$	$3 \leq i \leq n-1$
B_n	$2\omega_n$	$\omega_1 + \omega_{n-1}$	$\alpha_1 + \dots + \alpha_{n-1}$	$\alpha_{n-1} + 2\alpha_n$	$n > 2$
C_n	ω_j	$\omega_1 + \omega_{j-1}$	$\alpha_1 + \dots + \alpha_{j-1}$	$\alpha_{i-1} + 2\alpha_i + \dots + 2\alpha_{n-1} + \alpha_n$	$3 \leq i \leq n-1$
D_n	ω_j	$\omega_1 + \omega_{j-1}$	$\alpha_1 + \dots + \alpha_{j-1}$	$\alpha_{i-1} + 2\alpha_i + \dots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$	$3 \leq i \leq n-2$
E_6	ω_3	$\omega_1 + \omega_2$	$\alpha_2 + \alpha_4 + \alpha_5 + \alpha_6$	$\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5$	
E_6	ω_5	$\omega_2 + \omega_6$	$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$	$\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6$	
E_6	ω_4	$\omega_5 + \omega_6$	$\alpha_1 + \alpha_2$	$\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5$	
E_6	$2\omega_2$	$2\omega_2$	α_2	$\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$	
E_7	ω_3	$2\omega_1$	α_1	$\alpha_1 + 2\omega_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$	
E_7	ω_3	$\omega_2 + \omega_7$	$\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7$	$\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5$	
E_7	ω_2	$\omega_1 + \omega_7$	$\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7$	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$	
E_7	ω_4	$\omega_1 + \omega_3$	$\alpha_1 + \alpha_3$	$\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5$	
E_7	ω_4	$2\omega_2$	α_2	$\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$	
E_7	ω_4	$\omega_5 + \omega_7$	$\alpha_5 + \alpha_6 + \alpha_7$	$\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5$	
E_7	ω_5	$\omega_1 + \omega_2$	$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$	$\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6$	
E_7	ω_5	$\omega_6 + \omega_7$	$\alpha_6 + \alpha_7$	$\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6$	
E_7	ω_6	ω_3	$\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5$	$\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5$	
E_8	ω_1	ω_7	$\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_7 + \alpha_8$	$\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6$	
E_8	ω_3	$\omega_2 + \omega_8$	$\alpha_2 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8$	$\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6$	
E_8	ω_2	$\omega_1 + \omega_8$	$\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8$	$\alpha_1 + \alpha_2 + 2\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8$	
E_8	ω_k	$\omega_{k+1} + \omega_8$	$\alpha_{k+1} + \dots + \alpha_8$	$\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_4 + \alpha_5$	
E_8	ω_7	$2\omega_8$	α_8	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$	
F_4	ω_2	$2\omega_1$	α_1	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 4\alpha_3 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + \alpha_8$	$4 \leq k \leq 6$
F_4	ω_2	$\omega_3 + \omega_4$	$\alpha_3 + \alpha_4$	$\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$	
F_4	$2\omega_3$	$\omega_1 + \omega_2$	$\alpha_1 + \alpha_2$	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4$	
F_4	ω_3	$2\omega_4$	α_4	$\alpha_2 + 2\alpha_3$	
F_4	$2\omega_4$	ω_2	$\alpha_1 + 2\alpha_2 + 2\alpha_3$	$\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4$	
G_2	$3\omega_1$	$2\omega_2$	α_2	$\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4$	
				$3\alpha_1 + \alpha_2$	

