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Rigidity at Infinity for Even-Dimensional Asymptotically Complex Hyperbolic Spaces

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Abstract. Any Kähler metric on the ball which is strongly asymptotic to complex hyperbolic space and whose scalar curvature is no less than the one of the complex hyperbolic space must be isometrically biholomorphic to it. This result has been known for some time in odd complex dimension and we provide here a proof in even dimension.

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1. – Introduction

Motivations. In [6], the second author proved that any asymptotically complex-hyperbolic spin and Kähler manifold *of odd complex dimension* whose scalar curvature is larger than the one of complex hyperbolic space cannot be too close at infinity to the model space. This stood as a Kählerian counterpart of the analogous rigidity statements proved by L. Andersson and M. Dahl [1], M. Min-Oo [9] and M. C. Leung [8] for the case of the real hyperbolic space. It is also reminiscent of well-known rigidity results for asymptotically flat spaces, known as positive mass theorems. The purpose of this short note is to extend the Kählerian rigidity statement to the (up to now missing) even-dimensional (complex) case.

Notations. We consider in this paper a complete Kähler manifold (M, g, J) of even complex dimension $m = 2n$. Its Kähler form will always be denoted by ω. We will denote by R its Riemann curvature tensor, Ric its Ricci tensor, with Scal its scalar curvature and Ric_0 its trace-free part, and ρ the Ricci 2-form, ρ_0 being its trace-free part with respect to the Kähler form.

An example is the complex hyperbolic space $\mathbb{C}\mathbf{H}^{2n}$ which is the unique simply connected Kähler manifold with constant negative holomorphic sectional

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curvature (it will be normalized to -4 throughout this paper). It is diffeomorphic to the unit ball $\mathbb{B}$ in $\mathbb{C}^n$ and is Einstein with scalar curvature $-4m(m+1)$.

Any Kähler manifold (M, g, J) is said to be *strongly asymptotic* to complex hyperbolic space if

(i) there are a compact set K of M and a diffeomorphism between $M \setminus K$ and the exterior of a ball in $\mathbb{C}\mathbf{H}^m$, so that we can consider $M \setminus K$ to be endowed with two Kähler structures: the original (g, J) and the standard (g_0, J_0);
(ii) if A is defined on $M \setminus K$ by $g(X, Y) = g_0(AX, AY)$ for all X, Y, then A, $J_0 J^{-1}$ and $J^{-1} J_0$ are uniformly bounded from below and above;
(iii) there is an $\varepsilon > 0$ such that, if $r = d_{g_0}(x_0, .)$ for an arbitrary basepoint x_0 in M, $|\nabla^{g_0} A| + |A - I| + |J^{-1} J_0 - I| + |J_0 J^{-1} - I| = O(e^{-(2m+2+\varepsilon)r})$;
(iv) the scalar curvature of (M, g) is uniformly bounded.

In this paper, we prove:

THEOREM. *Let* $(\mathbb{B}, g, J)$ *be a Kähler metric on the ball of even complex dimension m. If* (g, J) *is strongly asymptotic to the complex hyperbolic metric and* $\mathrm{Scal}_g \geqslant -4m(m+1)$*, then* $(\mathbb{B}, g, J)$ *is biholomorphically isometric to* $\mathbb{C}\mathbf{H}^m$.

The proof in [6] relied heavily on the existence on each complex hyperbolic space $\mathbb{C}\mathbf{H}^m$ of *odd* complex dimension of a special set of spinor fields, called *Kählerian Killing spinors*. The unfortunate non-existence of such spinor fields in *even* complex dimension (which is in a sense the non-compact counterpart of the well-known fact, in the realm of compact manifolds, that complex projective spaces $\mathbb{C}\mathbf{P}^{2n}$ are not spin) was the cause for the dimensional restriction.

