Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5)
Vol. I (2002), pp. 93-109

On Dicritical Foliations and Halphen Pencils

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Abstract. The aim of this article is to provide information on the number and on the geometrical position of singularities of holomorphic foliations of the projective plane. As an application it is shown that certain foliations are in fact Halphen pencils of elliptic curves. The results follow from Miyaoka's semipositivity theorem, combined with recent developments on the birational geometry of foliations.

Mathematics Subject Classification (2000): 37F75.

1. – Introduction

We consider in this paper holomorphic foliations of the complex projective plane and study the problem of giving bounds to the number of singularities (counted with multiplicities) in terms of the degree of the foliation.

There are different possibilities for the concept of multiplicity of a singularity. For example, if we take as multiplicity the Milnor number of the singularity, the classical Darboux's Theorem states that the sum of Milnor numbers of all the singularities of a foliation of degree d is equal to $d^2 + d + 1$. When we consider a different notion of multiplicity (cf. Definition 1), we obtain in Theorem 1 sharper estimations for the number of special types of singularities and information on their geometrical position in the plane, as a consequence of Miyaoka's Theorem ([Miy], [McQ1]).

The paper deals also with the question of finding global rational first integrals for dicritical foliations, i.e. foliations having singularities with infinite number of local separatrices. In other words, the question of deciding when a dicritical foliation is a pencil of algebraic curves. Of course a necessary condition is that all singularities of the foliation should have local first integrals, either holomorphic (for the non-dicritical singularities) or meromorphic (for the dicritical ones). Now, there exist examples of analytic families of foliations with the following properties: 1) the singularities move analytically with the foliation, in particular their number is constant; 2) the previous condition is

Pervenuto alla Redazione il 19 ottobre 2000 e in forma definitiva il 19 ottobre 2001.

always verified. Nevertheless, there are foliations in the family with rational first integral and foliations without rational first integral (cf. [LN]). A closer look into these examples reveals the presence of particular algebraic invariant curves which we call *special invariant curves* (cf. Definition 2). The main idea of this paper is that when such curves are absent we can get information about the questions we raised. Let us then proceed with the definitions and statements of results.

The following definition is analogous to the definition of algebraic multiplicity of a curve at a point.

DEFINITION 1. Let $\mathcal{G}$ be a foliation on the surface M induced locally by $\eta = 0$ at the singularity p, where η is a holomorphic 1-form with an isolated zero. Let σ be the blow up of M at p. Then we define $l(p) = l(p, \mathcal{G})$ as the vanishing order of $\sigma^*(\eta)$ along the exceptional curve $\sigma^{-1}(p)$.

This number can be easily computed from the vanishing order of η at the point $p \in M$. We remark that $l(p) = 1$ for *reduced* singularities (in Seidenberg's sense [Se]), although the Milnor number may be arbitrarily high.

Let us consider a foliation $\mathcal{F}$ on $\mathbb{C}P^2$ and perform a sequence of blow-up's $\sigma_1, \ldots, \sigma_k$ in order to have only reduced singularities for $\sigma^*\mathcal{F}$, where $\sigma = \sigma_1 \circ \cdots \circ \sigma_k$. Starting with some singularity p_0 of $\mathcal{F}_0 = \mathcal{F}$, let p_j be the singularity of $\mathcal{F}_j = (\sigma_1 \circ \cdots \circ \sigma_j)^*\mathcal{F}$ blown up by σ_{j+1} and $l(p_j) = l(p_j, \mathcal{F}_j)$. In this paper only non-reduced singularities are blown up (in particular we do not blow up regular points); the reduction of singularities of the foliation $\mathcal{F}$ produces a unique map σ (consequently a unique foliation $\sigma^*\mathcal{F}$), although the order the blow-up's are performed may change. We refer to $\sigma^*\mathcal{F}$ as *the reduction of singularities of $\mathcal{F}$*. These remarks of course apply to locally defined foliations (cf. [Br2], pg. 12).

We define

$$L(\mathcal{F}) := \sum_{j=0}^{k-1} (l(p_j) - 1)^2 .$$

Given a reduced algebraic curve $C \subset \mathbb{C}P^2$, let us define also

$$L(\mathcal{F}, C) := \sum_{j=0}^{k-1} \nu_{p_j} \cdot (l(p_j) - 1) ,$$

where ν_{p_j} is the multiplicity of (the strict transform of) C at p_j.

These integers will be used to obtain information on the number and on the position of the singularities of $\mathcal{F}$. We refer to Section 3 for a discussion about both number numbers $L(\mathcal{F})$ and $L(\mathcal{F}, C)$; in particular, we explain why they do not depend on the order the blow-up's are performed.

DEFINITION 2. A curve $S \subset M$ is a *special invariant curve* for a foliation $\mathcal{G}$ when S is smooth, isomorphic to $\mathbb{C}P^1$, $\mathcal{G}$-invariant, $\mathrm{Sing}(\mathcal{G}) \cap S = \{q\}$ and there are local coordinates for which $q = (0, 0)$ and $f(x, y) = x^m \cdot y$ $(m \in \mathbb{N})$ is a local holomorphic first integral for $\mathcal{G}$.

The reduction of singularities of a plane foliation has a special invariant curve when it is the strict transform of a rational plane curve or a curve introduced after blow-up's. For instance, special invariant curves appear in the reduction of certain types of singularities, e.g. cusps $d(y^2 - x^3) = 0$.

THEOREM 1. *Let $\mathcal{F}$ be a foliation of $\mathbb{C}P^2$ of degree $d(\mathcal{F})$ and let $C \subset \mathbb{C}P^2$ be an algebraic curve of degree $d(C)$. Suppose that the reduction of singularities $\sigma^*\mathcal{F}$ is free of special invariant curves. Then*

$$L(\mathcal{F}) \leq (d(\mathcal{F}) - 1)^2$$

and, $L(\mathcal{F}, C) \leq d(C) \cdot (d(\mathcal{F}) - 1)$ for $d(\mathcal{F}) \geq 1$.