TABLE 2

\mathcal{G}_0	\mathcal{G}_{-1}	\mathcal{G}_1	x	y
A_n	ω_s	$\omega_{n-s+2} + \omega_h$	$[[F_\Lambda, e_{-\alpha_s}], [e_{-\alpha_{s-1} - \alpha_s - \alpha_{s+1}}, F_\Lambda]]$	$[[E_M, e_{\alpha_s}], [e_{\alpha_{s-1}}, E_M]]$
A_n	ω_s	$\omega_1 + \omega_{h-s}$	$[[F_\Lambda, e_{-\alpha_s}], [e_{-\alpha_{s-1} - \alpha_s - \alpha_{s+1}}, F_\Lambda]]$	$[[E_M, e_{\alpha+\alpha_s}], [e_{\alpha_{s+1}}, E_M]]$
A_n	$\omega_s + \omega_t$	$\omega_{n-s} + \omega_{h-t+2}$	$[[F_\Lambda, e_{-\alpha_s}], [e_{-\alpha_t}, F_\Lambda]]$	$[[E_M, e_{\alpha+\alpha_s}], [e_{\alpha_{s+1}}, E_M]]$
B_n	ω_s	ω_{s+2}	$[[F_\Lambda, e_{-\alpha_s}], [e_{-\alpha_{s-1} - \alpha_s - \alpha_{s+1}}, F_\Lambda]]$	$[[E_M, e_{\alpha+\alpha_s}], [e_{\alpha_{s+1} + \alpha_{s+2}}, E_M]]$
B_n	ω_{h-2}	$2\omega_h$	$[[F_\Lambda, e_{-\alpha_{h-2}}], [e_{-\alpha_{h-3} - \alpha_{h-2} - \alpha_{h-1}}, F_\Lambda]]$	$[[E_M, e_{\alpha+\alpha_{h-2}}], [e_{\alpha_{h-1} + \alpha_{h-2}}, E_M]]$
B_n	ω_{h-1}	$2\omega_h$	$[[F_\Lambda, e_{-\alpha_{h-1}}], [e_{-\alpha_{h-2} - \alpha_{h-1} - \alpha_h}, F_\Lambda]]$	$[[E_M, e_{\alpha_{h-1}}], [e_{\alpha_{h-1} + 2\alpha_h}, E_M]]$
C_n	$2\omega_j$	$2\omega_{j+1}$	$[[F_\Lambda, e_{-\alpha_j}], [e_{-\alpha_j}, F_\Lambda]]$	$[[E_M, e_{\alpha+\alpha_j}], [e_{\alpha_{j+1}}, E_M]]$
C_n	ω_h	$\omega_1 + \omega_{h-1}$	$[[F_\Lambda, e_{-\alpha_{h-1} - \alpha_h}], [e_{-\alpha_{h-1} - \alpha_h}, F_\Lambda]]$	$[[E_M, e_{\alpha+\alpha_h}], [e_{\alpha_{h-1}}, E_M]]$
D_n	ω_i	ω_{i+2}	$[[F_\Lambda, e_{-\alpha_i}], [e_{-\alpha_{i-1} - \alpha_i - \alpha_{i+1}}, F_\Lambda]]$	$[[E_M, e_{\alpha+\alpha_i}], [e_{\alpha_{i+1} + \alpha_{i+2}}, E_M]]$
D_n	ω_{h-3}	$\omega_{h-1} + \omega_h$	$[[F_\Lambda, e_{-\alpha_{h-3}}], [e_{-\alpha_{h-4} - \alpha_{h-3} - \alpha_{h-2}}, F_\Lambda]]$	$[[E_M, e_{\alpha+\alpha_{h-3}}], [e_{\alpha_{h-2} + \alpha_{h-1}}, E_M]]$
D_n	ω_h	$\omega_1 + \omega_h$	$[[F_\Lambda, e_{-\alpha_h}], [e_{-\alpha_{h-3} - 2\alpha_{h-2} - \alpha_{h-1} - \alpha_h}, F_\Lambda]]$	$[[E_M, e_{\alpha+\alpha_h}], [e_{\alpha_{h-2} + \alpha_{h-1}}, E_M]]$
D_n	ω_{h-1}	$\omega_1 + \omega_{h-1}$	$[[F_\Lambda, e_{-\alpha_{h-1}}], [e_{-\alpha_{h-3} - 2\alpha_{h-2} - \alpha_{h-1} - \alpha_h}, F_\Lambda]]$	$[[E_M, e_{\alpha+\alpha_{h-1}}], [e_{\alpha_{h-2} + \alpha_{h-1}}, E_M]]$
E_6	ω_1	ω_3	$[[F_\Lambda, e_{-\alpha_1}], [e_{-(\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5)}, F_\Lambda]]$	$[[E_M, e_{\alpha+\alpha_1}], [e_{\alpha_3 + \alpha_4 + \alpha_5}, E_M]]$
E_6	ω_6	ω_5	$[[F_\Lambda, e_{-\alpha_6}], [e_{-(\omega_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6)}, F_\Lambda]]$	$[[E_M, e_{\alpha+\alpha_6}], [e_{\alpha_3 + \alpha_4 + \alpha_5}, E_M]]$
E_6	ω_2	$\omega_1 + \omega_6$	$[[F_\Lambda, e_{-\alpha_2}], [e_{-(\omega_2 + \alpha_3 + 2\alpha_4 + \alpha_5)}, F_\Lambda]]$	$[[E_M, e_{\alpha+\alpha_2}], [e_{\alpha_4 + \alpha_5 + \alpha_6}, E_M]]$
E_7	ω_1	ω_6	$[[F_\Lambda, e_{-\alpha_1}], [e_{-(\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5)}, F_\Lambda]]$	$[[E_M, e_{\alpha+\alpha_1}], [e_{\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6}, E_M]]$
E_7	ω_7	ω_2	$[[F_\Lambda, e_{-\alpha_7}], [e_{-\gamma}, F_\Lambda]]$	$[[E_M, e_{\alpha+\alpha_7}], [e_{\alpha_2 + \alpha_4 + \alpha_5 + \alpha_6}, E_M]]$
E_8	ω_8	ω_1	$[[F_\Lambda, e_{-\alpha_8}], [e_{-\gamma}, F_\Lambda]]$	$[[E_M, e_{\alpha+\alpha_8}], [e_{\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7}, E_M]]$
E_8	ω_7	ω_2	$[[F_\Lambda, e_{-\alpha_7}], [e_{-\gamma}, F_\Lambda]]$	$[[E_M, e_{\alpha+\alpha_7}], [e_{\alpha_2 + \alpha_4 + \alpha_5 + \alpha_6}, E_M]]$
F_4	ω_1	$2\omega_4$	$[[F_\Lambda, e_{-\alpha_1}], [e_{-\alpha_1 - 2\alpha_2 - 2\alpha_3}, F_\Lambda]]$	$[[E_M, e_{\alpha+\alpha_1}], [e_{\alpha_2 + \alpha_3 + \alpha_4}, E_M]]$
G_2	Ad	$2\omega_1$	$[[F_\Lambda, e_{-\alpha_1 - \alpha_2}], [e_{-\alpha_1 - \alpha_2}, F_\Lambda]]$	$[[E_M, e_{2\alpha_1 + \alpha_2}], [e_{\alpha_1}, E_M]]$

$n \geq 5, s \neq 1, 2, n$
 $n \geq 5, s \neq 1, n-1, n$
 $s \geq 1, s+2 < t \leq n$
 $2 \leq s \leq n-3$
 $n \geq 3$
 $1 \leq i \leq n-2$
 $2 \leq i \leq n-4$
 $n > 4$
 $n > 4$

$\gamma = \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7$
 $\gamma = \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8$
 $\gamma = \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7$

TABLE 3

\mathcal{G}_0	\mathcal{G}_1	x	y
A_n ($n \geq 4$)	$\omega_s + \omega_{s+2}$ ($1 \leq s \leq n-2$)	$[[F_\Lambda, e_{-\alpha_s}], [e_{-\alpha_{s+2}}, F_\Lambda]]$	$[[E_M, e_{\alpha_s}], [e_{\alpha_{s+1} + \alpha_{s+2}}, E_M]]$
C_n ($n \geq 3$)	$2\omega_{n-1}$	$[[F_\Lambda, e_{-\alpha_{n-1}}], [e_{-\alpha_{n-1}}, F_\Lambda]]$	$[[E_M, e_{\alpha_{n-1} + \alpha_n}], [e_{\alpha_{n-1} + \alpha_n}, E_M]]$
D_n ($n > 4$)	ω_{n-2}	$[[F_\Lambda, e_{-\alpha_{n-3} - \alpha_{n-2} - \alpha_n}], [e_{-\alpha_{n-2}}, F_\Lambda]]$	$[[E_M, e_{\alpha_{n-3} + \alpha_{n-2} + \alpha_{n-1}}], [e_{\alpha_{n-2} + \alpha_{n-1} + \alpha_n}, E_M]]$
D_n ($n > 4$)	ω_{n-2}	$[[F_\Lambda, e_{-\alpha_{n-3} - \alpha_{n-2} - \alpha_{n-1}}], [e_{-\alpha_{n-2}}, F_\Lambda]]$	$[[E_M, e_{\alpha_{n-3} + \alpha_{n-2} + \alpha_n}], [e_{\alpha_{n-2} + \alpha_{n-1} + \alpha_n}, E_M]]$
E_6	ω_3	$[[F_\Lambda, e_{-\alpha_3}], [e_{-\alpha_2 - \alpha_3 - 2\alpha_4 - \alpha_5}, F_\Lambda]]$	$[[E_M, e_{\alpha_1}], [e_{\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5}, E_M]]$
E_6	ω_5	$[[F_\Lambda, e_{-\alpha_5}], [e_{-\alpha_2 - \alpha_3 - 2\alpha_4 - \alpha_5}, F_\Lambda]]$	$[[E_M, e_{\alpha_6}], [e_{\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6}, E_M]]$
E_7	ω_6	$[[F_\Lambda, e_{-\alpha_6}], [e_{-\alpha_2 - \alpha_3 - 2\alpha_4 - 2\alpha_5 - \alpha_6}, F_\Lambda]]$	$[[E_M, e_{\alpha_7}], [e_{\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7}, E_M]]$