The main contribution of this paper is that an analogous proof for the even dimensional case may be available, *once the right objects are used*. It is based on a simple (but crucial) fact, proven below, that seems to have escaped notice so far: whereas distinguished spinor fields do not exist for the classical Spin-structure on $\mathbb{C}\mathbf{H}^{2n}$, there does exist a lot of distinguished sections of spinor bundles issued from well-chosen Spin^c-structures. This remark paves the way for using the techniques already developed in [6], through a similar, but different, Weitzenböck-type formula. However, shifting from the Spin to the Spin^c-context renders the proofs more involved and also unfortunately makes the technique slightly less adequate.

Although we restricted ourselves to the ball in the Theorem above, we will prove below a slightly stronger statement, which is our main technical result and implies immediately the previous one.

THEOREM (technical version). *Let* (M, g, J) *be a Kähler manifold of even complex dimension m, such that* $\left[\frac{\omega}{i\pi}\right]$ *is an integral class and its associated line bundle defines a* Spin^c*-structure. If* (M, g, J) *is strongly asymptotic to the complex hyperbolic metric and* $\mathrm{Scal}_g \geqslant -4m(m+1)$*, then* (M, g, J) *is biholomorphically isometric to* $\mathbb{C}\mathbf{H}^m$.

REMARKS 1. Although the topological conditions on the Kähler class appear essentially as *ad hoc* assumptions for the Theorem to hold, they are satisfied in a large number of situations. They are for instance preserved by blow-up.

2. Apart from the analogy with the odd-dimensional case, another motivation is the construction by C. R. Graham and J. Lee [5] and more recently by O. Biquard [4] of a large family of complete Einstein metrics on the ball. These are deformations of the standard metrics of non-compact rank-one symmetric spaces and are in one-to-one correspondance with a neighbourhood of deformations of the structure at infinity of each symmetric space. This favours the idea that each admissible structure at infinity can be filled by a unique Einstein manifold (possibly not topologically trivial). Graham-Lee and Biquard results settle the *local* (close to the standard structure) existence and uniqueness problem, whereas results as in [6], [8], [9] provide *global* uniqueness, at least when the structure at infinity is standard enough. Unfortunately, even the technical version of our result brings only very partial information in this context.

CONTENTS OF THE NOTE. In Section 1, we describe the model spinors on the complex hyperbolic space and begin the proof of the theorem (in its technical version): we show that any manifold satisfying the assumptions bears Kählerian Killing spinors for an adequate Spin^c-structure. Section 2 is then devoted to the geometrical constructions leading to the final rigidity statement: any such manifold must be the complex hyperbolic space. Since parts of the proof are simple reproduction of arguments given in [6], we have chosen to be quite short for these steps and to stress only the points that differ from the previously existing proof.

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2. – The fundamental formula and its consequences

From now one, we will study a Kähler manifold (M, g, J) satisfying all the assumptions of the theorem above. If $\frac{\omega}{i\pi}$ is in the image of $H^2(M, \mathbb{Z})$ in $H^2(M, \mathbb{R})$, $F = -2i\omega$ is the curvature form of a hermitian connection on a complex line bundle L. Its associated S^1-bundle will be denoted by $\pi : P \to M$. We consider the Spin^c-structure on M induced by this choice as an auxiliary bundle. Its spinor bundle will be denoted by Σ^c and it splits under the action of the Kähler form as

$$\Sigma^c = \bigoplus_{0 \leqslant q \leqslant m} \mathrm{Ker}\left(\omega \cdot +i(m-2q)\,\mathrm{id}\right) = \oplus\ \Sigma^c_q\,.$$

A Kählerian Killing spinor is a section $\Psi = \psi_{r-1} + \psi_r$ (for some r in $\{1, \dots, m\}$) solving

$$\nabla_X \psi_{r-1} + iX^{0,1} \cdot \psi_r = 0, \quad \nabla_X \psi_r + iX^{1,0} \cdot \psi_{r-1} = 0 .$$