Let us give a simple example where the bound given by Theorem 1 fails due to the presence of special invariant curves. Consider the pencil of rational curves generated by two smooth conics which cross each other transversely in 4 points a, b, c, d; there are 4 *radial* points (exactly a, b, c, d), that is, singularities locally induced by $\eta(x, y) = xdy - ydx + \text{h.o.t.} = 0$, which are the base points of the pencil, and 3 reduced singularities. Let us consider the straight line S joining a to b; it crosses the straight line joining c to d at a reduced singularity q which locally has a holomorphic first integral of the type $d(x \cdot y) = 0$. When we blow up at the base points the pencil becomes a reduced foliation and the strict transform of S becomes a special invariant curve (with one singularity q). Theorem 1 when applied to a foliation of degree d whose singular set is composed by a) k radial singularities and b) reduced singularities, yields $k \leq (d - 1)^2$, since an easy computation shows that for radial points $l(q) = 2$. But in the example, $k = 4$ and $d = 2$.

We consider again the problem of deciding when a foliation (without special invariant curves) is in fact a pencil of algebraic curves once its singularities have local first integrals. Let us fix the number of singularities and their local analytic type; since Theorem 1 gives only a lower bound for the degree of the foliation, we could in fact have too many possibilities. We then add another information to our problem, namely, a relation between the degree of the foliation and the data coming from the singularities. In order to have a guess about the possible relations, we may use particular examples which have first integrals. The relation depends on the type of pencil we use; in this paper we work with ("generic") pencils of elliptic curves; the reason is that we use the well known theory of elliptic surfaces.

Any pencil of elliptic plane curves gives rise to an elliptic fibration on the *rational* surface obtained from the projective plane after reduction of the base points. Conversely, any elliptic rational surface is birationally equivalent to the projective plane and its fibration comes from a pencil of elliptic curves; this implies that a pencil of elliptic curves is in fact equivalent to a *Halphen pencil*. Recall that a *Halphen pencil of index* $l - 1$ ($l \in \mathbb{N}$, $l \geq 2$) is a pencil of elliptic curves of degree $3(l - 1)$ with 9 base points (possibly infinitely near). A most interesting property is that when $l - 1 \geq 2$, the pencil contains a unique (reduced) cubic curve, which is the support of a non-reduced curve of the pencil of multiplicity $l-1$. This property corresponds to the fact that any elliptic rational surface has at most one multiple fiber (see e.g. [C-D], pg. 348 and [Mir], pg. 10).

To construct examples of Halphen pencils, we fix a smooth cubic given by a polynomial $P=0$. Given an integer $3(l-1)$, there exists an irreducible curve $Q=0$ with degree $3(l-1)$ which intersects $P=0$ in 9 differents points; we may choose $Q=0$ as to have $l-1$ branches through each of these points intersecting each other and also $P=0$ transversely. From the genus formula it follows that $Q=0$ is an elliptic curve; a Halphen pencil is defined by the level sets of the rational function P^{l-1}/Q. For more details see [Mir], pg. 10. This pencil, seen as a foliation, has degree $3l-2$; any base point p_i has $l(p_i)=l$, for $i=1,\ldots,9$.

In order to state Theorem 2, we need to restrict the type of dicritical singularities we allow. The simplest dicritical singularity allowed is the radial one; in general we will consider the following singularities:

Definition 3. A dicritical singularity is *removable by one blow up* if the exceptional divisor introduced by blowing it up is not invariant for the transformed foliation and contains only singularities which admit local holomorphic first integral.

It is not strictly true that these singularities have meromorphic first integrals a priori (see [K] and [Su]); this is not relevant to us because of the next theorem.

Theorem 2. *Let $l \geq 2$. Consider a foliation $\mathcal{F}$ of $\mathbb{C}P^2$ whose reduction of singularities is free of special invariant curves. Suppose also that the singular set of $\mathcal{F}$ is composed by singularities $p_1,\ldots,p_k$ which are removable by one blow up and singularities which have local holomorphic first integral. If $d(\mathcal{F})=3l-2$ and $l(p_i)=l$ for all $1\leq i\leq k$, then $\mathcal{F}$ is a Halphen pencil of index $l-1$.*

Along the proof of Theorem 2, it will become clear that the hypotheses lead to generic Halphen pencils, in the sense that there are 9 distinct base points and the singular fibers of the elliptic fibration associated to the pencil are cycles of rational curves.

We think it to be a very interesting problem to prove similar results using relations suggested by other kinds of pencils.

The first author thanks Ivan Pan for useful conversations. He also thanks the support of the “Instituto de Matemática Pura e Aplicada-IMPA” and the “Pontifícia Universidade Católica do Rio de Janeiro-PUC”, where he started working on the present paper.

2. – Preliminaries

We will collect some facts involving line bundles associated to foliations (see [Br1] and [Br2] for more information).

Let S be an irreducible compact curve and $\mathcal{G}$ a foliation of a compact complex surface M.

If S is *not* invariant by $\mathcal{G}$, then

$$T^*_{\mathcal{G}} \cdot S = \mathrm{tang}(\mathcal{G}, S) - S \cdot S\,,$$

where $T^*_{\mathcal{G}}$ is the cotangent bundle of $\mathcal{G}$, $\mathrm{tang}(\mathcal{G}, S) \geq 0$ is the sum of the orders of tangencies between the foliation and the curve and $S \cdot S$ is the self-intersection number of S.

When S is $\mathcal{G}$-invariant, we get

$$T^*_{\mathcal{G}} \cdot S = \mathrm{GSV}(\mathcal{G}, S) + 2p_a(S) - 2\,,$$

where $\mathrm{GSV}(\mathcal{G}, S)$ is the sum of the local indices defined in [G-S-V] and $p_a(C)$ is the arithmetical genus.

Next comes the formula for the variation of the cotangent bundle $T^*_{\mathcal{G}}$ under a blow up $\pi : N \to M$ at a point p:

$$T^*_{\pi^*\mathcal{G}} = \pi^*(T^*_{\mathcal{G}}) \otimes \mathcal{O}_N(-(l(p) - 1)E_p)\,,$$

where E_p is the exceptional divisor. We will need also to deal later with the normal bundle $N_{\mathcal{G}}$ of the foliation $\mathcal{G}$; its variation under blow up's is given by

$$N_{\pi^*\mathcal{G}} = \pi^*(N_{\mathcal{G}}) \otimes \mathcal{O}_N(-l(p)E_p)\,.$$

Finally, given a foliation $\mathcal{F}$ of the projective plane $\mathbb{C}P^2$, there are isomorphisms

$$T^*_{\mathcal{F}} = \mathcal{O}_{\mathbb{C}P^2}(d(\mathcal{F}) - 1) \quad \text{and} \quad N_{\mathcal{F}} = \mathcal{O}_{\mathbb{C}P^2}(d(\mathcal{F}) + 2)\,.$$

Recall that a line bundle (divisor) D over a surface is *nef* (*numerically effective*) if $D \cdot S \geq 0$ for any irreducible curve S of the surface.