We finally analyze and rule out the remaining cases:

- \mathcal{G}_0 of type A_n , $\mathcal{G}_{-1} = V(\omega_s)$:
 - (i) if $s = 1$ (or, equivalently, $s = n$) then $\mathcal{G}_1 = V(\omega_1 + \omega_{n-1})$ and $\mathcal{G}_{-2} \subset S^2\mathcal{G}_{-1} = S^2V(\omega_1) = 0$ since $S^2V(\omega_1) = V(2\omega_1)$ and $[F_\Lambda, F_\Lambda] = 0$, therefore \mathcal{G} has finite depth;
 - (ii) if $s = 2$, $\mathcal{G}_1 = V(2\omega_n)$, $\alpha = \alpha_1$, (or, equivalently, $s = n - 1$, $\mathcal{G}_1 = V(2\omega_1)$, $\alpha = \alpha_n$) then \mathcal{G} is isomorphic to the finite-dimensional Lie superalgebra $p(n)$ (for the definition of $p(n)$ see [K2]);
 - (iii) if $n = 4$ and $s = 3$, i.e. $\mathcal{G}_{-1} \cong \Lambda^2 sl_5^*$, $\mathcal{G}_1 = V(\omega_3 + \omega_4)$, $\alpha = \alpha_1 + \alpha_2$ (or, equivalently, $\mathcal{G}_{-1} = V(\omega_2)$, $\mathcal{G}_1 = V(\omega_1 + \omega_2)$, $\alpha = \alpha_3 + \alpha_4$), then \mathcal{G} is isomorphic to the infinite-dimensional Lie superalgebra $E(5, 10)$ (for the definition of $E(5, 10)$ see [K3]).

- \mathcal{G}_0 of type B_n ($n \geq 2$), $\mathcal{G}_{-1} = V(\omega_1)$ and:
 - (i) $\mathcal{G}_1 = V(\omega_3)$ if $n > 3$ ($\alpha = \alpha_2 + 2\alpha_3 + \dots + 2\alpha_n$),
 - (ii) $\mathcal{G}_1 = V(2\omega_3)$ if $n = 3$ ($\alpha = \alpha_2 + 2\alpha_3$),
 - (iii) $\mathcal{G}_1 = V(\omega_2)$ if $n = 2$ ($\alpha = \alpha_2$).

For all these cases $\mathcal{G}_{-2} \subset S^2\mathcal{G}_{-1} = S^2V(\omega_1) = V(2\omega_1) + 1 = 1$ since $[F_\Lambda, F_\Lambda] = 0$. Thus \mathcal{G} has finite depth.

- \mathcal{G}_0 of type D_n ($n \geq 4$):
 - (i) $\mathcal{G}_{-1} = V(\omega_1)$, $\mathcal{G}_1 = V(\omega_1 + \omega_3)$, then $\mathcal{G}_{-2} \subset S^2\mathcal{G}_{-1} = S^2V(\omega_1) = V(2\omega_1) + 1 = 1$ hence \mathcal{G} has finite depth.
 - (ii) $n = 4$, $\mathcal{G}_{-1} = V(\omega_4)$, $\mathcal{G}_1 = V(\omega_1 + \omega_4)$ ($\alpha = \alpha_1 + \alpha_2 + \alpha_3$) (or, equivalently, $\mathcal{G}_{-1} = V(\omega_3)$, $\mathcal{G}_1 = V(\omega_1 + \omega_3)$), then we can use the same argument as in (i) and conclude. □

2.2. – Case $(\Lambda, \alpha) \neq 0$

In the following we assume $(\Lambda, \alpha) \neq 0$.

REMARK 2.11. Under the hypothesis $(\Lambda, \alpha) \neq 0$ the vector $[F_\Lambda, F_\Lambda]$ is different from 0: $[E_M, [F_\Lambda, F_\Lambda]] = 2[e_{-\alpha}, F_\Lambda] \neq 0$. Nevertheless, $[E_M, E_M] = 0$ and therefore $[[E_M, e_\beta], E_M] = 0$ for every positive root β (see Lemma 2.2).

COROLLARY 2.12. *If $(\Lambda, \alpha) \neq 0$ then either $(\Lambda, \alpha) = (\alpha, \alpha)$ or $(\Lambda, \alpha) = (\alpha, \alpha)/2$.*

PROOF. It is enough to apply Lemma 1.14 to the following vectors:

$$x_\lambda = F_\Lambda, \quad x_\mu = E_M. \quad \square$$

LEMMA 2.13. *Suppose that α is not simple. Then there exists j such that: $\alpha - \alpha_j$ is a root and $\alpha + \alpha_j$, $2\alpha - \alpha_j$, $\alpha - 2\alpha_j$ are not roots, in all cases except those in the following list:*

- \mathcal{G}_0 of type B_n and $\alpha = \alpha_i + \alpha_{i+1} + \cdots + \alpha_n$, $\alpha = \alpha_{n-1} + 2\alpha_n$;
- \mathcal{G}_0 of type C_n and $\alpha = 2\alpha_i + \cdots + 2\alpha_{n-1} + \alpha_n$, $\alpha = \alpha_{n-1} + \alpha_n$;
- \mathcal{G}_0 of type F_4 and $\alpha = \alpha_1 + \alpha_2 + \alpha_3$, $\alpha = \alpha_2 + \alpha_3$, $\alpha = \alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4$,
 $\alpha = \alpha_2 + 2\alpha_3$, $\alpha = \alpha_2 + 2\alpha_3 + 2\alpha_4$, $\alpha = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4$;
- \mathcal{G}_0 of type G_2 and $\alpha = 2\alpha_1 + \alpha_2$, $\alpha = \alpha_1 + \alpha_2$, $\alpha = 3\alpha_1 + \alpha_2$.