For future reference, we note that, equivalently [7],

$$\nabla_X \Psi + \frac{i}{2} X \cdot \Psi + \frac{(-1)^n}{2} JX \cdot \overline{\Psi} \ = \ 0 \qquad \forall X \in TM . \tag{1.1}$$

We begin with a quick tour of the model space. Letting K be the canonical bundle of the complex hyperbolic space, we denote by K^q ($q \in \mathbb{Q}$) its rational powers. Since K is trivial on the ball, they always exist and are still isomorphic to the trivial bundle; the notation then means we are using a modified connection on the trivial bundle. The elementary but fundamental remark is the following:

Proposition 1.1. *Let $m = 2n$. Then the spinor subbundles $\Sigma^c_{n-1} \oplus \Sigma^c_n$ of the* Spin^c*-structure on $\mathbb{C}\mathbf{H}^m$ induced by the choice $L = K^{-\frac{1}{m+1}}$ are trivialized by Kählerian Killing spinors.*

Proof. We rely on the explicit computations done in [7]. Any spinor field $\psi \in \Sigma^c_q$ can be written as $\bar{\omega} \otimes \psi_0$ where ω is a form of type $(q, 0)$ on $\mathbb{C}\mathbf{H}^m$ and ψ_0 is a holomorphic section of $\Sigma^c_0 = K^{\frac{n}{m+1}}$. Then

$$\nabla_{X^{0,1}} \left(|\psi_0|^2 \omega \right) = 0, \quad \text{and} \quad \nabla_{X^{1,0}} \omega = \frac{1}{n}\, \iota_{X^{1,0}} (\partial \omega) \qquad \text{for any } X , \tag{1.2}$$

is equivalent to

$$\nabla_X \psi_{n-1} + iX^{0,1} \cdot \psi_n = 0, \quad \nabla_{X^{1,0}} \psi_n + iX^{1,0} \cdot \psi_{n-1} = 0, \qquad \text{for any } X , \tag{1.3}$$

for the pair $(\psi_{n-1} = \bar{\omega} \otimes \psi_0, \psi_n = \frac{1}{2in} \mathcal{D} \psi_{n-1})$, and, whenever $n \neq 1$, this is actually equivalent to the Kählerian Killing spinor equations [7]. Let $\psi_0 = (dz_1 \wedge \dots \wedge dz^m)^{n/(m+1)}$ and $f = 1 - \sum |z^i|^2$ on $\mathbb{C}\mathbf{H}^m$ seen as the unit ball. Then, Kirchberg's Ansatz [7] shows that, for any choice of multi-index $\alpha = (\alpha_{i_1}, \dots, \alpha_{i_{n-1}})$, $\omega = f^{-n} dz^{\alpha_{i_1}} \wedge \dots \wedge dz^{\alpha_{i_{n-1}}}$ provides a Kählerian Killing spinor, as does $\omega = f^{-n}\, \iota_{\sum (z^j \partial_j)} (dz^{\beta_{i_1}} \wedge \dots \wedge dz^{\beta_{i_n}})$ for any choice of multi-index $\beta = (\beta_{i_1}, \dots, \beta_{i_n})$.

In complex dimension $m = 2$ ($n = 1$), Kirchberg's Ansatz above does not provide *a priori* Kählerian Killing spinors (the second set of equations above is not *in general* equivalent to the Kählerian Killing condition) but it is easily checked that the spinor fields given a few lines above are *indeed* Kählerian Killing spinors. $\square$

REMARK 1.2. Proposition 1.1 is part of a more general phenomenon: for each r in $\{1,\dots,m\}$, there exists on the complex hyperbolic space a well chosen Spin^c-structure, built from a root of the canonical bundle, endowed with distinguished spinor fields living in $\Sigma^c_{r-1}\oplus\Sigma^c_r$. This remark can be easily substantiated by using Kirchberg's Ansatz.