We say that a line bundle (divisor) L is *pseudo-effective* if $L \cdot D \geq 0$ for any nef line bundle D. A fundamental fact in the study of holomorphic foliations is Miyaoka's Theorem ([Miy], [McQ1]), which can be stated for algebraic surfaces as: *if $\mathcal{G}$ is not a foliation by rational curves then $T^*_{\mathcal{G}}$ is pseudo-effective*.

We will need the notions of *Kodaira dimension* $k(\mathcal{G})$ and *numerical Kodaira dimension* $\nu(\mathcal{G})$ of a reduced foliation $\mathcal{G}$. The reader may consult [Men], [McQ2] and [Br2] for main properties.

We assign to $\mathcal{G}$ a *non negative* Kodaira dimension if $h^0(M, T^{*\otimes m}_{\mathcal{G}}) \geq 1$ for some $n \in \mathbb{N}$; we define $k(\mathcal{G}) = k \geq 0$ if $h^0(M, T^{*\otimes n}_{\mathcal{G}}) \sim n^k$; if $k \geq 1$, there are global sections $s_0, \cdots s_k$ of $T^{*\otimes n}_{\mathcal{G}}$ giving rise to k algebraically independent rational functions of M:

$$f_1 = \frac{s_1}{s_0}, \cdots, f_k = \frac{s_k}{s_0}\,.$$

For completeness, $k(\mathcal{G}) := -\infty$ if $h^0(M, T_{\mathcal{G}}^{*\otimes n}) = 0 \ \forall n \geq 1$.

We recall the *Zariski decomposition* of a pseudo-effective divisor. Supposing that $T_{\mathcal{G}}^*$ is pseudo-effective, there exists a numerical decomposition

$$T_{\mathcal{G}}^* = P + N\,,$$

where i) some integral multiple of P is a nef divisor; ii) N is a uniquely defined $\mathbb{Q}^+$-divisor, whose support is contractible to a normal singularity of a surface and iii) $P \cdot N = 0$. This enable us to define $\nu(\mathcal{G})$ (without ambiguities) as

1) $\nu(\mathcal{G}) = -\infty$ if $\mathcal{G}$ is a rational fibration.
2) $\nu(\mathcal{G}) = 0$ if P is numerically trivial.
3) $\nu(\mathcal{G}) = 1$ if P is not numerically trivial but $P \cdot P = 0$.
4) $\nu(\mathcal{G}) = 2$ if $P \cdot P > 0$.

Both dimensions are invariants of reduced foliations under birational transformations between smooth surfaces (cf. [Men]) or normal surfaces (cf. [McQ2]). It follows from a Theorem of [McQ2] that, when $k(\mathcal{G}) \geq 0$ one has

$$k(\mathcal{G}) = \nu(\mathcal{G})\,.$$

3. – Proof of Theorem 1

Let us start giving an alternative description for the special invariant curves.

Lemma 1. *Let $\mathcal{G}$ be a foliation with reduced singularities on a projective surface, whose cotangent bundle $T_{\mathcal{G}}^*$ is pseudo-effective. Then, an irreducible curve S is a special invariant curve if and only if $T_{\mathcal{G}}^* \cdot S < 0$.*

Proof. The only if part is immediate from the intersection formula given in the Preliminaries.

Let us suppose that $T_{\mathcal{G}}^* \cdot S < 0$ and prove that S is a special invariant curve for $\mathcal{G}$. Since $T_{\mathcal{G}}^*$ is pseudo-effective, then $S \cdot S < 0$.

We claim that S is $\mathcal{G}$-invariant. Otherwise we would have

$$T_{\mathcal{G}}^* \cdot S = \operatorname{tang}(\mathcal{G}, S) - S \cdot S$$

and $T_{\mathcal{G}}^* \cdot S > 0$. Now we assert that S is smooth and isomorphic to $\mathbb{C}P^1$. One has

$$T_{\mathcal{G}}^* \cdot S = \operatorname{GSV}(\mathcal{G}, S) + 2p_a(S) - 2\,,$$

and $\operatorname{GSV}(\mathcal{G}, S) \geq 0$ because $\mathcal{G}$ has reduced singularities; thus $p_a(S) \leq 0$. Since S is an irreducible curve, $p_a(S) \geq 0$, that is, $p_a(S) = 0$ and this implies that S is in fact a smooth curve, isomorphic to $\mathbb{C}P^1$ (cf. [B-P-V], pg. 68). It follows from $T_{\mathcal{G}}^* \cdot S < 0$ and $p_a(S) = 0$ that $\operatorname{GSV}(\mathcal{G}, S) \in \{0, 1\}$. Let us remark that

$\mathrm{GSV}(\mathcal{G}, S) = 0$ occurs only when there are no singularities of the foliation along the *smooth* curve S, which is impossible in the present situation because $S \cdot S < 0$.

We consider finally the case $\mathrm{GSV}(\mathcal{G}, S) = 1$, that is, $\mathrm{Sing}(\mathcal{G}) \cap S = \{q\}$ and the vanishing order of $\mathcal{G}$ along C is one. Since the leaf $S \setminus \{q\}$ is the complex line $\mathbb{C}$, the group of holonomy of this leaf is trivial. There are only two possibilities for the singularity q since it is a reduced singularity: either it is a saddle-node singularity or it has two different eigenvalues whose quotient is a negative rational number (the singularity is said to belong to the Siegel domain). If q is a saddle-node singularity, the separatrix contained in S can not be the strong one because such a separatrix has a non periodic local holonomy diffeomorphism. It follows that the vanishing order of $\mathcal{G}$ along S is greater than 1, contradiction. Finally, when q belongs to the Siegel domain, since the local holonomy diffeomorphism of the separatrix contained in S is the identity map, we may invoke [M-M] (Teorema 2, pg. 482): if the local holonomy diffeomorphism is conjugated to a linear one (which is obviouly our case), the foliation is locally equivalent to $mydx + xdy = 0$ near the singularity, so that it admits a holomorphic first integral equivalent to $f(x, y) = x^m \cdot y$ ($m \in \mathbb{N}$). Consequently, S is a special invariant curve for the foliation $\mathcal{G}$. □

Before proving Theorem 1, let us do a little digression to show that the numbers $L(\mathcal{F})$ and $L(\mathcal{F}, C)$ (cf. Introduction for notation and Definitions) are well defined, since they could depend on the order we perform blow-up's to reach the reduction of singularities. We consider only $L(\mathcal{F})$, the argument for $L(\mathcal{F}, C)$ being analogous.