PROOF. Case by case check. □

LEMMA 2.14. *Let α be a positive root of \mathcal{G}_0 and suppose that it is not simple. If α_j is a simple root of \mathcal{G}_0 such that $\alpha - \alpha_j$ is a root and $\alpha + \alpha_j$, $2\alpha - \alpha_j$, $\alpha - 2\alpha_j$ are not roots, then either*

$$\tilde{x} := [[[[E_M, e_{\alpha_j}], e_{\alpha - \alpha_j}], E_M]]$$

is a lowest weight vector in \mathcal{G}_{-2} or $\tilde{x} = 0$ and

$$x := [[[[E_M, e_{\alpha_j}], e_\alpha], E_M]]$$

is a lowest weight vector in \mathcal{G}_{-2} .

PROOF. If $\tilde{x} \neq 0$ then, using the transitivity of \mathcal{G} , one can show that it is a lowest weight vector in \mathcal{G}_{-2} . If $\tilde{x} = 0$ then $[x, e_{-k}] = 0$ for every $k = 1, \dots, n$. □

PROPOSITION 2.15. *If α is not a simple root and the growth of \mathcal{G} is finite then either (\mathcal{G}_0, α) belongs to the list in Lemma 2.13 or $(\mathcal{G}_0, \alpha) = (A_n, \text{longest root})$ and $\tilde{x} := [[[[E_M, e_{\alpha_j}], e_{\alpha - \alpha_j}], E_M]] \neq 0$.*

PROOF. Suppose that (\mathcal{G}_0, α) is not in the list in Lemma 2.13. Since α is not a simple root we can apply Lemma 2.14: in the case $\tilde{x} = 0$ we take $y = [F_\Lambda, F_\Lambda]$. Then $[x, y] = 2\Lambda(h_\alpha)e_{-\alpha + \alpha_j} \neq 0$, and, by Theorem 1.16, we get infinite growth.

If $\tilde{x} := [[[[E_M, e_{\alpha_j}], e_{\alpha - \alpha_j}], E_M]] \neq 0$ then, by bitransitivity, $[\tilde{x}, F_\Lambda] = (\Lambda(h_\alpha) - 2\Lambda(h_j))E_M \neq 0$, thus $[\tilde{x}, y] = (\Lambda(h_\alpha) - 2\Lambda(h_j))e_{-\alpha}$ is different from zero. Then the thesis follows from Theorem 1.16. (Notice that the case \mathcal{G}_0 of type C_n , α its longest root, is in the list of Lemma 2.13 and is therefore excluded by the hypotheses.) □

LEMMA 2.16. *If the growth of \mathcal{G} is finite and β is a positive root such that $\alpha + \beta$ and $\alpha - \beta$ are not roots, then $(\Lambda, \beta) = 0$.*

PROOF. Suppose $(\Lambda, \beta) \neq 0$. We define:

$$\begin{aligned} E_1 &= [e_\alpha, E_M], & E_2 &= [[E_M, e_\alpha], e_\beta], \\ F_1 &= F_\Lambda, & F_2 &= \Lambda(h_\beta)^{-1}[F_\Lambda, e_{-\beta}], \\ H &= h_\alpha. \end{aligned}$$

It is easy to verify that the conditions of Lemma 1.13 are satisfied with $a_1 = a_2 = -\Lambda(h_\alpha)$, thus $r(\mathcal{G}) = \infty$. □

THEOREM 2.17. *Let $\mathcal{G} = \bigoplus_{i \in \mathbb{Z}} \mathcal{G}_i$ be a \mathbb{Z} -graded, consistent, simple, irreducible Lie superalgebra of finite growth. Assume that \mathcal{G}_0 is a simple Lie algebra, that \mathcal{G}_1 is an irreducible \mathcal{G}_0 -module which is not contragredient to \mathcal{G}_{-1} and that the local part generates \mathcal{G} . Let F_Λ be a highest weight vector in \mathcal{G}_{-1} and E_M a lowest weight vector in \mathcal{G}_1 so that $\Lambda + M = -\alpha$ for a positive root α . If $(\Lambda, \alpha) \neq 0$ then \mathcal{G}_0 has rank 1.*

PROOF. By Proposition 2.15 and its proof only the following cases may occur:

- α is a simple root;
- $(\mathcal{G}_0, \alpha) = (A_n, \text{longest root})$;
- (\mathcal{G}_0, α) is in the list of Lemma 2.13.

Let us analyze these possibilities case by case:

- 1) \mathcal{G}_0 of type A_n , $\alpha = \alpha_1 + \dots + \alpha_n$. If $n = 1$ we get the thesis. Now suppose $n \geq 2$. The proof of Proposition 2.15 shows that this possibility holds if

$$\tilde{x} = [[[E_M, e_j], e_{\alpha - \alpha_j}], E_M]$$

is a nonzero vector, thus either $j = 1$ or $j = n$. If we apply Lemma 2.16 to $\alpha = \alpha_1 + \dots + \alpha_n$ and $\beta = \alpha_2 + \dots + \alpha_{n-1}$ we deduce that $(\Lambda, \alpha_i) = 0$ for every $i = 2, \dots, n - 1$, therefore $(\Lambda, \alpha) = (\Lambda, \alpha_1) + (\Lambda, \alpha_n)$.