REMARK 1.3. Our special spinors can also be obtained by projecting (in the sense of [10], [11]) the parallel spinors of $\mathbb{C}^{2n+2}$, tensored with an adequate root of the complex volume form, over the complex hyperbolic space. This idea leads to the proofs developed in the next sections.

We now come back to our general manifold (M,g,J) of complex dimension $m=2n$ equipped with its distinguished Spin^c-structure. If ∇ and $\mathcal{D}$ denote the Levi-Civita connection and Dirac operator on Σ^c, we also define a modified connection $\widehat{\nabla}$ and a modified Dirac operator $\widetilde{\mathcal{D}}$ acting on a spinor $\psi=\sum\psi_q$ as

$$\begin{aligned}\widehat{\nabla}_X\psi &= \nabla_X\psi + iX^{1,0}\cdot\psi_{n-1} + iX^{0,1}\cdot\psi_n\,,\\ \widetilde{\mathcal{D}}\psi &= \mathcal{D}\psi - im\sum_{q-n\text{ even}}\psi_q - i(m+2)\sum_{q-n\text{ odd}}\psi_q\,.\end{aligned}$$

It is crucial to notice at this point that the modified Dirac operator $\widetilde{\mathcal{D}}$ *is not* the Dirac operator naturally issued from the modified connection $\widehat{\nabla}$ (hence the difference in the notation). The Dirac operator issued from $\widehat{\nabla}$ is generally not coercive on L^2 and is then useless, whereas the modified Dirac operator $\widetilde{\mathcal{D}}$ is coercive, at least in our situation. This discrepancy plays a major role in the arguments below. Note also that the operators defined above are different from the operators used in the proof of [6].

The main tool in this section is the Weitzenböck fomula for spinors proven in [6, Section 3]. We now rewrite it in the Spin^c context, with the above choice of coefficients adapted to our needs. We also correct two typing mistakes in [6].

LEMMA 1.4. *For any spinor ψ, let $\alpha^\psi(X)=\langle\widehat{\nabla}_X\psi+X\cdot\widetilde{\mathcal{D}}\psi,\psi\rangle$. Then*

$$\begin{aligned}-\operatorname{div}\alpha^\psi &= |\widehat{\nabla}\psi|^2-|\widetilde{\mathcal{D}}\psi|^2+2i\sum_{q=0}^{m}(-1)^{q-n}\langle(\widetilde{\mathcal{D}}\psi)_q,\psi_q\rangle\\ &\quad+\sum_{q=0}^{m}\left\langle\left(\frac{1}{4}\operatorname{Scal}+m(m+2)-2(m-q)(u_q)^2-2q(v_q)^2-\frac{1}{2}F\cdot\right)\psi_q,\psi_q\right\rangle\end{aligned}$$

where $u_{n-1}=1$, $v_n=1$, and all other u_q, v_q are set to zero.

LEMMA 1.5. *The bundle $\Sigma^c_{n-1}\oplus\Sigma^c_n$ is trivialized by $\widehat{\nabla}$-parallel spinors.*

PROOF. The first step is to prove that there exists a full set of $\widetilde{\mathcal{D}}$-harmonic spinors on M, asymptotic to the model spinors on $\mathbb{C}\mathbf{H}^m$ described in the previous section; once these spinors have been obtained, the proof will go along the arguments of [6]. The key point in this first step is to show that the zeroth order terms in the Weitzenböck formula in Lemma 3.1 are always nonnegative if the curvature assumptions of the theorem are satisfied. Taking into account $F = -2i\omega$, and letting $\mathrm{Scal} = -4m(m+1)\kappa$ and $p = q - n$, the zeroth order term acting on sections of Σ_q^c is

$$-m(m+1)\kappa + m(m+2) - 2\,(n-p)\,u_{n+p}^2 - 2\,(n+p)\,v_{n+p}^2 - 2p\,.$$

A case by case check yields that this is always equal to $m(m+1)(1-\kappa)$, hence nonnegative. It then implies that the modified Dirac operator $\widetilde{\mathcal{D}}$ is coercive in L^2, so that we may find $\widetilde{\mathcal{D}}$-harmonic spinors for the Spin^c-structure on M which are L^2 perturbations of the model Kählerian Killing spinors at infinity.