First of all, if q is the singularity of a (locally defined) foliation $\mathcal{G}$, we may take its reduction of singularities and associate $L(\mathcal{G}, q)$ as above. To the singularity q we attach the number $b = b(\mathcal{G}, q)$ of blow-up's needed to reach its reduction; this number does not depend on the order the blow-up's are applied. We claim that the association L is well defined; we proceed by induction on b. The claim is trivially true if $b = 0$; suppose it is true for singularities of foliations such that $b \leq n$. Let us take a singularity q of a foliation $\mathcal{G}$ with $b = n+1$ and blow it up once; we get a new foliation $\mathcal{G}'$ with singularities $q_1, \ldots, q_l$ along the exceptional divisor. Now $L(\mathcal{G}, q) = (l(q) - 1)^2 + \sum_j L(\mathcal{G}', q_j)$, and since $b(\mathcal{G}', q_j) \leq n$, for $j = 1, \ldots, l$, the induction hypothesis can be used to grant that each $L(\mathcal{G}', q_l)$ independs on the order blow-up's are applied. Therefore the claim is also true for q.

All this can be applied to $\mathcal{F}$ by noticing that $L(\mathcal{F}) = \sum_p L(\mathcal{F}, p)$, where p is a singularity of $\mathcal{F}$.

Let us then prove Theorem 1. Consider a foliation $\mathcal{F}$ of the projective plane which has a reduction of singularities $\sigma^*\mathcal{F}$ free of special invariant curves.

If $d(\mathcal{F}) = 0$ then $\mathcal{F}$ is the pencil of lines passing by one point, which is a radial singularity, and then the first assertion of Theorem 1 is a trivial equality.

We claim now that Miyaoka's Theorem implies that $T^*_{\sigma^*\mathcal{F}}$ is a pseudoeffective line bundle, if $d(\mathcal{F}) \geq 1$. The reason is that $\sigma^*\mathcal{F}$ is not a rational fibration:

in fact, the reduction of singularities of any pencil of plane rational curves $\mathcal{F}$ with $d(\mathcal{F}) \geq 1$ is a rational fibration having at least one singular fiber (see e.g. [Br1]). Now, a singular fiber of a rational fibration is a tree of rational curves (see [B-P-V], pg. 142), so in this fiber there exists necessarily a special invariant curve, contradicting our hypothesis on $\sigma^*\mathcal{F}$.

We may therefore apply Lemma 1 and conclude that $T^*_{\sigma^*\mathcal{F}}$ is a nef line bundle. Since $T^*_{\sigma^*\mathcal{F}}$ is simultaneously nef and pseudo-effective divisor, we have

$$T^*_{\sigma^*\mathcal{F}} \cdot T^*_{\sigma^*\mathcal{F}} \geq 0\,.$$

In order to compute this number from data of the foliation, we use the line bundle isomorphism on $\mathbb{C}P^2$, $T^*_{\mathcal{F}} = \mathcal{O}(d(\mathcal{F}) - 1)$ and the formula for the variation of the cotangent bundle under a blow up (cf. Preliminaries). Applying repeatedly this formula and using the definition of $L(\mathcal{F})$, one gets

$$T^*_{\sigma^*\mathcal{F}} \cdot T^*_{\sigma^*\mathcal{F}} = (d(\mathcal{F}) - 1)^2 - L(\mathcal{F})\,,$$

so that the inequality $L(\mathcal{F}) \leq (d(\mathcal{F} - 1)^2$ follows.

Let now $\overline{C} := \sigma^*(C) - \sum_j \nu_{p_j} E_j$ be the strict transform of a plane curve C by the reduction of singularities; E_j stands for the exceptional divisor introduced at the $(j-1)$-th step of the desingularization. We have then

$$T^*_{\sigma^*\mathcal{F}} \cdot \overline{C} = d(C) \cdot (d(\mathcal{F}) - 1) - L(\mathcal{F}, C)\,.$$

Since $T^*_{\sigma^*\mathcal{F}}$ is a nef divisor, we conclude that $L(\mathcal{F}, C) \leq d(C) \cdot (d(\mathcal{F}) - 1)$. □

4. – Relation to Halphen pencils

In order to prove Theorem 2, we will need some Lemmata. Let us take once more a foliation $\mathcal{F}$ of the projective plane whose reduction of singularities $\sigma^*\mathcal{F}$ is free of special invariant curves; we assume also that its singularities are *a*) dicritical singularities $p_1, \ldots, p_k$ removable by one blow up and *b*) singular points with local holomorphic first integral.

Under these assumptions:

Lemma 2. *Let $l \geq 2$. The following conditions are equivalent.*

i) $d(\mathcal{F}) = 3l - 2$ *and* $l(p_i) = l$ *for* $1 \leq i \leq k$.

ii) $k = 9$, $\sum_{i=1}^{9} (l(p_i) - 1)^2 = (d(\mathcal{F}) - 1)^2$ *and* $\sum_{i=1}^{9} (l(p_i) - 1) = 3(d(\mathcal{F}) - 1)$.

Proof. Suppose that *i*) holds. The reduction of singularities $\sigma : N \to \mathbb{C}P^2$ of $\mathcal{F}$ can be regarded as a composition $\sigma = \sigma_2 \circ \sigma_1$, where $\sigma_1 : N' \to \mathbb{C}P^2$ denotes the composition of k blow ups at $p_1, \cdots, p_k$.