As we already noticed in the proof of Proposition 2.15, for every $k = 1, \dots, n$, $[\tilde{x}, e_{-k}] = 0$ thus, since we assume $\tilde{x} \neq 0$, transitivity implies $[\tilde{x}, F_\Lambda] \neq 0$. Since $[\tilde{x}, F_\Lambda] = (\Lambda(h_\alpha) - 2\Lambda(h_j))E_M$, it turns out that $\Lambda(h_1) \neq \Lambda(h_n)$. Corollary 2.12 now implies that either $(\Lambda, \alpha_1) = 0$ or $(\Lambda, \alpha_n) = 0$. But this hypothesis contradicts Theorem 1.16, since if we take the highest weight vector $y = [F_\Lambda, F_\Lambda]$ in \mathcal{G}_{-2} , then $[\tilde{x}, y] \neq 0$ but the irreducible submodule of \mathcal{G}_{-2} generated by $[F_\Lambda, F_\Lambda]$ is not the standard A_n -module.

- 2) \mathcal{G}_0 of type A_n , α simple, $n \geq 2$.

- 2a) $n \geq 3$, $\alpha = \alpha_j$ with $j \neq 1, n$

If we apply Lemma 2.16 with $\alpha = \alpha_j$ and $\beta = \alpha_{j-1} + \alpha_j + \alpha_{j+1}$ we find a contradiction.

- 2b) $\alpha = \alpha_1$ (or, equivalently, $\alpha = \alpha_n$).

Again, by applying Lemma 2.16 with $\beta = \alpha_3 + \dots + \alpha_n$, we find $(\Lambda, \alpha_i) = 0$ for every $i \geq 3$. On the other hand, $(\Lambda, \alpha_2) \neq 0$ since $[E_M, [F_\Lambda, e_{-\alpha_2}]] = e_{-\alpha_1 - \alpha_2} \neq 0$. We distinguish two cases:

CASE 1: $(\Lambda, \alpha_2) \neq 1$

Under this hypothesis let us consider the following vectors:

$$x_\mu = [[[E_M, e_1], [E_M, e_2]], [E_M, e_1]],$$

$$x_\lambda = \Lambda(h_1)^{-1}(1 - \Lambda(h_2))^{-1}(3 + \Lambda(h_1))^{-1}[F_\Lambda, [F_\Lambda, [F_\Lambda, e_{-2}]]].$$

Then x_λ and x_μ satisfy the hypotheses of Lemma 1.14 with $\delta = \alpha_1$. Since $(3\Lambda - \alpha_2, \alpha_1) = 3(\Lambda, \alpha_1) + 1 \geq 4$ we find a contradiction.

CASE 2: $(\Lambda, \alpha_2) = 1$

By Corollary 2.12, either $\Lambda(h_1) = 1$ or $\Lambda(h_1) = 2$. Notice that $x := [F_\Lambda, F_\Lambda]$ is a highest weight vector in \mathcal{G}_{-2} and $y := [[E_M, e_1], [E_M, e_1]]$ is a lowest weight vector in \mathcal{G}_2 . Since $[x, y] = -4\Lambda(h_1)h_1$, \mathcal{G}_0 contains a \mathbb{Z} -graded Lie subalgebra with local part $s_{-2} \oplus \mathcal{G}_0 \oplus s_2$, where s_{-2} is the irreducible submodule of \mathcal{G}_{-2} generated by x and s_2 is the irreducible submodule of \mathcal{G}_2 generated by y . The classification of Kac-Moody Lie algebras immediately allows us to rule out the case $\Lambda(h_1) = 2$ and the case $\Lambda(h_1) = 1, n > 2$.

Now suppose $n = 2, \Lambda(h_1) = 1 = \Lambda(h_2)$. Under these hypotheses \mathcal{G}_{-2} contains the highest weight vector

$$z := -4[[F_\Lambda, e_{-\alpha_1-\alpha_2}], F_\Lambda] + 5[[[F_\Lambda, e_{-\alpha_1}], e_{-\alpha_2}], F_\Lambda] - 3[[[F_\Lambda, e_{-\alpha_2}], F_\Lambda], e_{-\alpha_1}]$$

of weight Λ . Besides, $[z, y] = -24e_{-\alpha_1-\alpha_2}$ and this contradicts Theorem 1.16 since the irreducible \mathcal{G}_0 -submodule of \mathcal{G}_{-2} containing z is the adjoint module and not the standard one.

3) \mathcal{G}_0 of type B_n ($n \geq 2$), $\alpha = \alpha_i + \dots + \alpha_n$ ($1 \leq i \leq n-1$).

3a) If $i > 1$ take $\beta = \alpha_{i-1} + \alpha_i + 2\alpha_{i+1} + \dots + 2\alpha_n$, then $\alpha + \beta$ and $\alpha - \beta$ are not roots and, by Lemma 2.16, $(\Lambda, \beta) = 0$, i.e. $(\Lambda, \alpha_j) = 0$ for every $j \geq i-1$ which contradicts the hypothesis $(\Lambda, \alpha) \neq 0$.

3b) If $i = 1$ and $n \geq 3$ take $\beta = \alpha_2 + \dots + 2\alpha_n$. Then, by Lemma 2.16, $(\Lambda, \alpha_i) = 0$ for every $i \neq 1$. This implies the following contradiction:

$$0 = [E_M, [F_\Lambda, e_{-\alpha_n}]] = [e_{-\alpha}, e_{-\alpha_n}] \neq 0.$$

3c) Let $i = 1$ and $n = 2$, i.e. $\alpha = \alpha_1 + \alpha_2$.

If $(\Lambda, \alpha_2) = 0$, as above we have:

$$0 = [E_M, [F_\Lambda, e_{-\alpha_2}]] = [e_{-\alpha}, e_{-\alpha_2}] \neq 0.$$

Thus suppose $(\Lambda, \alpha_2) \neq 0$. Since α and α_2 have both length 1, Corollary 2.12 implies $(\Lambda, \alpha_1) = 0$ and either $\Lambda(h_2) = 1$ or $\Lambda(h_2) = 2$. Notice that \mathcal{G}_{-2} contains the highest weight vector $x := [F_\Lambda, F_\Lambda]$. Now, if $\Lambda(h_2) = 1$ then \mathcal{G}_2 contains the lowest weight vector $y := [[E_M, e_{\alpha_1}], [E_M, e_{\alpha_2}]]$ and $[x, y] = 2e_{-\alpha}$ thus \mathcal{G}_0 has infinite growth according to Theorem 1.16.

If $\Lambda(h_2) = 2$, by bitransitivity, then $y = 0$ and the vector

$$z := [[E_M, e_{\alpha_1+\alpha_2}], [E_M, e_{\alpha_2}]]$$

is a lowest weight vector in \mathcal{G}_2 . Again, since $[x, z] = -8e_{-\alpha_1}$, this contradicts Theorem 1.16.