As a second step, we apply the Weitzenböck formula to any such spinor field Ψ and get

$$\lim_{r\to\infty} \int_{S_r} \alpha^{\Psi}(\nu_r) \geqslant \int_M |\widehat{\nabla}\Psi|^2\,.$$

Arguing as in [6], our asymptotic conditions yield that the limit is zero, so that Ψ is $\widehat{\nabla}$-parallel.

We now write $\Psi = \sum \psi_q$ with ψ_q in Σ_q^c. As $\widehat{\nabla}$ respects the splitting of Σ^c into its $\{\psi_{n-1}, \psi_n\}$-components (where it is a modified connection) and the remaining components $\{\psi_q, q \neq n-1, n\}$ (where it equals the Levi-Civita connection), one gets that $\widehat{\nabla}(\psi_{n-1} + \psi_n) = 0$ and that each component ψ_q is parallel if $q \neq n-1, n$. These remaining components are zero since they are in L^2 (model Kählerian Killing spinors live in $\Sigma_{n-1}^c \oplus \Sigma_n^c$). We finally get that each solution $\Psi = \psi_{n-1} + \psi_n$ solves the Kählerian Killing equation. As model spinors trivialize on $\mathbb{C}\mathbf{H}^m$, the solutions trivialize $\Sigma_{n-1}^c \oplus \Sigma_n^c$ as well. □

LEMMA 1.6. *(M, g, J) is Kähler-Einstein, with Ricci form $\rho = -2(m+1)\omega$.*

PROOF. Let $\Psi = \psi_{n-1} + \psi_n$ be any of our special spinors and X a tangent vector. We compute in two different ways:

$$\begin{aligned} mX^{1,0}\cdot\Psi + (m+2)X^{0,1}\cdot\Psi &= \sum_{i=1}^{2m} e_i \cdot \left(-\nabla_{e_i}\nabla_X + \nabla_X\nabla_{e_i}\right)\Psi \\ &= -\frac{1}{2}\,\mathrm{Ric}(X)\cdot\Psi - i\sum_{i=1}^{2m}\omega(e_i, X)e_i\cdot\Psi \\ &= -\frac{1}{2}\,\mathrm{Ric}(X)\cdot\Psi + iJX\cdot\Psi\,. \end{aligned} \tag{1.4}$$

Letting $Z = (\mathrm{Ric}(X) + 2(m+1)X)^{1,0}$, then $Z\cdot\psi_{n-1} = 0$ and $\overline{Z}\cdot\psi_n = 0$. Since, at any point, either ψ_{n-1} or ψ_n is non zero, this forces $Z = 0$. □

REMARK 1.7. A direct consequence is a proof of the theorem in complex dimension $m = 2$. The preceding arguments provide a full set of spinors trivializing $\Sigma_0^c \oplus \Sigma_1^c$ and the metric is Kähler-Einstein. There is on M another (elementary) Spin^c-structure, associated with the choice of determinant bundle $L' = K_M$, the canonical bundle of M. The associated spinor bundle Σ'^c has a trivial parallel section: the volume form, that generates Σ'^c_2. One may now argue algebraically and show that the metric has constant negative holomorphic sectional curvature.

This method does unfortunately not work in higher dimensions: when one seeks distinguished spinor fields living in $\Sigma_{r-1}^c \oplus \Sigma_r^c$ for arbitrary r, coercivity may fail in the above Weitzenböck formula, even if one looks for different Spin^c-structures.

3. – The circle bundle and its cone

We are now ready for the remaining parts of the proof: our aim is to show that our manifold has constant (negative) holomorphic sectional curvature. As in [6], we use the construction devised by Ch. Bär [2] and A. Moroianu [10], [11].