According to Baum-Bott's formula (cf. [B-B] or [Br1]):

$$N_{\sigma_1^*\mathcal{F}} \cdot N_{\sigma_1^*\mathcal{F}} = \sum_{q \in N'} BB(\sigma_1^*\mathcal{F}, q)\,,$$

where $BB(\sigma_1^*\mathcal{F}, q) \in \mathbb{C}$ is the Baum-Bott index at a singularity q of $\sigma_1^*\mathcal{F}$. By the hypothesis on the singular set of $\mathcal{F}$, all the singularities of $\sigma_1^*\mathcal{F}$ have local holomorphic first integrals. It is known that in this case $BB(\sigma_1^*\mathcal{F}, q)$ is a non-positive integer; we conclude that

$$N_{\sigma_1^*\mathcal{F}} \cdot N_{\sigma_1^*\mathcal{F}} \leq 0\,.$$

In order to compute this number, we proceed in the same way as in the proof of Theorem 1. One has the isomorphism

$$N_{\mathcal{F}} = \mathcal{O}_{\mathbb{C}P^2}(d(\mathcal{F}) + 2)$$

and the formula for the variation of the normal bundle of a foliation under a blow up (cf. Preliminaries). Applying repeatedly this formula one gets

$$N_{\sigma^*\mathcal{F}} \cdot N_{\sigma^*\mathcal{F}} = (d(\mathcal{F}) + 2)^2 - \sum_{i=1}^{k} l(p_i)^2\,,$$

so that $(d(\mathcal{F}) + 2)^2 - \sum_{i=1}^{k} l(p_i)^2 \leq 0$. Since $l(p_i) = l$ (for $i = 1, \cdots, k$), we get[(1)]:

$$(d(\mathcal{F}) + 2)^2 \leq k \cdot l^2\,,$$

that is, $9 \leq k$, since $d(\mathcal{F}) = 3l - 2$. By another side, according to Theorem 1:

$$\sum_{i=1}^{k} (l(p_i) - 1)^2 \leq (d(\mathcal{F}) - 1)^2\,,$$

that is, $k \cdot (l - 1)^2 \leq (3l - 3)^2$ and then $k \leq 9$, so that $k = 9$ as desired. The other two equalities of $ii)$ follow trivially.

Suppose now that $ii)$ holds. Let us show that the solutions of the equations

$$\sum_{i=1}^{9} (l(p_i) - 1)^2 = (d(\mathcal{F}) - 1)^2 \quad \text{and} \quad \sum_{i=1}^{9} (l(p_i) - 1) = 3(d(\mathcal{F}) - 1)$$

are given by $d(\mathcal{F}) = 3l - 2$ and $l(p_i) = l$ $(l \geq 2)$.

It is easy to show that $l(p_i) \geq 2$, if p_i is removable by one blow up. Hence $d(\mathcal{F}) \geq 4$ and we may define

$$r_i := \frac{l(p_i) - 1}{d(\mathcal{F}) - 1}$$

and consider the equations

$$\sum_{s=1}^{9} r_i^2 = 1 \quad \text{and} \quad \sum_{s=1}^{9} r_i = 3\,.$$

The unique real solution to this pair of equations is

$$(r_1, \cdots, r_9) = \left(\frac{1}{3}, \cdots, \frac{1}{3}\right)\,.$$

Then $i)$ is proved. □

(1) This inequality is already found in [Po] in the particular case when $\mathcal{F}$ is a pencil of plane curves.

Let us remark at this point that the equality

$$L(\mathcal{F}) := \sum_{i=1}^{9} (l(p_i) - 1)^2 = (d(\mathcal{F}) - 1)^2$$

and Theorem 1 imply that $l(q) = 1$ for all the singularities of $\sigma_1^* \mathcal{F}$. Since q has a holomorphic first integral, its reduction is the same as the reduction of its separatrix; we conclude that the only possibilities are (modulo local change of coordinates) $d(x \cdot y) = 0$, $d(y^2 - x^{2m}) = 0$ and $d(y^2 - x^{2m+1}) = 0$, where $m \in \mathbb{N}$. The last two cases give rise to special invariant curves, so that we are left only with the first case, that is, all the singularities of $\sigma_1^* \mathcal{F}$ are of Morse type. In particular, the reduction σ is given, up to biholomorphisms, by $\sigma = \sigma_1$.

LEMMA 3. *The Kodaira dimension of $\mathcal{F}$ is equal to* 1.

PROOF. First of all let us prove that $T^*_{\sigma^* \mathcal{F}}$ admits a global holomorphic section. As remarked above, $l(q) = 1$ for all the singularities having local holomorphic first integral that appear along the reduction of $\mathcal{F}$; thus, there exists a line bundle isomorphism

$$T^*_{\sigma^* \mathcal{F}} = \sigma^*(\mathcal{O}_{\mathbb{C}P^2}(3(l-1))) \otimes \mathcal{O}_N \left(- \sum_{i=1}^{9} (l-1) E_i \right),$$

where E_i is the exceptional line introduced by the blow up of $p_i, 1 \leq i \leq 9$. Now we observe that a curve $(l-1)C$, where C is a plane cubic curve (not necessarily smooth) passing through the points $p_1, \cdots, p_9$, corresponds to a global holomorphic section of $T^*_{\sigma^* \mathcal{F}}$ (more precisely, consider a section s of $T^*_{\sigma^* \mathcal{F}}$ which vanishes along the transform of a straight line with multiplicity $3(l-1)$ and has poles of order $(l-1)$ along each E_i, $1 \leq i \leq 9$; the global holomorphic section we look for is $\overline{s_1} = (P^{l-1} \circ \sigma) \cdot s$, where $P = 0$ is a reduced affine equation for C). Hence, by definition,

$$k(\mathcal{F}) \geq 0 .$$

Notice at this point that thanks to the inequality

$$\sum_{i=1}^{9} \nu_{p_i}(C) \cdot (l-1) \leq 3(l-1) \cdot d(C) ,$$

stated in Theorem 1, any cubic curve C passing by $p_1 \cdots p_9$ is reduced (that is, C_{red} is not one or two straight lines) and C must be smooth at all the points $p_1, \cdots, p_9$.