- 4) \mathcal{G}_0 of type B_n , α simple.
 - 4a) If $\alpha = \alpha_i$ with $i \neq 1, n$, we proceed as for A_n .
 - 4b) If $\alpha = \alpha_1$ we take $\beta = \alpha_1 + 2\alpha_2 + \dots + 2\alpha_n$ and apply Lemma 2.16.
 - 4c) If $\alpha = \alpha_n$ and $n \geq 3$ we take $\beta = \alpha_{n-2} + 2\alpha_{n-1} + 2\alpha_n$. Then Lemma 2.16 holds and we get a contradiction.
 - 4d) $n = 2$, $\alpha = \alpha_2$. In this case relation $[E_M, [F_\Lambda, e_{-\alpha_1}]] = e_{-\alpha_2-\alpha_1}$ implies $(\Lambda, \alpha_1) \neq 0$. This possibility is therefore ruled out by the classification of Kac-Moody Lie algebras once we have noticed that since \mathcal{G}_{-2} contains the highest weight vector $x := [F_\Lambda, F_\Lambda]$ and \mathcal{G}_2 contains the lowest weight vector $y := [[E_M, e_{\alpha_2}], [E_M, e_{\alpha_2}]]$, with $[x, y] \neq 0$, \mathcal{G}_0 contains an affine Kac-Moody, \mathbb{Z} -graded Lie subalgebra with local part $s_{-2} \oplus \mathcal{G}_0 \oplus s_2$, where s_{-2} is the \mathcal{G}_0 -irreducible module with highest weight 2Λ and s_2 is the \mathcal{G}_0 -module contragredient to s_{-2} .
- 5) \mathcal{G}_0 of type B_n , $\alpha = \alpha_{n-1} + 2\alpha_n$.
 - 5a) If $n \geq 3$ take $\beta = \alpha_{n-2} + \alpha_{n-1} + \alpha_n$ and use Lemma 2.16.
 - 5b) Let $n = 2$, $\alpha = \alpha_1 + 2\alpha_2$. If we take $\beta = \alpha_1$ then Lemma 2.16 implies $(\Lambda, \alpha_1) = 0$ thus $\Lambda(h_\alpha) = \Lambda(h_2)$ is either 1 or 2. One can easily verify, using the bitransitivity of \mathcal{G} , that the vector $z := [[E_M, e_{\alpha_1+\alpha_2}], [E_M, e_{\alpha_2}]]$ is equal to 0, the vector $y := [[E_M, e_{\alpha_1+2\alpha_2}], [E_M, e_{\alpha_2}]]$ is a lowest weight vector in \mathcal{G}_2 and, as in the previous cases, $x := [F_\Lambda, F_\Lambda]$ is a highest weight vector in \mathcal{G}_{-2} . Since $[x, y] = 24(\Lambda(h_2) + 1)e_{-\alpha_1-\alpha_2}$ this contradicts Theorem 1.16.
- 6) \mathcal{G}_0 of type C_n ($n \geq 3$), $\alpha = 2\alpha_i + \dots + 2\alpha_{n-1} + \alpha_n$ ($1 \leq i \leq n - 1$).
 - If $i \neq 1$ we apply Lemma 2.16 to $\beta = \alpha_{i-1} + \alpha_i + 2\alpha_{i+1} + \dots + 2\alpha_{n-1} + \alpha_n$ and get a contradiction.
 - If $i = 1$ take $\beta = 2\alpha_2 + \dots + 2\alpha_{n-1} + \alpha_n$. Then Lemma 2.16 implies $(\Lambda, \alpha_i) = 0$ for every $i \geq 2$. Thus $(\Lambda, \alpha) = 2(\Lambda, \alpha_1)$. Consider the following vectors:

$$x = [F_\Lambda, F_\Lambda]$$

$$y = [[[E_M, e_1], e_\alpha], E_M].$$

Then x is a highest weight vector in \mathcal{G}_{-2} and y is a lowest weight vector in \mathcal{G}_2 . Besides, $[x, y] = 2\Lambda(h_\alpha)e_{\alpha_1-\alpha}$. This contradicts Theorem 1.16 since $\alpha - \alpha_1$ is not the highest root of \mathcal{G}_0 .

- 7) \mathcal{G}_0 of type C_n ($n \geq 3$), $\alpha = \alpha_{n-1} + \alpha_n$.
 - If we take $\beta = 2\alpha_{n-2} + 2\alpha_{n-1} + \alpha_n$, by Lemma 2.16, we get a contradiction.

- 8) \mathcal{G}_0 of type C_n ($n \geq 3$), α simple.
- 8a) If $\alpha = \alpha_i$ with $i \neq 1, n-1, n$ then we proceed as for A_n , case 2a).
- 8b) If $\alpha = \alpha_{n-1}$, take $\beta = 2\alpha_{n-2} + 2\alpha_{n-1} + \alpha_n$ and apply Lemma 2.16.
- 8c) If $\alpha = \alpha_1$, $[E_M, [F_\Lambda, e_{-\alpha_2}]] = e_{-\alpha_1 - \alpha_2}$ implies $(\Lambda, \alpha_2) \neq 0$. Thus we apply the same argument as in case 4d) with $x = [F_\Lambda, F_\Lambda]$ and $y = [[E_M, e_{\alpha_1}], [E_M, e_{\alpha_1}]]$.
- 8d) If $\alpha = \alpha_n$ we take $\beta = 2\alpha_{n-1} + \alpha_n$. By Lemma 2.16 we find a contradiction.
- 9) \mathcal{G}_0 of type D_n ($n \geq 4$), α simple.
- 9a) If $\alpha = \alpha_i$, $i \neq 1, n-1, n$ we proceed as for A_n , case 2a).
- 9b) If $\alpha = \alpha_1$ we apply Lemma 2.16 to $\beta = \alpha_1 + 2\alpha_2 + \dots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$ and find a contradiction.
- 9c) If $\alpha = \alpha_n$ (or, equivalently, $\alpha = \alpha_{n-1}$) we apply Lemma 2.16 to $\beta = \alpha_{n-3} + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$.
- 10) \mathcal{G}_0 of type E_6 , α simple, $\alpha = \alpha_i$.
 If $i \neq 1, 2, 6$ we proceed as for A_n , case 2a).
 Otherwise we apply Lemma 2.16 as follows:
 if $i = 1$ we take $\beta = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5$;
 if $i = 6$ we take $\beta = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$;
 if $i = 2$ we take $\beta = \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5$.
- 11) \mathcal{G}_0 of type E_7 or E_8 .
 The situation is analogous to case 10).
- 12) \mathcal{G}_0 of type F_4 and α in the list.
 We apply Lemma 2.16 with the following roots α and β :
- $\alpha = \alpha_1 + \alpha_2 + \alpha_3$, $\beta = \alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$;
 - $\alpha = \alpha_2 + \alpha_3$, $\beta = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$;
 - $\alpha = \alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4$, $\beta = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4$;
 - $\alpha = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4$, $\beta = \alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4$;
 - $\alpha = \alpha_2 + 2\alpha_3$, $\beta = \alpha_2 + \alpha_3 + \alpha_4$;
 - $\alpha = \alpha_2 + 2\alpha_3 + 2\alpha_4$, $\beta = \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4$.
- 13) \mathcal{G}_0 of type F_4 , α simple.
 We apply Lemma 2.16 with the following roots α and β :
- $\alpha = \alpha_1$, $\beta = \alpha_1 + 2\alpha_2 + 2\alpha_3$;
 - $\alpha = \alpha_2$, $\beta = \alpha_1 + \alpha_2 + \alpha_3$;
 - $\alpha = \alpha_3$, $\beta = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$;
 - $\alpha = \alpha_4$, $\beta = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$.