LEMMA 2.1. *The total space of the auxiliary bundle P has a Lorentz Einstein metric, invariant by the S^1-action. The* Spin^c*-structure on M lifts to a* Spin*-structure on P.*

PROOF. We define the metric g_P on P such that horizontal and vertical spaces given by the Levi-Civita connection are orthogonal, the bundle projection π becomes a metric-preserving submersion and the vertical vector field V (induced from the S^1-action) has norm -1. From Remark 9.78 in [3] it is an Einstein metric. We now follow the argument in [11]. We extend the Spin^c frame bundle $P_{\mathrm{Spin}^c}M$ to a Spin$^c(n, 1)$ principal bundle Q by letting $Q = P_{\mathrm{Spin}^c}M \times_{\mathrm{Spin}^c(n)} \mathrm{Spin}^c(n, 1)$. The bundle π^*Q is a Spin^c frame bundle defining a Spin^c-structure on P. Moreover, the auxiliary bundle of this structure is the pull-backed bundle π^*P which is trivial. Any choice of a global section then induces a Spin structure on P. □

Let $C = \mathbb{R}_+^* \times P$ be the cone over P, endowed with the metric $g_C = -ds^2 + s^2 g_P$ and complex structure J_C given by the lift of the complex structure J on C and by the relation $J_C V = (-1)^n s \frac{\partial}{\partial s}$. Its Kähler form will be denoted by ω_C.

LEMMA 2.2. *(C, g_C, J_C) is a Ricci-flat Kähler spin manifold of signature $(2m, 2)$. It bears a C_{2n+1}^n-dimensional space of parallel spinors that trivialize the eigen-subbundle $\mathcal{S}_n = \mathrm{Ker}(\omega_C \cdot +i)$ of its spinor bundle.*

Proof. The first claim follows from straightforward computations, using the O'Neill and Gauss-Codazzi-Mainardi formulas. We must now understand how our Kählerian Killing spinors transform when pulled back to P, then to C.

We apply the computations of [10], [11] to our case, *cum grano salis* due to the Lorentz signature. The pulled-back bundle $\pi^*\Sigma^c$ is made into a Dirac bundle over P if one fixes $V \cdot \varphi = (-1)^n \overline{\varphi}$ (conjugation of spinors on an even-dimensional manifold). The connection $\bar{\nabla}$ induced by the Levi-Civita connection on P and the pull-backed connection on the auxiliary bundle of the Spin^c-structure acts then as follows: for any pull-backed spinor field $\pi^*\varphi$ and any horizontal lift X^H on P of a vector X on M,

$$\bar{\nabla}_{X^H}\pi^*\varphi = \pi^*\left(\nabla_X\varphi + \frac{(-1)^n}{2}\pi_*\nabla_{X^H}V \cdot \overline{\varphi}\right) .$$

As the metric g_P is defined from the natural connection on P whose curvature is $F = -2i\omega$, the usual 6-term formula yields $\pi_*\nabla_{X^H}V = -JX$ and finally

$$\bar{\nabla}_{X^H}\pi^*\varphi = \pi^*\left(\nabla_X\varphi + \frac{(-1)^n}{2} JX \cdot \overline{\varphi}\right) .$$

Similarly, one computes

$$\bar{\nabla}_V\pi^*\varphi = -\frac{1}{2}\pi^*\left(\omega \cdot \varphi\right) .$$

Using the canonical global section on π^*P over P, we can now turn any spinor field for the Spin^c-structure into a section for the Spin structure whose existence has been already remarked. Hence, if φ is any spinor field, it may be seen as associated to the Spin^c-structure (and is acted upon by the connection $\bar{\nabla}$ induced from the Levi-Civita connection tensorized with the pull-back connection) or to the Spin-structure (and is then acted upon by the Levi-Civita connection ∇^P which is better seen as Levi-Civita tensorized by the trivial connection). But the trivial connection on π^*P differs from the pulled back connection by a factor which is exactly the connection 1-form α of $P \to M$ and this shows that $\nabla^P_{X^H}\varphi = \bar{\nabla}_{X^H}\varphi$ and $\nabla^P_V\varphi = \bar{\nabla}_V\varphi - \frac{i}{2}\varphi$.