Since $T^*_{\sigma^* \mathcal{F}}$ is a pseudo-effective divisor which is also a non-trivial nef divisor with $T^*_{\sigma^* \mathcal{F}} \cdot T^*_{\sigma^* \mathcal{F}} = 0$, by definition the numerical Kodaira dimension is

$$\nu(\mathcal{F}) = 1$$

and we conclude (cf. Preliminaries) that $k(\mathcal{F}) = 1$. □

REMARK 1. As a consequence of Lemma 3 we have that $h^0(N, T^{*\otimes n}_{\sigma^*\mathcal{F}}) \geq 2$ for some $n \geq 1$. Taking independent global sections $s_1 = \overline{s_1}^{\,n}$ and s_2 of $T^{*\otimes n}_{\sigma^*\mathcal{F}}$, we define the rational mapping

$$\mathcal{H}' : N \to \mathbb{C}P^1 ,$$
$$\mathcal{H}'(x) := (s_1(x) : s_2(x)) .$$

This map is free of indetermination points, that is, it defines a fibration $\mathcal{H}' : N \to \mathbb{C}P^1$; the reason is that the divisors $(s_1)_0$ and $(s_2)_0$ do not intersect:

$$(s_1)_0 \cdot (s_2)_0 = n^2 \cdot T^*_{\sigma^*\mathcal{F}} \cdot T^*_{\sigma^*\mathcal{F}} = 0 .$$

By construction, $(\mathcal{H}'^{-1}(\infty))_{\mathrm{red}}$ is the strict transform $\bar{C}$ of C by σ (the fact that the curve C is smooth at the points $p_1, \cdots, p_9$ is crucial here), so that it is connected and the exceptional divisors $E_1, \cdots, E_9$ are multisections to $\mathcal{H}'$. Also, $\mathcal{H}'^{-1}(\infty)$ has multiplicity $n(l-1)$ for the map $\mathcal{H}'$. Using Stein Factorization ([B-P-V]), we may change $\mathcal{H}'$ by $\mathcal{H}$ in order to have connected fibers (notice that $(\mathcal{H}'^{-1}(\infty))_{\mathrm{red}} = (\mathcal{H}^{-1}(\infty))_{\mathrm{red}}$ although the multiplicity may change); a fiber H' of $\mathcal{H}'$ becomes linearly equivalent to mH, where $m \in \mathbb{N}$ and H is a fiber of $\mathcal{H}$ (in particular we have $T^*_{\sigma^*\mathcal{F}} \cdot H = 0$).

LEMMA 4. *$\mathcal{H}$ is an elliptic fibration which is relatively minimal, that is, the fibers of $\mathcal{H}$ are free from exceptional curves.*

PROOF. Since mH is linearly equivalent to $(s_1)_0$, the well-known formula $K_N = \sigma^*(K_{\mathbb{C}P^2}) \otimes \mathcal{O}_N(\sum_i^9 E_i)$ for the variation of canonical line bundles gives:

$$\begin{aligned}
\chi(mH) = \chi((s_1)_0) &= -T^{*\otimes n}_{\sigma^*\mathcal{F}} \cdot (T^{*\otimes n}_{\sigma^*\mathcal{F}} \otimes K_N) \\
&= -T^{*\otimes n}_{\sigma^*\mathcal{F}} \cdot K_N \\
&= -[T^{*\otimes n}_{\mathcal{F}} \cdot K_{\mathbb{C}P^2} + \sum_{i=1}^{9} (l(p_i) - 1)n] \\
&= -[-9n(l-1) + 9n(l-1)] = 0
\end{aligned}$$

and also $\chi(H) = 0$. Therefore $\mathcal{H}$ is an elliptic fibration.

One has also that $\mathcal{H}$ is relatively minimal. In fact, if we suppose that there are exceptional curves contained in fibers, after contracting all such curves we get a rational surface N' with a relatively minimal elliptic fibration. The Euler number of N' verifies $3 \leq e(N') < 12$ (the first inequality comes from $e(\mathbb{C}P^2) = 3$ and the second from the fact that $e(N) = 12$). But $e(N')$ must be a multiple of 12, because of *a*) Noether's formula $12 \cdot \chi(\mathcal{O}_{N'}) = K_{N'}.K_{N'} + e(N')$, where $\chi(\mathcal{O}_{N'})$ is the holomorphic Euler characteristic and $K_{N'}$ is the canonical bundle of N' and *b*) $K_{N'}.K_{N'} = 0$, which is an immediate consequence of Kodaira's formula for the canonical bundle of an elliptic surface, cf. [F-M], Proposition 3.21. We have then a contradiction. □

After contracting the exceptional divisors $E_1, \cdots, E_9$, we conclude that $\sigma_*(\mathcal{H})$ is a Halphen pencil of index $l'-1$, $l' \geq 2$, whose base points are $p_1, \cdots, p_9$. It is worth noticing that $\mathcal{H}$ has at least one singular fiber, since the Euler characteristic of N is positive.

LEMMA 5. *Any irreducible component of a singular fiber of $\mathcal{H}$ is $\sigma^*\mathcal{F}$-invariant.*

PROOF. Let us consider an irreducible component $H_{j,s}$ of a singular fiber $H_s = \sum_j n_{j,s} H_{j,s}$, with $n_{j,s} > 0$. Kodaira's classification of singular fibers of a relatively minimal elliptic fibrations (see [B-P-V], pg. 150) tell us that $H_{j,s} \cdot H_{j,s} = -2$ when $H_{j,s}$ is smooth; in any case, if we suppose that $H_{j,s}$ is not $\sigma^*\mathcal{F}$-invariant, we get

$$0 < \operatorname{tang}(\sigma^*\mathcal{F}, H_{j,s}) - H_{j,s}^2 = T^*_{\sigma^*\mathcal{F}} \cdot H_{j,s}\,.$$

Since $T^*_{\sigma^*\mathcal{F}}$ is a nef divisor, $T^*_{\sigma^*\mathcal{F}} \cdot H_{j,s} \leq T^*_{\sigma^*\mathcal{F}} \cdot H_s$. In order to finish the proof, we use that $T_{\sigma^*\mathcal{F}} \cdot H_s = 0$ (see Remark 1), contradiction. $\square$

Let us see how Lemma 5 restricts the possibilities for a singular fiber of $\mathcal{H}$. We use again Kodaira's classification of singular fibers of a (relatively minimal) elliptic fibration.