14) \mathcal{G}_0 of type G_2 , α in the list.

14a) $\alpha = 2\alpha_1 + \alpha_2$

If we apply Lemma 2.16 with $\beta = \alpha_2$ we find $(\Lambda, \alpha_2) = 0$ thus $(\Lambda, \alpha) = 2(\Lambda, \alpha_1)$. Besides, Corollary 2.12 implies $\Lambda(h_\alpha) = 2$, i.e. $\Lambda(h_1) = 1$.

Consider the vector $x := [[E_M, e_\alpha], [E_M, e_{\alpha_1}]]$. Then one can verify that x is a lowest weight vector. Now, if we take $y := [F_\Lambda, F_\Lambda]$ in \mathcal{G}_{-2} , then $[x, y] \neq 0$ and this contradicts Theorem 1.16.

14b) $\alpha = \alpha_1 + \alpha_2$

In this case we apply Lemma 2.16 with $\beta = 3\alpha_1 + \alpha_2$ and find a contradiction.

14c) $\alpha = 3\alpha_1 + \alpha_2$

We proceed as in 14b) with $\beta = \alpha_1 + \alpha_2$.

15) \mathcal{G}_0 of type G_2 , α simple.

If $\alpha = \alpha_1$ apply Lemma 2.16 with $\beta = 3\alpha_1 + 2\alpha_2$.

If $\alpha = \alpha_2$ apply Lemma 2.16 with $\beta = 2\alpha_1 + \alpha_2$. □

3. – The classification theorem

Let L be a finite-dimensional Lie superalgebra and let σ be an automorphism of L of finite order k . Then

$$(3) \quad L = \bigoplus_{i=0}^{k-1} L_i$$

where $L_i = \{x \in L \mid \sigma(x) = \epsilon^i x\}$, $\epsilon = e^{2\pi i/k}$. Notice that (3) is a mod- k gradation of L .

Consider the Lie superalgebra $\mathbf{C}[x, x^{-1}] \otimes L = \bigoplus_{i=-\infty}^{+\infty} x^i \otimes L$ and its subalgebra

$$G^k(L, \sigma) := \bigoplus_{i=-\infty}^{+\infty} x^i \otimes L_{i \pmod k}$$

called the *covering superalgebra* of L . Then $G^k(L, \sigma)$ is a \mathbb{Z} -graded Lie superalgebra of infinite depth and growth 1.

EXAMPLE 1 (The Lie superalgebra $S_1(n)$). We recall that $sl(m, n)$ is the Lie superalgebra of $(m+n) \times (m+n)$ matrices with supertrace equal to 0, i.e., in suitable coordinates, the set of matrices $\left\{ \left(\begin{array}{c|c} a & b \\ \hline c & d \end{array} \right) \mid \text{tr}(a) = \text{tr}(d) \right\}$.

Let $\tilde{Q}(n)$ ($n \geq 2$) be the subalgebra of $sl(n+1, n+1)$ consisting of matrices of the form $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$, where $\text{tr}(b) = 0$. Then $\tilde{Q}(n)$ has a one-dimensional centre $C = \langle I_{2n+2} \rangle$ and we define $Q(n) = \tilde{Q}(n)/C$. Notice that $Q(n)$ has even part isomorphic to the Lie algebra of type A_n and odd part isomorphic to

$ad\,sl_{n+1}$ and has therefore dimension $2(n^2 + 2n)$. We consider the following automorphism σ of $Q(n)$:

$$\sigma \begin{pmatrix} a & b \\ b & a \end{pmatrix} = \begin{pmatrix} -a^t & ib^t \\ ib^t & -a^t \end{pmatrix}.$$

Then σ has order 4 and $Q(n) = \bigoplus_{i=0}^3 Q(n)_i$ where

$$Q(n)_0 \cong so_{n+1},$$

$$Q(n)_1 = \{b \in sl_{n+1} | b = b^t\},$$

$$Q(n)_2 = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a = a^t \right\} / C,$$

$$Q(n)_3 = \{b \in sl_{n+1} | b = -b^t\}.$$

Let us suppose $n \neq 3$ and denote by $S_1(n)$ the covering superalgebra $G^4(Q(n), \sigma)$. Notice that $Q(n)_3$ is isomorphic to the adjoint module of so_{n+1} and if $n > 2$ then $Q(n)_1$ and $Q(n)_2$ are isomorphic, as so_{n+1} -modules, to the highest weight module $V(2\omega_1)$, while if $n = 2$ $Q(n)_1$ and $Q(n)_2$ are $sl(2)$ -irreducible modules of dimension 5.