We now apply the preceding remarks to our set of Kählerian Killing spinors Ψ trivializing $\Sigma^c_{n-1} \oplus \Sigma^c_n$ on M. From formula (1.1), we get that the pull back spinors Ψ^P solve

$$\nabla^P_W\Psi^P + \frac{i}{2}W \cdot \Psi^P \;=\; 0 \qquad \forall\, W \in TP . \tag{2.1}$$

And it is now straightforward to show that these can be extended as parallel spinors on the cone C living in the desired subbundle. □

Lemma 2.3. *C is flat and M has constant negative holomorphic sectional curvature.*

PROOF. The previous lemma provides parallel spinors trivializing the component $\mathcal{S}_n$ (with respect to the action of the Kähler form of N) of the spin bundle $\mathcal{S}$ of C. The Weyl tensor W^C acts trivially on $\mathcal{S}_n$, i.e. for each X, Y in TC,

$$W^C_{X,Y} \cdot \psi = 0\,. \tag{2.2}$$

Mimicking the representation-theoretic argument in [6] shows that the only 2-form in $\Lambda^{1,1}_0 C$ whose action on $\mathcal{S}_n$ is zero is the zero form itself. This proves C is a flat manifold; applying O'Neill formulas yields that M has constant negative holomorphic sectional curvature [3, Corollary 9.29]. □

LEMMA 2.4. *(M, g, J) is biholormophically isometric to $\mathbb{C}\mathbf{H}^m$.*

PROOF. We only need to check that M is simply-connected. Assume the contrary: let $M_0 = \mathbb{C}\mathbf{H}^m$ be the universal covering of M. If the covering is not trivial, M_0 has at least two ends since the unique end of M is simply-connected. This is of course a contradiction. □

REFERENCES

[1] L. ANDERSSON – M. DAHL, *Scalar curvature rigidity for asymptotically locally hyperbolic manifolds*, Ann. Global Anal. Geom. **16** (1998), 1-27.

[2] C. BÄR, *Real Killing spinors and holonomy*, Comm. Math. Phys. **154** (1993), 509-521.

[3] A. L. BESSE, "Einstein manifolds", Ergeb. Math. Grenzgeb., Band 10, Springer, Berlin, 1981.

[4] O. BIQUARD, "Métriques d'Einstein asymptotiquement symétriques", Astérisque, vol. 265, Soc. math. France, 2000.

[5] C. R. GRAHAM – J. LEE, *Einstein metrics with prescribed conformal infinity on the ball*, Adv. Math. **87** (1991), 186-225.

[6] M. HERZLICH, *Scalar curvature and rigidity for odd-dimensional complex hyperbolic spaces*, Math. Ann. **312** (1998), 641-657.

[7] K. D. KIRCHBERG, *Killing spinors on Kähler manifolds*, Ann. Global Anal. Geom. **11** (1993), 141-164.

[8] M. C. LEUNG, *Pinching theorem on asymptotically hyperbolic spaces*, Internat. J. Math. **4** (1993), 841-857.

[9] M. MIN-OO, *Scalar curvature rigidity of asymptotically hyperbolic spin manifolds*, Math. Ann. **285** (1989), 527-539.

[10] A. MOROIANU, *La première valeur propre de l'opérateur de Dirac sur les variétés kähleriennes compactes*, Comm. Math. Phys. **169** (1995), 373-384.

[11] A. MOROIANU, Spin^c *manifolds and complex contact structures*, Comm. Math. Phys. **193** (1998), 661-674.

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