LEMMA 6. *$\mathcal{H} = \sigma^*\mathcal{F}$ (as foliations); the singular fibers are of type I_b, $b > 0$ (nodal rational curve or cycle of rational curves).*

PROOF. Let H_s be a singular fiber of $\mathcal{H}$.

If H_s is of type II, III, or IV, we conclude that $\sigma^*\mathcal{F}$ has a non-reduced singularity, a contradiction.

If H_s is of type I_b^*, II^*, III^* or IV^*, there exists a smooth component H_{s_0} of H_s which intersects three other components; in particular, there are at least three singular points of $\sigma^*\mathcal{F}$ along H_{s_0}. It follows that $\mathrm{GSV}(\sigma^*\mathcal{F}, H_{s_0}) \geq 3$ and hence

$$T^*_{\sigma^*\mathcal{F}} \cdot H_{s_0} = \mathrm{GSV}(\sigma^*\mathcal{F}, H_{s_0}) - 2 > 0\,.$$

Since $T^*_{\sigma^*\mathcal{F}}$ is a nef divisor, then:

$$T^*_{\sigma^*\mathcal{F}} \cdot H_s \geq T^*_{\sigma^*\mathcal{F}} \cdot H_{s_0} > 0\,,$$

contradicting $T^*_{\sigma^*\mathcal{F}} \cdot H_s = 0$.

We are then left only with critical fibers of type $I_b, b > 0$. To finish the proof we have to show that $\mathcal{H} = \sigma^*\mathcal{F}$ (as foliations). The Kodaira's functional invariant of a fiber of type $I_b, b > 0$ is ∞ (cf. [B-P-V], pg. 159), meaning in particular that the analytical type of elliptic fibers in a neighborhood of I_b varies. Suppose by absurd that $\mathcal{H} \neq \sigma^*\mathcal{F}$. Then $\sigma^*\mathcal{F}$ is transverse to $\mathcal{H}$ at all points of the (generic) fiber H of $\mathcal{H}$; in fact, $T^*_{\sigma^*\mathcal{F}} \cdot H = 0$ and $H \cdot H = 0$ imply that

$$0 = T^*_{\sigma^*\mathcal{F}} \cdot H = \operatorname{tang}(\sigma^*\mathcal{F}, H)\,.$$

The transverse foliation can be used to define holomorphic diffeomorphisms between smooth fibers; consequently, the analytic type of the generic fibers of $\mathcal{H}$ is constant, contradiction. $\square$

We may finish now the proof of Theorem 2.

We have shown before that $\sigma_*\mathcal{H}$ a Halphen pencil of index $l'-1$ having 9 distinct base-points in the plane; we wish to prove that $l'=l$. We know that the generic fiber of the pencil has an ordinary singularity of multiplicity $l'-1$ at each base-point. Therefore, at each such a point we have a local expression for the pencil as the family of level curves of the meromorphic function $x^{-(l'-1)}(\prod_{j=1}^{l'-1}(y-c_jx)+\dots)$, where $x=0$ denotes the local (reduced) equation of the multiple fiber and $(\prod_{j=1}^{l'-1}(y-c_jx)+\dots)=0$ gives the local expression of the generic fiber. Or, as a differential equation

$$d\left[x^{-(l'-1)}\left(\prod_{j=1}^{l'-1}(y-c_jx)+\dots\right)\right]=0$$

which is also (for $A(x,y):=\prod_{j=1}^{l'-1}(y-c_jx)$)

$$A(x,y)\{(l'-1-x\sum_{j=1}^{l'-1}c_j(y-c_jx)^{-1})dx+x\sum_{j=1}^{l'-1}(y-c_jx)^{-1}dy\}+\dots=0\,;$$

this equation has vanishing order $l'-1$ at the base point, and since we have already proven that $\mathcal{F}=\sigma_*\mathcal{H}$, it follows that $l=l'$. This ends the proof.

It is not difficult to find examples of elliptic fibrations with the properties we have just described, although we are not able to give explicit formulae. The Hesse pencil in the plane is given in homogeneous coordinates by:

$$X_0^3+X_1^3+X_2^3+\lambda X_0X_1X_2=0,\quad \lambda\in\mathbb{C}P^1\,,$$

produces after reduction of singularities an elliptic fibration with 4 singular fibers of type I_3; it has no multiple fibers. In order to create a multiple fiber, we may for example select a smooth fiber of the fibration (type I_0) and apply to it a logarithmic transformation (cf. [B-P-V], pg. 164) to become a multiple fiber with smooth reduction (type ${}_mI_0$). Although logarithmic transformations are quite different from birational transformations, in our case we still end up with a *rational* surface, more precisely the projective plane blown up at 9 points.

In order to see this, we use some facts from the theory of elliptic surfaces (cf. [F-M], sections 1.3.5 and 1.3.6). Let M be a relatively minimal elliptic surface of Euler number $e(M)$ that fibers over $\mathbb{C}P^1$. Suppose there is one multiple fiber F_1 of multiplicity $m\in\mathbb{N}$. Then Kodaira's formula for the canonical bundle K_M is

$$K_M=\pi^*(K_{\mathbb{C}P^1}+\mathcal{L})+(m-1)F_1$$

where $\pi:M\to\mathbb{C}P^1$ is the fibration, $K_{\mathbb{C}P^1}$ is the canonical bundle of $\mathbb{C}P^1$ and $\mathcal{L}$ is a line bundle over $\mathbb{C}P^1$ of degree equal to $\chi(\mathcal{O}_M)$, the holomorphic Euler characteristic of M; in particular, $K_M.K_M=0$, as we have used in

Lemma 4. In the case of our construction $e(M) = 12$, so that by Noether's formula $12 \cdot \chi(\mathcal{O}_M) = K_M.K_M + e(M)$ we get $\chi(\mathcal{O}_M) = 1$. Consequently

$$K_M = -F + (m-1)F_1 = -mF_1 + (m-1)F_1 = -F_1\,,$$

where F is the generic fiber of the fibration. We conclude that

$$h^0(M, K_M^{\otimes n}) = h^0(M, -nF_1) = 0$$

and the Kodaira dimension $k(M) = -\infty$. The surface M is then the projective plane blown up at 9 points (Proposition 3.23, [F-M]).

We finish this section with some comments about this construction:

1) It may be applied to any of the critical fibers of the Hesse fibration.

2) If we introduce two multiple fibers in the construction we no longer get a rational surface (the Kodaira dimension is no longer $-\infty$).