EXAMPLE 2 (The Lie superalgebra $S_2(m)$). Suppose $m = 2n - 1$ and consider the following automorphism τ of $Q(m)$:

$$\tau \left(\begin{array}{cc|cc} a & b & r & s \\ c & d & v & w \end{array} \right) = \left(\begin{array}{cc|cc} -d^t & b^t & -iw^t & is^t \\ c^t & -a^t & iv^t & -ir^t \\ \hline -iw^t & is^t & -d^t & b^t \\ iv^t & -ir^t & c^t & -a^t \end{array} \right)$$

where a, b, c, d, r, s, v, w are $n \times n$ -blocks and $\text{tr}(r) + \text{tr}(w) = 0$. Then $\tau^4 = 1$ and $Q(m) = \bigoplus_{i=0}^3 Q(m)_i$ where

$$Q(m)_0 \cong sp(2n),$$

$$Q(m)_1 = \left\{ \begin{pmatrix} r & s \\ v & w \end{pmatrix} \mid r = -w^t, s = s^t, v = v^t \right\},$$

$$Q(m)_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid b^t = -b, c^t = -c, a^t = d \right\} / C,$$

$$Q(m)_3 = \left\{ \begin{pmatrix} r & s \\ v & w \end{pmatrix} \mid w^t = r, s^t = -s, v^t = -v, \text{tr}(r) = 0 \right\}.$$

Let us denote by $S_2(m)$ the covering superalgebra $G^4(Q(m), \tau)$. Notice that $Q(m)_1$ is isomorphic to the adjoint module of the Lie algebra $sp(2n)$ and $Q(m)_2, Q(m)_3$ are isomorphic to the $sp(2n)$ -module $\Lambda_0^2 sp_{2n}$.

EXAMPLE 3 (The Lie superalgebra S_3). Let $D(2, 1; \alpha)$ be the one-parameter family of 17-dimensional Lie superalgebras with even part isomorphic to $A_1 \oplus A_1 \oplus A_1$ and odd part isomorphic to $sl_2 \otimes sl_2 \otimes sl_2$. We recall that two members $D(2, 1; \alpha)$ and $D(2, 1; \beta)$ of this family are isomorphic if and only if α and

β lie in the same orbit of the group V of order 6 generated by $\alpha \mapsto -1 - \alpha$, $\alpha \mapsto 1/\alpha$.

$D(2, 1; \alpha)$ is the contragredient Lie superalgebra associated to the matrix

$$\begin{pmatrix} 0 & 1 & -1 - \alpha \\ 1/\alpha & 0 & 1 \\ 1 & -\alpha/(1 + \alpha) & 0 \end{pmatrix}.$$

Suppose that $\alpha^2 + \alpha + 1 = 0$ and consider the following automorphism φ of $D(2, 1; \alpha)$:

$$\begin{aligned} \varphi(e_1) &= -e_2 & \varphi(f_1) &= -f_2 & \varphi(h_1) &= h_2 \\ \varphi(e_2) &= -e_3 & \varphi(f_2) &= -f_3 & \varphi(h_2) &= h_3 \\ \varphi(e_3) &= -e_1 & \varphi(f_3) &= -f_1 & \varphi(h_3) &= h_1. \end{aligned}$$

Then φ has order 6 and $D(2, 1; \alpha) = \bigoplus_{i=0}^5 V_i$ where

- V_0 is isomorphic to the Lie algebra of type A_1 ;
- V_1 is isomorphic, as a V_0 -module, to the $sl(2)$ -irreducible module of dimension 4;
- V_2 is isomorphic, as a V_0 -module, to the adjoint module of $sl(2)$;
- V_3 is isomorphic to the $sl(2)$ -irreducible module of dimension 2;
- V_4 is isomorphic to the adjoint module of $sl(2)$;
- V_5 is isomorphic to the $sl(2)$ -irreducible module of dimension 2.

We denote by S_3 the covering superalgebra $G^6(D(2, 1; \alpha), \varphi)$.

THEOREM 3.1. *Let $\mathcal{G} = \bigoplus_{i \in \mathbb{Z}} \mathcal{G}_i$ be an infinite-dimensional \mathbb{Z} -graded Lie superalgebra. Suppose that:*

- \mathcal{G} is simple and generated by its local part,
- the \mathbb{Z} -gradation is consistent and has infinite depth,
- \mathcal{G}_0 is simple,
- \mathcal{G}_{-1} and \mathcal{G}_1 are irreducible \mathcal{G}_0 -modules which are not contragredient.

Then \mathcal{G} has finite growth if and only if it is isomorphic to one of the Lie superalgebras S_i for some $1 \leq i \leq 3$.

PROOF. Theorems 2.10 and 2.17 show that under our hypotheses either \mathcal{G}_0 has rank 1 or one of the following possibilities occur:

- a) \mathcal{G}_0 is of type A_3 , \mathcal{G}_{-1} is its adjoint module and $\mathcal{G}_1 = V(2\omega_2)$;
- b) \mathcal{G}_0 is of type B_n , \mathcal{G}_{-1} is its adjoint module and $\mathcal{G}_1 = V(2\omega_1)$;
- c) \mathcal{G}_0 is of type C_n ($n \geq 3$), \mathcal{G}_1 is its adjoint module and $\mathcal{G}_{-1} \cong \Lambda_0^2 sp_{2n}$;
- d) \mathcal{G}_0 is of type D_n ($n \geq 4$), \mathcal{G}_{-1} is its adjoint module and $\mathcal{G}_1 = V(2\omega_1)$.

Besides, if \mathcal{G}_0 has rank 1, by Corollary 2.12, either

- e) $\mathcal{G}_{-1} \cong V(\omega)$ and $\mathcal{G}_1 \cong V(3\omega)$ or
- f) \mathcal{G}_{-1} is isomorphic to the adjoint module of A_1 and $\mathcal{G}_1 \cong V(4\omega)$.

By Propositions 1.7 and 1.9 we conclude that \mathcal{G} is isomorphic to the Lie superalgebra $S_1(m) = G^4(Q(m), \sigma)$ with $m = 5$ in case a), $m = 2n$ in case b), $m = 2$ in case f) and $m = 2n - 1$ in case d); in case c) \mathcal{G} is isomorphic to the Lie superalgebra $S_2(m) = G^4(Q(m), \tau)$ with $m = 2n - 1$. Finally, in case e) \mathcal{G} is isomorphic to the Lie superalgebra S_3 . □

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