3) The same kind of ideas used in the example above allows us to show that the Halphen pencils that appear in Theorem 2 have singular fibers of reduced type I_b for $b = 1, 2$ or 3. In fact, if a singular fiber H_s is the multiple fiber of the pencil, it comes from a cubic curve in the plane, so that the statement is trivially true. If not, we start by turning the multiple fiber of the pencil (whose reduction is of type I_b for $0 \leq b \leq 3$) into a fiber without multiplicity by means of a logarithmic transformation; the new surface is still rational. Then we make H_s into the multiple fiber of an elliptic fibration of a rational surface, once more using a logarithmic transformation. We are back to the initial case, therefore the reduced type is I_b for $b = 1, 2$ or 3.

5. – Final remarks

Let us discuss a more direct approach for finding the elliptic fibration $\mathcal{H}$ of the last section (cf. Lemmata 3, 4).

We take any cubic curve C that passes through the dicritical singularities $p_1, \cdots, p_9$; we know from Theorem 1 that C is reduced, so that the possibilities are (modulo change of coordinates and leaving aside for the moment the smooth case):

1) a cubic curve with a cusp $y^2 - x^3 = 0$;
2) a conic tangent to a straight line $y \cdot (y - x^2) = 0$;
3) three concurrent lines $x \cdot y \cdot (y - x) = 0$;
4) a rational curve with a node $y^2 = x^2 \cdot (x - 1)$;
5) a conic and a line meeting at two points $(y - 1) \cdot (y - x^2) = 0$;
6) three lines forming a triangle $x \cdot y \cdot (x + y - 1) = 0$.

Theorem 1 implies that C is smooth at the points $p_1, \cdots, p_9$. Also, for cases 2), 3), 5), 6) above, Theorem 1 implies that there are at most 3 dicritical singularities in components of C having degree one and at most 6 dicritical singularities in components of C having degree two. Since the cubic C passes by all 9 dicritical points, then there are exactly 3 dicritical points along the components of degree 1 and exactly 6 dicritical singularities along the components of degree 2. Let us denote a component of C by C' and let $\overline{C'}$ be the strict transform by σ. It follows that $\overline{C'} \cdot \overline{C'} = -2$ and if C' has degree one:

$$T^*_{\sigma^*\mathcal{F}} \cdot \overline{C'} = 3(l-1) - 3(l-1) = 0\,;$$

also if C' has degree two:

$$T^*_{\sigma^*\mathcal{F}} \cdot \overline{C'} = 6(l-1) - 6(l-1) = 0\,,$$

If C' were not $\mathcal{F}$-invariant, we would find in both cases:

$$\operatorname{tang}(\sigma^*\mathcal{F}, \overline{C'}) = -2\,,$$

which is impossible. Therefore all the possibilities 2), 3), 5), 6) correspond to $\mathcal{F}$-invariant cubics.

As for the cases 1) and 4), one has that $T^*_{\sigma^*\mathcal{F}} \cdot \overline{C} = 0$ and $\overline{C} \cdot \overline{C} = 0$, so that $\operatorname{tang}(\sigma^*\mathcal{F}, \overline{C}) = 0$ if C is not invariant, which again is impossible due to the presence of a cusp or a node of $\overline{C}$. Therefore we may then assume in all cases above that (each component of) the cubic curve C is $\mathcal{F}$-invariant; let us allow the possibility of C being smooth and $\mathcal{F}$-invariant. Using that the non-dicritical singularities of $\mathcal{F}$ are reduced (cf. remark after the proof of Lemma 2), we exclude cases 1), 2) and 3).

Let us look now to a global rational vector field $Z_{\mathcal{F}}$ inducing $\mathcal{F}$, having isolated zeros at the singularities of $\mathcal{F}$ and pole of order $d(\mathcal{F}) - 1 = 3 \cdot (l-1)$ along the line at infinity L_∞ (L_∞ can be supposed transverse to C). Let $Z_{\mathcal{P}}$ be a rational vector field which extends (with a pole of order one along L_∞) a polynomial vector field inducing the foliation $dP = 0$, where $P = 0$ is a reduced affine equation of C along $\mathbb{C}P^2 - L_\infty$. The singularities $p_1, \cdots, p_9$ of $\mathcal{F}$ being removable by one blow up, then the restriction $(Z_{\mathcal{F}})_{|\,C}$ is a rational vector field with zeroes of multiplicity $l-1$ at these points; also, the zeroes of $(Z_{\mathcal{F}})_{|\,C}$ at each branch of the singular points of C (when they exist) have multiplicity one. By another side, the vector field $(Z_{\mathcal{P}})_{|\,C}$ has zeroes with multiplicity one at the same branches. There is a meromorphic function f over C verifying:

$$(Z_{\mathcal{F}})_{|\,C} = f \cdot (Z_{\mathcal{P}})_{|\,C}$$

and we observe that f has zeroes of order $l-1$ at the points $p_1, \cdots, p_9$ and poles of order $3(l-1)$ at the points q_1, q_2 and q_3 of $C \cap L_\infty$. Therefore the divisor $D = \sum_{k=1}^{9} p_k - \sum_{j=1}^{3} q_j$ of C has torsion: $(l-1)D = (f)$ is principal.

The same is true, although by a different reason, when C is smooth but not $\mathcal{F}$-invariant. We consider the meromorphic function $f = (dP \cdot Z_{\mathcal{F}})_{|\,C}$ along C, where $P = 0$ is a reduced affine equation of C; we show easily that $p_1, \cdots, p_9$

are zeroes of order $l-1$ (this is the vanishing order of $Z_{\mathcal{F}}$ at each of these singularities). Again if we consider the divisor $D=\sum_{k=1}^{9} p_k - \sum_{j=1}^{3} q_j$, then $(l-1)D=(f)$ is principal. It is important to notice that D is special in the sense that its support is disjoint from the singular points of C.

Starting from the meromorphic function f and the torsion divisor D above, there exists a well known way to move C inside a pencil of elliptic curves as to have that C appears as the support of a $(l-1)$-multiple curve. The reader may consult [S] (Theorem 7), for a smooth support; it extends naturally to our situation, because of the property $\mathrm{supp}(D) \cap \mathrm{sing}(C) = \emptyset$.

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